APPLIED MECHANICS.

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PREFACE.

THIS book is the result of the experience of the writer in teaching the subject of Applied Mechanics for the last twelve years at the Massachusetts Institute of Technology.

The immediate object of publishing it is, to enable him to dispense with giving to the students a large amount of notes. As, however, it is believed that it may be found useful by others, the following remarks in regard to its general plan are submitted.

The work is essentially a treatise on strength and stability; but, inasmuch as it contains some other matter, it was thought best to call it "Applied Mechanics," notwithstanding the fact that a number of subjects usually included in treatises on applied mechanics are omitted.

It is primarily a text-book; and hence the writer has endeavored to present the different subjects in such a way as seemed to him best for the progress of the class, even though it be at some sacrifice of a logical order of topics. While no attempt has been made at originality, it is believed that some features of the work are quite different from all previous efforts; and a few of these cases will be referred to, with the reasons for so treating them.

In the discussion upon the definition of "force," the object is, to make plain to the student the modern objections to the usual ways of treating the subject, so that he may have a clear conception of the modern aspect of the question, rather than to support the author's definition, as he is fully aware that this, as well as all others that have been given, is open to objection.

In connection with the treatment of statical couples, it was thought best to present to the student the actual effect of the action of forces on a rigid body, and not to delay this subject until dynamics of rigid bodies is treated, as is usually done.

In the common theory of beams, the author has tried to make plain the assumptions on which it is based. A little more prominence than usual has also been given to the longitudinal shearing of beams.

In that part of the book that relates to the experimental results on strength and elasticity, the writer has endeavored to give the most reliable results, and to emphasize the fact, that, to obtain constants suitable for use in practice, we must deduce them from tests on full-size pieces. This principle of being careful not to apply experimental results to cases very different from those experimented upon, has long been recognized in physics, and therefore needs no justification.

The government reports of tests made at the Watertown Arsenal have been extensively quoted from, as it is believed that they furnish some of our most reliable information on these subjects.

The treatment of the strength of timber will be found to be quite different from what is usually given; but it speaks for itself, and will not be commented upon here.

In the chapter on the "Theory of Elasticity," a combination is made of the methods of Rankine and of Grashof.

In preparing the work, the author has naturally consulted the greater part of the usual literature on these subjects; and, whenever he has drawn from other books, he has endeavored to acknowledge it. He wishes here to acknowledge the assistance furnished him by Professor C. H. Peabody of the Massachusetts Institute of Technology, who has read all the proofs, and has aided him materially in other ways in getting out the work.

GAETANO LANZA.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
April, 1885.

PREFACE TO THE FOURTH EDITION.

THE principal differences between this and the earlier editions consist in the introduction of the results of a large amount of the experimental work that has been done during the last five years upon the strength of materials.

The other changes that have been made in the book are not a great many, and have been suggested as desirable by the author's experience in teaching.

September, 1890.

PREFACE TO THE SEVENTH EDITION.

THE principal improvements in this edition consist in the introduction, in Chapter VII, of the results of a considerable amount of the experimental work on the strength of materials that has been done during the last six years. A few changes have also been made in other parts of the book.

October, 1896.

PREFACE TO THE EIGHTH EDITION.

IN this edition a considerable number of additional results of recent tests, especially upon full-size pieces, have been introduced, some of the older ones having been omitted to make room for them.

September, 1900.

PREFACE TO THE NINTH EDITION.

THE principal improvements in the Ninth Edition consist in very extensive changes in Chapter VII, in order to bring the account of the experimental work that has been performed in various places up to date.

Some changes have also been made in the mathematical portion of the book, especially in the Theory of Columns.

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APPLIED MECHANICS.

CHAPTER I.

COMPOSITION AND RESOLUTION OF FORCES.

§ 1. **Fundamental Conceptions.** — The fundamental conceptions of Mechanics are Force, Matter, Space, Time, and Motion.

§ 2. **Relativity of Motion.** — The limitations of our natures are such that all our quantitative conceptions are relative. The truth of this statement may be illustrated, in the case of motion, by the fact, that, if we assume the shore as fixed in position, a ship sailing on the ocean is in motion, and a ship moored in the dock is at rest; whereas, if we assume the sun as our fixed point, both ships are really in motion, as both partake of the motion of the earth. We have, moreover, no means of determining whether any given point is absolutely fixed in position, nor whether any given direction is an absolutely fixed direction. Our only way of determining direction is by means of two points assumed as fixed; and the straight line joining them, we are accustomed to assume as fixed in direction. Thus, it is very customary to assume the straight line joining the sun with any fixed star as a line fixed in direction; but if the whole visible universe were in motion, so as to change the absolute direction of this line, we should have no means of recognizing it.

§ 3. **Rest and Motion.** — In order to define rest and motion, we have the following; viz., —

When a single point is spoken of as having motion or rest, some other point is always expressed or understood, which is for the time being considered as a fixed point, and some direction is assumed as a fixed direction: and we then say that the first-named point is at rest relatively to the fixed point, when the straight line joining it with the fixed point changes neither in length, nor in direction; whereas it is said to be in motion relatively to the fixed point, when this straight line changes in length, in direction, or in both.

If, on the other hand, we had considered the first-named point as our fixed point, the same conditions would determine whether the second was at rest, or in motion, relatively to the first.

A body is said to be at rest relatively to a given point and to a given direction, when all its points are at rest relatively to this point and this direction.

§ 4. **Velocity and Acceleration.**—When the motion of one point relatively to another, or of one body relatively to another, is such that it describes equal distances in equal times, however small be the parts into which the time is divided, the motion is said to be uniform and the velocity constant.

The velocity, in this case, is the space passed over in a unit of time, and is to be found by dividing the space passed over in any given time by the time; thus, if s represent the space passed over in time t, and v represent the velocity, we shall have

$$v = \frac{s}{t}.$$

When the motion is not uniform, if we divide the time into small parts, and then divide the space passed over in one of these intervals by the time, and then pass to the limit as these intervals of time become shorter, we shall obtain the velocity

Thus, if Δs represent the space passed over in the interval of time Δt, then we shall have

$$v = \text{limit of } \frac{\Delta s}{\Delta t} \text{ as } \Delta t \text{ diminishes,}$$

or

$$v = \frac{ds}{dt}.$$

In this case the rate of change of velocity per unit of time is called the *Acceleration*, and if we denote it by f, we have

$$f = \frac{dv}{dt} = \frac{d^2 s}{dt^2}.$$

§ 5. **Force.** — We shall next attempt to obtain a correct definition of force, or at least of what is called force in mechanics.

It may seem strange that it should be necessary to do this; as it would appear that clear and correct definitions must have been necessary in order to make correct deductions, and therefore that there ought to be no dispute whatever over the meaning of the word *force.* Nevertheless, it is a fact in mechanics, as well as in all those sciences which attempt to deal with the facts and laws of nature, that correct definitions are only gradually developed, and that, starting with very imperfect and often erroneous views of natural laws and phenomena, it is only after these errors have been ascertained and corrected by a long range of observation and experiment, and an increased range of knowledge has been acquired, that exactness and perspicuity can be obtained in the definitions.

Now, this is precisely what has happened in the case of force.

In ancient times *rest* was supposed to be the natural state of bodies; and it was assumed that, in order to make them move, force was necessary, and that even after they had been set in motion their own innate inertia or sluggishness would cause them to come to rest unless they were constantly urged on by the application of some force, the bodies coming to rest whenever the force ceased acting.

It was under the influence of these vague notions that such terms arose as *Force of Inertia*, *Moment of Inertia*, *Vis Viva* or *Living Force*, etc.

A number of these terms are still used in mechanics; but in all such cases they have been re-defined, such new meanings having been attached to them as will bring them into accord with the more advanced ideas of the present time. Such definitions will be given in the course of this work, as the necessity may arise for the use of the terms.

NEWTON'S FIRST LAW OF MOTION.

Ideas becoming more precise, in course of time there was framed Newton's first law of motion; and this law is as follows: —

A body at rest will remain at rest, and a body in motion will continue to move uniformly and in a straight line, unless and until some external force acts upon it.

The assumed truth of this law was based upon the observed facts of nature; viz., —

When bodies were seen to be at rest, and from rest passed into a state of motion, it was always possible to assign some cause; i.e., they had been brought into some new relationship, either with the earth, or with some other body: and to this cause could be assigned the change of state from rest to motion. On the other hand, in the case of bodies in motion, it was seen, that, if a body altered its motion from a uniform rectilinear motion, there was always some such cause that could be assigned. Thus, in the case of a ball thrown from the hand, the attraction of the earth and the resistance of the air soon caused it to come to rest. In the case of a ball rolled along the ground, friction (i.e., the continual contact and collision with the ground) gradually destroyed its motion, and brought it to rest; whereas, when such resistances were diminished by rolling it on glass or on the ice, the motion always continued longer: hence it was inferred, that, were these resistances entirely removed, the motion would continue forever.

In accordance with these views, the definition of force usually given was substantially as follows: —

Force is that which causes, or tends to cause, a body to change its state from rest to motion, from motion to rest, or to change its motion as to direction or speed.

Under these views, uniform rectilinear motion was recognized as being just as much a condition of equilibrium, or of the action of no force or of balanced forces, as rest; and the recognition of this one fact upset many false notions, destroyed many incorrect conclusions, and first rendered possible a science of mechanics. Along with the above-stated definition of force is ordinarily given the following proposition; viz., —

Forces are proportional to the velocities that they impart, in a unit of time (i.e. to the accelerations that they impart), to the same body. The reasoning given is as follows: —

Suppose a body to be moving uniformly and in a straight line, and suppose a force to act upon it for a certain length of time t in the direction of the body's motion: the effect of the force is to alter the velocity of the body; and it is only by this alteration of velocity that we recognize the action of the force. Hence, as long as the alteration continues at the same rate, we recognize the same force as acting.

If, therefore, f represent the amount of velocity which the force would impart in one unit of time, the total increase in the velocity of the body will be ft; and, if the force now stop acting, the body will again move uniformly and in the same direction, but with a velocity greater by ft.

Hence, if we are to measure forces by their effects, it will follow that —

The velocity which a force will impart to a given (or standard) *body in a unit of time is a proper measure of the force.* And we shall have, that two forces, each of which will impart the same velocity to the same body in a unit of time, are equal to each other; and a force which will impart to a given body twice the velocity per unit of time that another force will impart to the same body, is itself twice as great, or, in other words, —

Forces are proportional to the velocities that they impart, in a unit of time (*i.e. to the accelerations that they impart*), *to the same body.*

MODERN CRITICISM OF THE ABOVE.

The scientists and the metaphysicians of the present time are recognizing two other facts not hitherto recognized, and the result is a criticism adverse to the above-stated definition of force. Other definitions have, in consequence, been proposed; but none are free from objection on logical grounds, and at the same time capable of use in mechanics in a quantitative way.

The two facts referred to are the following; viz., —

1°. That all our ideas of space, time, rest, motion, and even of direction, are relative.

2°. That, because two effects are identical, it does not follow that the causes producing those effects are identical.

Hence, in the light of these two facts, it is plain, that, inasmuch as we can only recognize motion as relative, we can only recognize force as acting when at least two bodies are concerned in the transaction; and also that if the forces are simply the causes of the motion in the ordinary popular sense of the word *cause*, we cannot assume, that, when the effects are equal, the causes are in every way identical, although we have, of course, a perfect right to say that they are identical so far as the production of motion is concerned.

I shall now proceed, in the light of the above, to deduce a definition of force, which, although not free from objection, seems as free as any that has been framed.

It is one of the facts of nature, that, when bodies are by any means brought under certain relations to each other, certain tendencies are developed, which, if not interfered with, will exhibit themselves in the occurrence of certain definite phenomena. What these phenomena are, depends upon the nature of the bodies concerned, and on the relationships into which they are brought.

As an illustration, we know that if an apple is placed at a certain height above the surface of the earth, there is developed between the two bodies a tendency to approach each other; and if there is no interference with this tendency, it exhihits itself in the fall of the apple. If, on the other hand, the apple were hung on the hook of a spring balance in the same position as before, the spring would stretch, and there would be developed a tendency of the spring to make the apple move upwards. This tendency to make the apple move upwards would be just equal to the tendency of the earth and apple to approach each other. This would be expressed by saying that the pull of the spring is just equal and opposite to the weight of the apple.

As other illustrations of these tendencies developed in bodies when placed in certain relations to each other, we have the following cases: —

(*a*) When two bodies collide.

(*b*) When two substances, coming together, form a chemical union, as sodium and water.

(*c*) When the chemical union is entered into only by raising the temperature to some special point.

Any of these tendencies that are developed by bringing about any of these special relationships between bodies might properly be called a force; and the term might properly be, and is, used in the same sense in the mental and moral world, as well as in the physical. In mechanics, however, we have to deal only with the relative motion of bodies; and hence we give the name *force* only to tendencies to change the relative motion of the bodies concerned; and this, whether these tendencies are unresisted, and exhibit themselves in the actual occurrence of a change of motion, or whether they are resisted by equal and opposite tendencies, and exhibit themselves in the production of a tensile, compressive, or other stress in the bodies concerned, instead of motion.

DEFINITION OF FORCE.

Hence our definition of force, as far as mechanics has to deal with it or is capable of dealing with it, is as follows; viz., —

Force is a tendency to change the relative motion of the two bodies between which that tendency exists.

Indeed, when, as in the illustration given a short time ago, the apple is hung on the hook of a spring balance, there still exists a tendency of the apple and the earth to approach each other; i.e., they are in the act of trying to approach each other; and it is this tendency, or *act of trying*, that we call the force of gravitation. In the case cited, this tendency is balanced by an opposite tendency on the part of the spring; but, were the spring not there, the force of gravitation would cause the apple to fall.

Professor Rankine calls force "an action between two bodies, either causing or tending to cause change in their relative rest or motion;" and if the *act of trying* can be called an *action*, my definition is equivalent to his.

For the benefit of any one who wishes to follow out the discussions that have lately taken place, I will enumerate the following articles that have been written on the subject: —

(*a*) "Recent Advances in Physical Science," by P. G. Tait, Lecture XIV.

(*b*) Herbert Spencer, "First Principles of Philosophy" (certain portions of the book).

(*c*) Discussion by Messrs. Spencer and Tait, "Nature," Jan. 2, 9, 16, 1879.

(*d*) Force and Energy, "Nature," Nov. 25, Dec. 2, 9, 16, 1880.

§ 6. **External Force.** — We thus see, that, in order that a force may be developed, there must be two bodies concerned in the transaction; and we should speak of the force as that developed or existing between the two bodies.

But we may confine our attention wholly to the motion or condition of one of these two bodies; and we may refer its motion either to the other body as a fixed point, or to some body different from either; and then, in speaking of the force, we should speak of it as the force acting on the body under consideration, and call it an external force. It is the tendency of the other body to change the motion of the body under consideration relatively to the point considered as fixed.

§ 7. **Relativity of Force.** — In adopting the above-stated definition of force, we acknowledge our incapacity to deal with it as an absolute quantity; for we have defined it as a tendency to change the relative motion of a pair of bodies. Hence it is only through relative motion that we recognize force; and hence force is relative, as well as motion.

§ 8. **Newton's First Law of Motion.** — In the light of the above discussion, we might express Newton's first law of motion as follows: —

A body at rest, or in uniform rectilinear motion relatively to a given point assumed as fixed, will continue at rest, or in uniform motion in the same direction, unless and until some external force acts either on the body in question, or on the fixed point, or on the body which furnishes us our fixed direction. This law is really superfluous, as it has all been embodied in the definition.

§ 9. **Measure of Force.** — We next need some means of comparing forces with each other in magnitude; and, subsequently, we need to select one force as our unit force, by means of which to estimate the magnitude of other forces.

Let us suppose a body moving uniformly and in a straight line, relatively to some fixed point; as long as this motion continues, we recognize no unbalanced force acting on it; but, if the motion changes, there must be a tendency to change that motion, or, in other words, an unbalanced force is acting on the body from the instant when it begins to change its motion.

Suppose a body to be moving uniformly, and a force to be applied to it, and to act for a length of time t, and to be so applied as not to change the direction of motion of the body, but to increase its velocity; the result will be, that the velocity will be increased by equal amounts in equal times, and if f represent the amount of velocity the force would impart in one unit of time, the total increase in velocity will be ft. This results merely from the definition of a force; for if the velocity produced in one (a standard) body by a given force is twice as great as that produced by another given force, then is the tendency to produce velocity twice as great in the first case as in the second, or, in other words, the first force is twice as great as the second. Hence —

Forces are proportional to the velocities which they will impart to a given (or standard) body in a unit of time.

We may thus, by using one standard body, determine a set of equal forces, and also the proportion between different forces.

§ 10. **Measure of Mass.** — After having determined, as shown, a set of equal (unit) forces, if we apply two of them to different bodies, and let them act for the same length of time on each, and find that the resulting velocities are unequal, these bodies are said to have unequal *masses;* whereas, if the resulting velocities are equal, they are said to have equal *masses.*

Hence we have the following definitions: —

1°. *Equal forces are those which, by acting for equal times on the same or standard body, impart to it equal velocities.*

2°. *Equal masses are those masses to which equal forces will impart equal velocities in equal times.*

§ 11. Suppose two bodies of equal mass moving side by side with the same velocity, and uniformly, let us apply to one of them a force F in the direction of the body's motion: the effect of this force is to increase the velocity with which the body moves; and if we wish, at the same time, to increase the velocity of the other, so that they will continue to move side by side, it will be necessary to apply an equal force to that also.

We are thus employing a force $2F$ to impart to the two bodies the required increment of velocity.

If we unite them into one, it still requires a force $2F$ to impart to the one body resulting from their union the required increment of velocity: hence, if we double the *mass* to which we wish to impart a certain velocity, we must double the force, or, in other words, employ a force which would impart to the first mass alone a velocity double that required. Hence —

Forces are proportional to the masses to which they will impart the same velocity in the same time.

§ 12. **Momentum.** — The product obtained by multiplying the number of units of mass in a body by its velocity is called the momentum of the body.

§ 13. **Relation between Force and Momentum.** — The number of units of momentum imparted to a body in a unit of time by a given force, is evidently identical with the number of units of velocity that would be imparted by the same force, in the same time, to a unit mass. Hence —

Forces are proportional to the momenta (or velocities per unit of mass) which they will generate in a unit of time.

Hence, if F represent a force which generates, in a unit of time, a velocity f in a body whose mass is m, we shall have

$$F \propto mf;$$

and, inasmuch as the choice of our units is still under our control, we so choose them that

$$F = mf;$$

i.e., the force F contains as many units of force as mf contains units of momentum; in other words, —

The momentum generated in a body in a unit of time by a force acting in the direction of the body's motion, is taken as a measure of the force.

§ 14. **Statical Measure of Force.** — When the forces are prevented from producing motion by being resisted by equal and opposite forces, as is the case in that part of mechanics known as *Statics*, they must be measured by a direct comparison with other forces. An illustration of this has already been given in the case of an apple hung on the hook of a spring balance. In that case the pull of the spring is equal in magnitude to the weight of the apple: indeed, it is very customary to adopt for forces what is known as the *gravity measure*, in which case we take as our unit the gravitation, or *tendency to fall*, of a given piece of metal, at a given place on the surface of the earth; in other words, its weight at a given place.

The gravity unit may thus be the kilogram, the pound, or the ounce, etc.

It is evident, moreover, from our definition of force, and the subsequent discussion, that whatever we take as our unit of mass, the *statical measure* of a force is proportional to its *dynamical measure;* i.e., the numbers representing the magnitudes of any two forces, in pounds, are proportional to the momenta they will impart to any body in a unit of time.

§ 15. **Gravity Measure of Mass.** — If we assume one pound as our unit of force, one foot as our unit of length, and one second as our unit of time, the ratio between the number of pounds in any given force and the momentum it will impart to a body on which it acts unresisted for a unit of time, will depend on our unit of mass; and, as we are still at liberty to fix this as we please, it will be most convenient so to choose it that the above-stated ratio shall be unity, so that there shall be no difference in the measure of a force, whether it is measured statically or dynamically. Now, it is known that a body falling freely under the action of its own weight acquires, every second, a velocity of about thirty-two feet per second: this number is denoted by g, and varies for different distances from the centre of the earth, as does also the weight of the body.

Now, if W represent the weight of the body in pounds, and m the number of units of mass in its mass, we must have, in order that the statical and dynamical measures may be equal,

$$W = mg.$$

Hence

$$m = \frac{W}{g};$$

i.e., the number of units of mass in a body is obtained by dividing the weight in pounds, by the value of g at the place where the weight is determined.

The values of W and of g vary for different positions, but the value of m remains always the same for the same body.

UNIT OF MASS.

If $m = 1$, then $W = g$; or, in words, —

The weight in pounds of the unit of mass (when the gravity measure is used) is equal to the value of g *in feet per second for the same place.*

§ 16. **Newton's Second Law of Motion.** — Newton's second law of motion is as follows: —

"*Change of momentum is proportional to the impressed moving force, and occurs along the straight line in which the force is impressed.*"

Newton states further in his "Principia:"—

"If any force generate any momentum, a double force will generate a double, a triple force will generate a triple, momentum, whether simultaneously and suddenly, or gradually and successively impressed. And if the body was moving before, this momentum, if in the same direction as the motion, is added; if opposite, is subtracted; or if in an oblique direction, is annexed obliquely, and compounded with it, according to the direction and magnitude of the two."

Part of this law has reference to the proportionality between the force and the momentum imparted to the body; and this has been already embodied in our definition of force, and illustrated in the discussion on the measure of forces.

The other part is properly a law of motion, and may be expressed as follows:—

If a body have two or more velocities imparted to it simultaneously, it will move so as to preserve them all.

The proof of this law depends merely upon a proper conception of motion. To illustrate this law when two velocities are imparted simultaneously to a body, let us suppose a man walking on the deck of a moving ship: he then has two motions in relation to the shore, his own and that of the ship.

Suppose him to walk in the direction of motion of the ship at the rate of 10 feet per second, while the ship moves at 25 feet per second relatively to the shore: then his motion in relation to the shore will be $25 + 10 = 35$ feet per second. If, on the other hand, he is walking in the opposite direction at the same rate, his motion relatively to the shore will be $25 - 10 = 15$ feet per second.

Suppose a body situated at A (Fig. 1) to have two motions imparted to it simultaneously, one of which would carry it to B

in one second, and the other to C in one second; and that it is required to find where it will be at the end of one second, and what path it will have pursued.

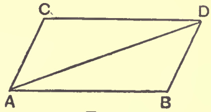


FIG. 1.

Imagine the body to move in obedience to the first alone, during one second: it would thus arrive at B; then suppose the second motion to be imparted to the body, instead of the first, it will arrive at the end of the next second at D, where BD is equal and parallel to AC. When the two motions are imparted simultaneously, instead of successively, the same point D will be reached in one second, instead of two; and by dividing AB and AC into the same (any) number of equal parts, we can prove that the body will always be situated at some point of the diagonal AD of the parallelogram, hence that it moves along AD. Hence follows the proposition known as the parallelogram of motions.

PARALLELOGRAM OF MOTIONS.

If there be simultaneously impressed on a body two velocities, which would separately be represented by the lines AB *and* AC, *the actual velocity will be represented by the line* AD, *which is the diagonal of the parallelogram of which* AB *and* AC *are the adjacent sides.*

§ 17. **Polygon of Motions.** — In all the above cases, the point reached by the body at the end of a second when the two motions take place simultaneously is the same as that which would be reached at the end of two seconds if the motions took place successively; and the path described is the straight line joining the initial position of the body, with its position at the end of one second when the motions are simultaneous.

The same principle applies whatever be the number of velocities that may be imparted to a body simultaneously. Thus, if we suppose the several velocities imparted to be (Fig. 2) AB, AC, AD, AE, and AF, and it be required to determine the resultant velocity, we first let the body move with the velocity AB for one second; at the end of that second it is found at B; then let it move with the velocity AC only, and at the end of another second it will be found at c; then with AD only, and at the end of the third second it will be found at d; at the end of the fourth at e; at the end of the fifth at f. Hence the resultant velocity, when all are imparted simultaneously, is Af, or the closing side of the polygon.

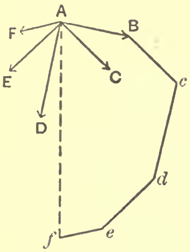


FIG. 2.

This proposition is known as the polygon of motions.

POLYGON OF MOTIONS.

If there be simultaneously impressed on a body any number of velocities, the resulting velocity will be represented by the closing side of a polygon of which the lines representing the separate velocities form the other sides.

§ 18. **Characteristics of a Force.** — A force has three characteristics, which, when known, determine it; viz., *Point of Application*, *Direction*, and *Magnitude*. These can be represented by a straight line, whose length is made proportional to the magnitude of the force, whose direction is that of the motion which the force imparts, or tends to impart, and one end of which is the point of application of the force; an arrow-head being usually employed to indicate the direction in which the force acts.

§ 19. **Parallelogram of Forces.**

PROPOSITION. — *If two forces acting simultaneously at the same point be represented, in point of application, direction, and magnitude, by two adjacent sides of a parallelogram, their resultant will be represented by the diagonal of the parallelogram, drawn from the point of application of the two forces.*

PROOF. — In the last part of § 16 was proved the proposition known as the *Parallelogram of Motions*, for the statement of which the reader is referred to the close of that section.

We have also seen that forces are proportional to the velocities which they impart, or tend to impart, in a unit of time, to the same body.

Hence the lines representing the two impressed forces are coincident in direction with, and proportional to, the lines representing the velocities they would impart in a unit of time to the same body; and moreover, since the resultant velocity is represented by the diagonal of the parallelogram drawn with the component velocities as sides, the resultant force must coincide in direction with the resultant velocity, and the length of the line representing the resultant force will bear to the resultant velocity the same ratio that one of the component forces bears to the corresponding velocity. Hence it follows, that the resultant force will be represented by the diagonal of the parallelogram having for sides the two component forces.

§ 20. **Parallelogram of Forces: Algebraic Solution.**

PROBLEM. — *Given two forces* F *and* F_1 *acting at the same point* A (*Fig.* 3), *and inclined to each other at an angle* θ; *required the magnitude and direction of the resultant force.*

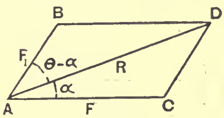


FIG. 3

Let AC represent F, AB represent F_1, and let angle $BAC = \theta$; then will $R = AD$ represent in magnitude and direction the resultant force. Also let angle $DAC = \alpha$; then from the triangle DAC we have

$$AD^2 = AC^2 + CD^2 - 2AC \, . \, CD \cos ACD.$$

But

$$ACD = 180^\circ - \theta \qquad \therefore \quad \cos ACD = -\cos\theta$$

$$\therefore \quad R^2 = F^2 + F_1^2 + 2FF_1 \cos\theta$$

$$\therefore \quad R = \sqrt{F^2 + F_1^2 + 2FF_1 \cos\theta}.$$

This determines the magnitude of R. To determine its direction, let angle $CAD = \alpha$ $\therefore$ angle $BAD = \theta - \alpha$, and we shall have from the triangle DAC

$$CD : AD = \sin CAD : \sin ACD,$$

or

$$F_1 : R = \sin \alpha : \sin \theta$$

$$\therefore \quad \sin \alpha = \frac{F_1}{R} \sin \theta,$$

and similarly

$$\sin(\theta - \alpha) = \frac{F}{R} \sin \theta.$$

EXAMPLES.

1°. Given $F = 47.34$, $F_1 = 75.46$, $\theta = 73° 14' 21''$; find R and α.
2°. Given $F = 5.36$, $F_1 = 4.27$, $\theta = 32° 10'$; find R and α.
3°. Given $F = 42.00$, $F_1 = 31.00$, $\theta = 150°$; find R and α.
4°. Given $F = 47.00$, $F_1 = 75.00$, $\theta = 253°$; find R and α.

§ 21. **Parallelogram of Forces when $\theta = 90°$.** — When the two given forces are at right angles to each other, the formulæ become very much simplified, since the parallelogram becomes a rectangle.

From Fig. 4 we at once deduce

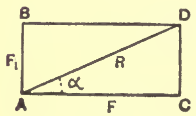


FIG. 4.

$$R = \sqrt{F^2 + F_1^2},$$

$$\sin \alpha = \frac{F_1}{R},$$

$$\cos \alpha = \frac{F}{R}.$$

EXAMPLES.

1°. Given $F = 3.0$, $F_1 = 5.0$; find R and α.
2°. Given $F = 3.0$, $F_1 = -5.0$; find R and α.
3°. Given $F = 5.0$, $F_1 = 12.0$; find R and α.
4°. Given $F = 23.2$, $F_1 = 21.3$; find R and α.

§ 22. **Triangle of Forces.** — *If three forces be represented, in magnitude and direction, by the three sides of a triangle taken in order, then, if these forces be simultaneously applied at one point, they will balance each other.*

Conversely, three forces which, when simultaneously applied at one point, balance each other, can be correctly represented in magnitude and direction by the three sides of a triangle taken in order.

These propositions, which find a very extensive application, especially in the determination of the stresses in roof and bridge trusses, are proved as follows: —

If we have two forces, AC and AB (see Fig. 3), acting at the point A, their resultant is, as we have already seen, AD; and hence a force equal in magnitude and opposite in direction to AD will balance the two forces AC and AB. Now, the sides of the triangle $ACDA$, if taken in order, represent in magnitude and direction the force AC, the force CD or AB, and a force equal and opposite to AD; and these three forces, if applied at the same point, would balance each other. Hence follows the proposition.

Moreover, we have

$$AC : CD : DA = \sin ADC : \sin CAD : \sin ACD,$$

or

$$F : F_1 : R = \sin(\theta - \alpha) : \sin \alpha : \sin \theta;$$

or each force is, in this case, proportional to the sine of the angle between the other two.

§ 23. **Decomposition of Forces in one Plane.** — It is often convenient to resolve a force into two components, in two given directions in a plane containing the force. Thus, suppose we have the force $R = AD$ (Fig. 3), and we wish to resolve it into two components acting respectively in the directions AC and AB; i.e., we wish to find two forces acting respectively in these directions, of which AD shall be the resultant: we determine these components graphically by drawing a parallelogram, of which AD shall be the diagonal, and whose sides shall have the directions AC and AB respectively. The algebraic values of the magnitudes of the components can be determined by solving the triangle ADC. In the case when the directions of the components are at right angles to each other, let the force R (Fig. 5), applied at O, make an angle α with OX. We may, by drawing the rectangle shown in the figure, decompose R into two components, F and F_1, along OX and OY respectively; and we shall readily obtain from the figure,

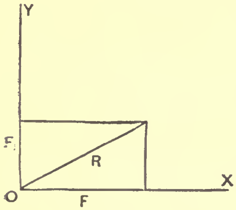


Fig. 5.

$$F = R \cos \alpha, \quad F_1 = R \sin \alpha.$$

EXAMPLES.

1°. The force exerted by the steam upon the piston of a steam-engine at the moment when it is in the position shown in the figure is $AB =$ 1000 lbs. The resistance of the guides upon the cross-head DE is vertical. Determine the force acting along the connecting-rod AC and the pressure on the guides; also resolve the force acting along the connecting-rod into two components, one along, and the other at right angles to, the crank OC.

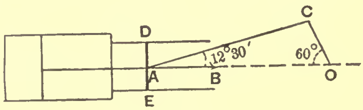


Fig. 6.

2°. A load of 500 lbs. is placed at the apex C of the frame ACB: find the stresses in AC and CB respectively.

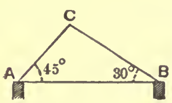


Fig. 7.

3°. A load of 4000 lbs. is hung at C, on the crane ABC: find the pressure in the boom BC, and the pull on the tie AC, where BC makes an angle of 60° with the horizontal, and AC an angle of 15°.

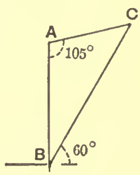


Fig. 8.

4°. A force whose magnitude is 7 is resolved into two forces whose magnitudes are 5 and 3: find the angles they make with the given force.

§ 24. Composition of any Number of Forces in One Plane, all applied at the Same Point.

(*a*) Graphical Solution. — Let the forces be represented (Fig. 2) by AB, AC, AD, AE, and AF respectively. Draw Bc ∥ and $= AC$, cd ∥ and $= AD$, de ∥ and $= AE$, and ef ∥ and $= AF$; then will Af represent the resultant of the five forces. This solution is to be deduced from § 17 in the same way as § 19 is deduced from § 16.

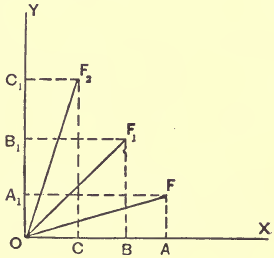


Fig. 9.

(*b*) Algebraic Solution. — Let the given forces (Fig. 9), of which three are represented in the figure, be F, F_1, F_2, F_3, F_4, etc.; and let the angles made by these forces with the axis OX be α, α_1, α_2, α_3, α_4, etc., respectively. Resolve each of these forces into two components, in the directions OX and OY respectively. We shall obtain for the components along OX

$$OA = F\cos\alpha, \quad OB = F_1\cos\alpha_1, \quad OC = F_2\cos\alpha_2, \quad \text{etc.};$$

and for those along OY

$$OA_1 = F\sin\alpha, \quad OB_1 = F_1\sin\alpha_1, \quad OC_1 = F_2\sin\alpha_2, \quad \text{etc.}$$

These forces are equivalent to the following two; viz., a force $F\cos\alpha + F_1\cos\alpha_1 + F_2\cos\alpha_2 + F_3\cos\alpha_3 +$ etc. along OX, and a force $F\sin\alpha + F_1\sin\alpha_1 + F_2\sin\alpha_2 + F_3\sin\alpha_3 +$ etc. along OY. The first may be represented by $\Sigma F\cos\alpha$, and the second by $\Sigma F\sin\alpha$, where Σ stands for *algebraic sum.* There remains only to find the resultant of these two, the magnitude of which is given by the equation

$$R = \sqrt{(\Sigma F\cos\alpha)^2 + (\Sigma F\sin\alpha)^2};$$

and, if we denote by α_r the angle made by the resultant with OX, we shall have

$$\cos \alpha_r = \frac{\Sigma F \cos \alpha}{R}, \quad \sin \alpha_r = \frac{\Sigma F \sin \alpha}{R}.$$

EXAMPLES.

1°. Given $\begin{cases} F = 47 & \alpha = 21^\circ \\ F_1 = 73 & \alpha_1 = 48^\circ \\ F_2 = 43 & \alpha_2 = 82^\circ \\ F_3 = 23 & \alpha_3 = 112^\circ \end{cases}$ Find the resultant force and its direction.

Solution.

F.	α.	$\cos \alpha$.	$\sin \alpha$.	$F \cos \alpha$.	$F \sin \alpha$.
47	21°	0.93358	0.35837	43.87826	16.84339
73	48°	0.66913	0.74315	48.84649	54.24995
43	82°	0.13917	0.99027	5.98431	42.58161
23	112°	−0.37461	0.92718	−8.61603	21.32414
				90.09303	134.99909

$$\therefore \quad \Sigma F \cos \alpha = 90.09303, \quad \Sigma F \sin \alpha = 134.99909,$$

$$\therefore \quad R = \sqrt{(\Sigma F \cos \alpha)^2 + (\Sigma F \sin \alpha)^2} = 162.2976.$$

$$\begin{aligned} \log \Sigma F \cos \alpha &= 1.954691 \\ \log R &= 2.210331 \\ \hline \log \cos \alpha_r &= 9.744360 \\ \alpha_r &= 56^\circ\ 17'. \end{aligned}$$

Observation. — It would be perfectly correct to use the minus sign in extracting the square root, or to call $R = -162.2976$; but then we should have

$$\cos \alpha_r = \frac{90.09303}{-162.2976}, \quad \text{and} \quad \sin \alpha_r = \frac{134.99909}{-162.2976},$$

or

$$\alpha_r = 180^\circ + 56^\circ\ 17' \qquad = 236^\circ\ 17';$$

a result which, if plotted, would give the same force as when we call

$$R = 162.2976 \quad \text{and} \quad \alpha_r = 56^\circ\, 17'.$$

Hence, since it is immaterial whether we use the plus or the minus sign in extracting the square root provided the rest of the computation be consistent with it, we shall, for convenience, use always plus.

2°.

$$\begin{aligned} F &= 4, & \alpha &= 77^\circ, \\ F_1 &= 3, & \alpha_1 &= 82^\circ, \\ F_2 &= 10, & \alpha_2 &= 163^\circ, \\ F_3 &= 5, & \alpha_3 &= 275^\circ. \end{aligned}$$

3°.

$$\begin{aligned} F &= 5, & \alpha &= \cos^{-1}\tfrac{4}{5}, \\ F_1 &= 4, & \alpha_1 &= 0, \\ F_2 &= 3, & \alpha_2 &= 90^\circ. \end{aligned}$$

§ 25. **Polygon of Forces.** — *If any number of forces be represented in magnitude and direction by the sides of a polygon taken in order, then, if these forces be simultaneously applied at one point, they will balance each other.*

Conversely, any number of forces which, when simultaneously applied at one point, balance each other, can be correctly represented in magnitude and direction by the sides of a polygon taken in order.

These propositions are to be deduced from § 24 (*a*) in the same way as the *triangle of forces* is deduced from the parallelogram of forces.

§ 26. **Composition of Forces all applied at the Same Point, and not confined to One Plane.** — This problem can be solved by the polygon of forces, since there is nothing in the demonstration of that proposition that limits us to a plane rather than to a *gauche* polygon.

The following method, however, enables us to determine algebraic values for the magnitude of the resultant and for its direction.

We first assume a system of three rectangular axes, OX, OY, and OZ (Fig. 10), whose origin is at the common point of the given forces. Now, let $OE = F$ be one of the given forces. First resolve it into two forces, OC and OD, the first of which lies in the z axis, and the second perpendicular to OZ, or, as it is usually called, in the z plane; the plane perpendicular to OX being the x plane, and that perpendicular to OY the y plane. Then resolve OD into two components, OA along OX, and OB along OY. We thus obtain three forces, OA, OB, and OC respectively, which are equivalent to the single force OE. These three components are the edges of a rectangular parallelopiped, of which $OE = F$ is the diagonal.

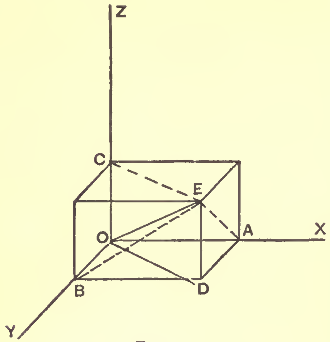


FIG. 10.

Let, now,

$$\text{angle } EOX = \alpha, \quad EOY = \beta, \quad \text{and} \quad EOZ = \gamma;$$

and we have, from the right-angled triangles EOA, EOB, and EOC respectively,

$$OA = F\cos\alpha, \quad OB = F\cos\beta, \quad OC = F\cos\gamma.$$

Moreover,

$$OA^2 + OB^2 = OD^2 \text{ and } OD^2 + OC^2 = OE^2$$
$$\therefore \quad OA^2 + OB^2 + OC^2 = OE^2,$$

and by substituting the values of OA, OB, and OC, given above, we obtain

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1;$$

a purely geometrical relation existing between the three angles that any line makes with three rectangular co-ordinate axes.

When two of the angles α, β, and γ are given, the third can be determined from the above equation.

Resolve, in the same way, each of the given forces into three components, along OX, OY, and OZ respectively, and we shall thus reduce our entire system of forces to the following three forces : —

1°. A single force $\Sigma F \cos \alpha$ along OX.
2°. A single force $\Sigma F \cos \beta$ along OY.
3°. A single force $\Sigma F \cos \gamma$ along OZ.

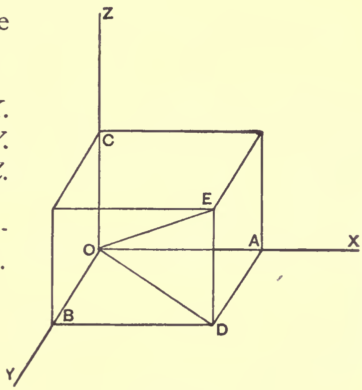


FIG. 11.

We next proceed to find a single resultant for these three forces.

Let (Fig. 11)

$$OA = \Sigma F \cos \alpha$$
$$OB = \Sigma F \cos \beta,$$
$$OC = \Sigma F \cos \gamma.$$

Compounding OA and OB, we find OD to be their resultant; and this, compounded with OC, gives OE as the resultant of the entire system. Moreover,

$$OE^2 = OD^2 + OC^2 = OA^2 + OB^2 + OC^2,$$

or

$$R^2 = (\Sigma F \cos \alpha)^2 + (\Sigma F \cos \beta)^2 + (\Sigma F \cos \gamma)^2$$

$$\therefore \quad R = \sqrt{(\Sigma F \cos \alpha)^2 + (\Sigma F \cos \beta)^2 + (\Sigma F \cos \gamma)^2};$$

and if we let $EOX = \alpha_r$, $EOY = \beta_r$, and $EOZ = \gamma_r$, we shall have

$$\cos \alpha_r = \frac{OA}{OE} = \frac{\Sigma F \cos \alpha}{R}, \quad \cos \beta_r = \frac{\Sigma F \cos \beta}{R}, \quad \text{and } \cos \gamma_r = \frac{\Sigma F \cos \gamma}{R}.$$

This gives us the magnitude and direction of the resultant.

The same observation applies to the sign of the radical for R as in the case of forces confined to one plane.

DETERMINATION OF THE THIRD ANGLE FOR ANY ONE FORCE.

When two of the angles α, β, and γ are given, the cosine of the third may be determined from the equation, —

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1\ ;$$

but, as we may use either the plus or the minus sign in extracting the square root, we have no means of knowing which of the two supplementary angles whose cosine has been deduced is to be used.

Thus, suppose $\alpha = 45°$, $\beta = 60°$, then

$$\cos \gamma = \pm \sqrt{1 - \tfrac{1}{2} - \tfrac{1}{4}} = \pm \tfrac{1}{2}$$
$$\therefore\ \gamma = 60°, \text{ or } 120°\ ;$$

but which of the two to use we have no means of deciding.

This indetermination will be more clearly seen from the following geometrical considerations : —

The angle α (Fig. 12), being given as 45°, locates the line representing the force on a right circular cone, whose axis is OX, and whose semi-vertical angle is $AOX = BOX = 45°$. On the other hand, the statement that $\beta = 60°$ locates the force on another right circular cone, having OY for axis, and a semi-vertical angle of 60°; both cones, of course, having their vertices at O. Hence, when α and β are given, we know that the line representing the force is an element of both cones ; and this is all that is given.

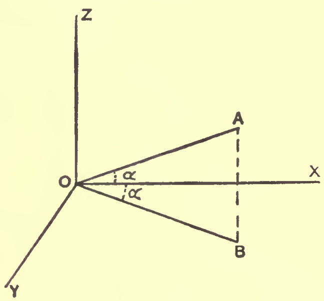


FIG. 12.

(a) Now, if the sum of the two given angles is less than 90°, the cones will not intersect, and the data are consequently inconsistent.

(*b*) If, on the other hand, one of the given angles being greater than 90°, their difference is greater than 90°, the cones will not intersect, and the data are again inconsistent.

(*c*) If $\alpha + \beta = 90°$, the cones are tangent to each other, and $\gamma = 90°$.

(*d*) If $\alpha + \beta > 90°$, and $\alpha - \beta$ or $\beta - \alpha < 90°$, the cones intersect, and have two elements in common; and we have no means of determining, without more data, which intersection is intended, this being the indetermination that arises in the algebraic solution.

EXAMPLES.

1. Given $\left\{\begin{array}{lll} F = 63 & \alpha = 53° & \beta = 42° \\ F_1 = 49 & \alpha = 87° & \gamma = 72° \\ F_2 = 2 & \beta = 70° & \gamma = 45° \end{array}\right\}$ Find the magnitude and direction of the resultant.

Solution.

F	α.	β.	γ.	$\cos\alpha$.	$\cos\beta$.	$\cos\gamma$.	$F\cos\alpha$.	$F\cos\beta$.	$F\cos\gamma$.
63	53°	42°		0.60182	0.74314	0.29250	37.91466	46.81782	18.42750
49	87°		72°	0.05234	0.94961	0.30902	2.56466	46.53089	15.14198
2		70°	45°	0.61888	0.34202	0.70711	1.23776	0.68404	1.41422
							41.71708 $\Sigma F\cos\alpha$	94.03275 $\Sigma F\cos\beta$	34.98370 $\Sigma F\cos\gamma$

$$R = \sqrt{(\Sigma F\cos\alpha)^2 + (\Sigma F\cos\beta)^2 + (\Sigma F\cos\gamma)^2} = 108.6569.$$

$\log \Sigma F\cos\alpha$	$= 1.620314$	$\log \Sigma F\cos\beta$	$= 1.973279$	$\log \Sigma F\cos\gamma$	$= 1.543866$
$\log R$	$= 2.036057$	$\log R$	$= 2.036057$	$\log R$	$= 2.036057$
$\log\cos\alpha_r$	$= 9.584257$	$\log\cos\beta_r$	$= 9.937222$	$\log\cos\gamma_r$	$= 9.507809$
α_r	$= 67° 25' 20''$	β_r	$= 30° 4' 14''$	γ_r	$= 71° 13' 5''$

	F.	α.	β.		F.	α.	β.	γ.
2.	4.3	47° 2′	65° 7′	3.	5	90°	90°	
	87.5	88° 3′	10° 5′		7	0°		
	6.4	68° 4′	83° 2′		4		0°	
					75	73°		45°

§ 27. **Conditions of Equilibrium for Forces applied at a Single Point.**

1°. When the forces are not confined to one plane, we have already found, for the square of the resultant,

$$R^2 = (\Sigma F \cos \alpha)^2 + (\Sigma F \cos \beta)^2 + (\Sigma F \cos \gamma)^2.$$

But this expression can reduce to zero only when we have

$$\Sigma F \cos \alpha = 0, \quad \Sigma F \cos \beta = 0, \quad \text{and} \quad \Sigma F \cos \gamma = 0;$$

for the three terms, being squares, are all positive quantities, and hence their sum can reduce to zero only when they are separately equal to zero.

Hence: *If a set of balanced forces applied at a single point be resolved into components along three directions at right angles to each other, the algebraic sum of the components of the forces along each of the three directions must be equal to zero, and conversely.*

2°. When the forces are all confined to one plane, let that plane be the z plane; then $\gamma = 90°$ in each case, and

$$\begin{aligned} \therefore \quad \beta &= 90° - \alpha \\ \therefore \quad \cos \beta &= \sin \alpha \\ \therefore \quad R^2 &= (\Sigma F \cos \alpha)^2 + (\Sigma F \sin \alpha)^2. \end{aligned}$$

Hence, for equilibrium we must have

$$(\Sigma F \cos \alpha)^2 + (\Sigma F \sin \alpha)^2 = 0;$$

and, since this is the sum of two squares,

$$\Sigma F \cos \alpha = 0, \text{ and } \Sigma F \sin \alpha = 0.$$

Hence: *If a set of balanced forces, all situated in one plane, and acting at one point, be resolved into components along two directions at right angles to each other, and in their own plane, the algebraic sum of the components along each of the two given directions must be equal to zero respectively; and conversely.*

§ 28. **Statics of Rigid Bodies.** — A rigid body is one that does not undergo any alteration of shape when subjected to the action of external forces. Strictly speaking, no body is absolutely rigid; but different bodies possess a greater or less degree of rigidity according to the material of which they are composed, and to other circumstances. When a force is applied to a rigid body, we may have as the result, not merely a rectilinear motion in the direction of the force, but, as will be shown later, this may be combined with a rotary motion; in short, the criterion by which we determine the ensuing motion is, that the effect of the force will distribute itself through the body in such a way as not to interfere with its rigidity.

What this mode of distribution is, we shall discuss hereafter; but we shall first proceed to some propositions which can be proved independently of this consideration.

§ 29. **Principle of Rectilinear Transferrence of Force in Rigid Bodies.** — If a force be applied to a rigid body at the point A (Fig. 13) in the direction AB, whatever be the motion that this force would produce, it will be prevented from taking place if an equal and opposite force be applied at A, B, C, or D, or at any point along the line of action of the force: hence we have the principle that —

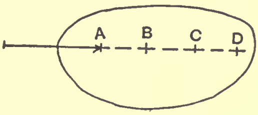


Fig. 13.

The point of application of a force acting on a rigid body, may be transferred to any other point which lies in the line of *action of the force, and also in the body, without altering the resulting motion of the body, although it does alter its state of stress.*

§ 30. **Composition of two Forces in a Plane acting at Different Points of a Rigid Body, and not Parallel to Each Other.** — Suppose the force F (Fig. 14) to be applied at A, and F_1 at B, both in the plane of the paper, and acting on the rigid body *abcdef*. Produce the lines of direction of the forces till they meet at O, and suppose both F and F_1 to act at O. Construct the parallelogram $ODHE$, where $OD = F$ and $OE = F_1$; then will $OH = R$ represent the resultant force in magnitude and in direction. Its point of application may be conceived at any point along the line OH, as at C, or any other point; and a force equal and opposite to OH, applied at any point of the line OH, will balance F at A, and F_1 at B.

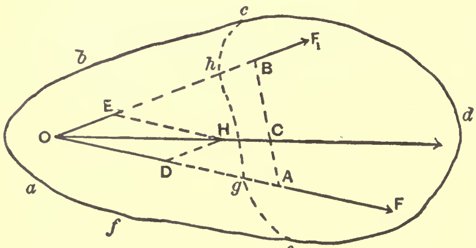


FIG. 14.

The above reasoning has assumed the points A, B, C and O, all within the body: but, since we have shown, that when this is the case, a force equal and opposite to R at C will balance F at A, and F_1 at B, it follows, that were these three forces applied, equilibrium would still subsist if we were to remove the part *bafeghc* of the rigid body; or, in other words, —

The same construction holds even when the point O *falls outside the rigid body.*

§ 31. **Moment of a Force with Respect to an Axis Perpendicular to the Force.**

DEFINITION. — The moment of a force with respect to an axis perpendicular to the force, and not intersecting it, is the product of the force by the common perpendicular to (shortest distance between) the force and the axis.

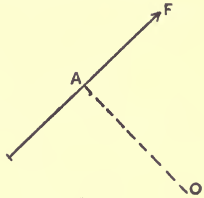


FIG. 15.

Thus, in Fig. 15 the moment of F about an axis through O and perpendicular to the plane of the paper is $F(OA)$. The sign of the moment will depend on the sign attached to the force and that attached to the perpendicular. These will be assumed in this book in such a manner as to render the following true; viz., —

The moment of a force with respect to an axis is called positive when, if the axis were supposed fixed, the force would cause the body on which it acts to rotate around the axis in the direction of the hands of a watch as seen by the observer looking at the face. It will be called negative when the rotation would take place in the opposite direction.

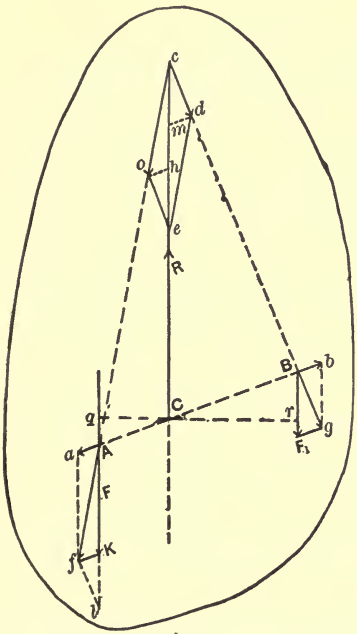


FIG. 16.

§ 32. **Equilibrium of Three Parallel Forces applied at Different Points of a Rigid Body.** — Let it be required to find a force (Fig. 16) that will balance the two forces F at A, and F_1 at B. Apply at A and B respectively, and in the line AB, the equal and opposite forces Aa and Bb. Their introduction will produce no alteration in the body's motion.

The resultant of F and Aa is Af, that of F_1 and Bb is Bg. Compound these by the method of § 30, and we obtain as resultant ce. A force equal in magnitude and opposite in direction to ce, applied at any point of the line cC, will be the force required to balance F at A and F_1 at B; and, as is evident from the construction, this line is in the plane of the two forces. Moreover, by drawing triangle fKl equal to Bbg, we can readily prove that triangles oce and Afl are equal: hence the angle oce equals the angle fAl, and R is parallel to F and F_1. Also

$$R = ce = ch + he = AK + Kl = F + F_1,$$

$$\frac{Cc}{AC} = \frac{AK}{fK} = \frac{F}{Aa},$$

and

$$\frac{Cc}{BC} = \frac{bg}{Bb} = \frac{F_1}{Bb};$$

$\therefore$ since $Aa = Bb$,

$$\frac{AC}{BC} = \frac{F_1}{F} \therefore \frac{F}{BC} = \frac{F_1}{AC} = \frac{F + F_1}{AB} \therefore \frac{F}{Cr} = \frac{F_1}{Cq} = \frac{F + F_1}{qr},$$

where qr is any line passing through C.

Hence we have the following propositions; viz., —

If three parallel forces balance each other, —

1°. *They must lie in one plane.*

2°. *The middle one must be equal in magnitude and opposite in direction to the sum of the other two.*

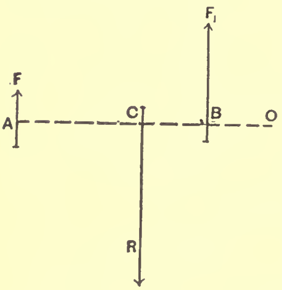


Fig. 17.

3°. *Each force is proportional to the distance between the lines of direction of the other two as measured on any line intersecting all of them.*

The third of the above-stated conditions may be otherwise expressed, thus: —

The algebraic sum of the moments of the three forces about any axis perpendicular to the forces must be zero.

PROOF. — Let F, F_{1}, and R (Fig. 17) be the forces; and let the axis referred to pass through O. Draw OA perpendicular to the forces. Then we have

$$\begin{aligned} F(OA) + F_{1}(OB) &= F(OC + CA) + F_{1}(OC - BC) \\ &= (F + F_{1})OC + F(AC) - F_{1}(BC). \end{aligned}$$

But, from what we have already seen,

$$F + F_{1} = -R$$

and

$$\frac{F}{BC} = \frac{F_{1}}{AC}$$

$$\therefore \quad F(AC) = F_{1}(BC)$$

$$\therefore \quad F(OA) + F_{1}(OB) = -R(OC) + 0$$

$$\therefore \quad F(OA) + F_{1}(OB) + R(OC) = 0,$$

or the algebraic sum of the moments of the forces about the axis through O is equal to zero.

§ 33. **Resultant of a Pair of Parallel Forces.** — In the preceding case, the resultant of any two of the three forces F, F_{1}, and R, in Fig. 16 or Fig. 17, is equal and opposite to the third force. Hence follow the two propositions: —

I. If two parallel forces act in the same direction, their resultant lies in the plane of the forces, is equal to their sum, acts in the same direction, and cuts the line joining their points of application, or any common perpendicular to the two forces, at a point which divides it internally into two segments inversely as the forces.

II. If two unequal parallel forces act in opposite directions, their resultant lies in the plane of the forces, is equal to their difference, acts in the direction of the larger force, and cuts the line joining their points of application, or any common perpendicular to them, at a point which (lying nearer the larger force) divides it externally into two segments which are inversely as the forces.

Another mode of stating the above is as follows: —

1°. The resultant of a pair of parallel forces lies in the plane of the forces.

2°. It is equal in magnitude to their algebraic sum, and coincides in direction with the larger force.

3°. The moment of the resultant about an axis perpendicular to the plane of the forces is equal to the algebraic sum of the moments about the same axis.

EXAMPLES.

1. Find the length of each arm of a balance such that 1 ounce at the end of the long arm shall balance 1 pound at the end of the short arm, the length of beam being 2 feet, and the balance being so proportioned as to hang horizontally when unloaded.

2. Given beam = 28 inches, 3 ounces to balance 15.

3. Given beam = 36 inches, 5 ounces to balance 25 ounces.

MODE OF DETERMINING THE RESULTANT OF A PAIR OF PARALLEL FORCES REFERRED TO A SYSTEM OF THREE RECTANGULAR AXES.

Let both forces (Fig. 18) be parallel to OZ; then we have, from what has preceded,

$$\frac{F}{bc} = \frac{F_1}{ab} = \frac{F + F_1}{ac} = \frac{R}{ac}, \text{ where } R = F + F_1.$$

But from the figure

$$\frac{bc}{x_3 - x_2} = \frac{ab}{x_2 - x_1} \quad \therefore \quad \frac{F}{x_3 - x_2} = \frac{F_1}{x_2 - x_1}$$

$$\therefore \quad Fx_2 - Fx_1 = F_1x_3 - F_1x_2$$

$$\therefore \quad (F + F_1)x_2 = Fx_1 + F_1x_3,$$

or

$$Rx_2 = Fx_1 + F_1x_3;$$

and similarly we may prove that

$$Ry_2 = Fy_1 + F_1y_3,$$

or

1°. The resultant of two parallel forces is parallel to the forces and equal to their algebraic sum.

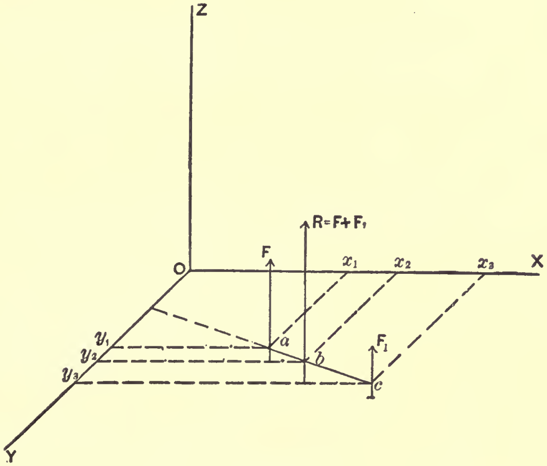


FIG. 18.

2°. The moment of the resultant with respect to OX is equal to the algebraic sum of their moments with respect to OX; and likewise when the moments are taken with respect to OY.

§ 34. **Resultant of any Number of Parallel Forces.** — Let it be required to find the resultant of any number of parallel forces.

In any such case, we might begin by compounding two of them, and then compounding the resultant of these two with a third, this new resultant with a fourth, and so on. Hence, for the magnitude of any one of these resultants, we simply add to the preceding resultant another one of the forces; and for the moment about any axis perpendicular to the forces, we add to the moment of the preceding resultant the moment of the new force.

Hence we have the following facts in regard to the resultant of the entire system : —

1°. *The resultant will be parallel to the forces and equal to their algebraic sum.*

2°. *The moment of the resultant about any axis perpendicular to the forces will be equal to the algebraic sum of the moments of the forces about the same axis.*

The above principles enable us to determine the resultant in all cases, except when the algebraic sum of the forces is equal to zero. This case will be considered later.

§ 35. **Composition of any System of Parallel Forces when all are in One Plane.** — Refer the forces to a pair of rectangular axes, OX, OY (Fig. 19), and assume OY parallel to the forces.

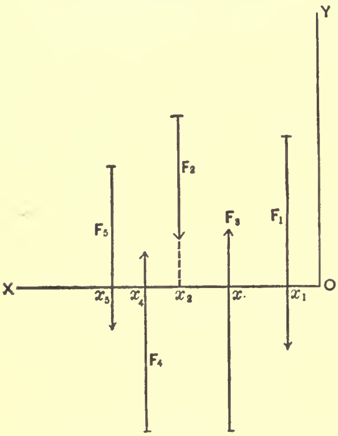


Fig. 19.

The forces and the co-ordinates of their lines of direction being as indicated in the figure, if we denote by R the resultant, and by x_0 the co-ordinate of its line of direction, we shall have, from the preceding,

$$R = \Sigma F; \qquad (1)$$

and if moments be taken about an axis through O, and perpendicular to the plane of the forces, we shall also have

$$Rx_0 = \Sigma Fx. \qquad (2)$$

Hence

$$R = \Sigma F \quad \text{and} \quad x_0 = \frac{\Sigma Fx}{\Sigma F}$$

determine the resultant in magnitude and in line of action, except when $\Sigma F = 0$, which case will be considered later.

§ 36. **Composition of any System of Parallel Forces not confined to One Plane.** — Refer the forces to a set of rectangular axes so chosen that OZ is parallel to their direction. If we denote the forces by F_1, F_2, F_3, F_4, etc., and the co-ordinates of their lines of direction by (x_1, y_1), (x_2, y_2), etc., and if we denote their resultant by R, and the co-ordinates of its line of direction by (x_0, y_0), we shall have, in accordance with what has been proved in § 34, —

1°. *The magnitude of the resultant is equal to the algebraic sum of the forces, or*

$$R = \Sigma F.$$

2°. *The moment of the resultant about* OY *is equal to the sum of the moments of the forces about* OY, *or*

$$x_0 \Sigma F = \Sigma F x.$$

3°. *The moment of the resultant about* OX *is equal to the sum of the moments about* OX, *or*

$$y_0 \Sigma F = \Sigma F y.$$

Hence

$$R = \Sigma F, \quad x_0 = \frac{\Sigma F x}{\Sigma F}, \quad y_0 = \frac{\Sigma F y}{\Sigma F},$$

determine the resultant in all cases, except when $\Sigma F = 0$.

§ 37. **Conditions of Equilibrium of any Set of Parallel Forces.** — If the axes be assumed as before, so that OZ is parallel to the forces, we must have

$$\Sigma F = 0, \quad \Sigma F x = 0, \quad \text{and} \quad \Sigma F y = 0.$$

To prove this, compound all but one of the forces. Then equilibrium will subsist only when the resultant thus obtained is equal and directly opposed to the remaining force; i.e, it must be equal, and act along the same line and in the opposite direction. Hence, calling R_a the resultant above referred to, and (x_a, y_a) the co-ordinates of its line of direction, and calling F_n the remaining force, and (x_n, y_n) the co-ordinates of its line of direction, we must have

$$
\begin{array}{llll}
 & R_a = -F_n, & x_a = x_n, & y_a = y_n, \\
\therefore & R_a + F_n = 0, & R_a x_a + F_n x_n = 0, & R_a y_a + F_n y_n = 0, \\
\therefore & \Sigma F = 0, & \Sigma F x = 0, & \Sigma F y = 0.
\end{array}
$$

When the forces are all in one plane, the conditions become

$$\Sigma F = 0, \quad \Sigma F x = 0.$$

§ 38. **Centre of a System of Parallel Forces.** — The resultant of the two parallel forces F and F_1 (Fig. 20), applied at A and B respectively, is a force $R = F + F_1$, whose line of action cuts the line AB at a point C, which divides it into two segments inversely as the forces. If the forces F and F_1 are turned through the same angle, and assume the positions AO and BO_1 respectively, the line of action of the resultant will still pass through C, which is called the centre of the two parallel forces F and F_1. Inasmuch as a similar reasoning will apply in the case of any number of parallel forces, we may give the following definition: —

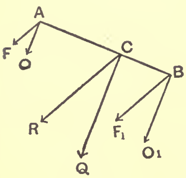


Fig. 20.

The centre of a system of parallel forces is the point through which the line of action of the resultant always passes, no matter how the forces are turned, provided only —

1°. *Their points of application remain the same.*

2°. *Their relative magnitudes are unchanged.*

3°. *They remain parallel to each other.*

Hence, in finding the centre of a set of parallel forces, we may suppose the forces turned through any angle whatever, and the centre of the set is the point through which the line of action of the resultant always passes.

§ 39. **Co-ordinates of the Centre of a Set of Parallel Forces.** — Let F_1 (Fig. 21) be one of the forces, and (x_1, y_1, z_1) the co-ordinates of its point of application. Let F_2 be another, and (x_2, y_2, z_2) co-ordinates of its point of application. Turn all the forces around till they are parallel to OZ, and find the line of direction of the resultant force when they are in this position. The co-ordinates of this line are

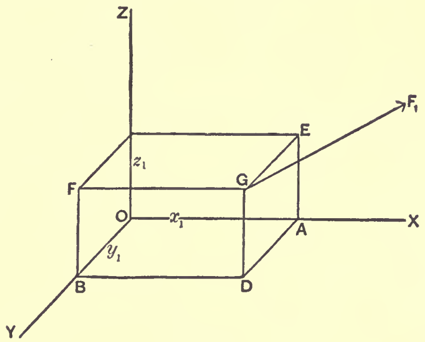


Fig. 21.

$$x_0 = \frac{\Sigma Fx}{\Sigma F}, \qquad y_0 = \frac{\Sigma Fy}{\Sigma F};$$

and, since the centre of the system is a point on this line, the above are two of the co-ordinates of the centre. Then turn the forces parallel to OX, and determine the line of action of the resultant. We shall have for its co-ordinates

$$y_0 = \frac{\Sigma Fy}{\Sigma F}, \qquad z_0 = \frac{\Sigma Fz}{\Sigma F}.$$

Hence, for the co-ordinates of the centre of the system, we have

$$x_0 = \frac{\Sigma Fx}{\Sigma F}, \qquad y_0 = \frac{\Sigma Fy}{\Sigma F}, \qquad z_0 = \frac{\Sigma Fz}{\Sigma F}.$$

When $\Sigma F = 0$ the co-ordinates would be ∞, therefore such a system has no centre.

§ 40. **Distributed Forces.** — While we have thus far assumed our forces as acting at single points, no force really acts at a single point, but all are distributed over a certain surface or through a certain volume; nevertheless, the propositions already proved are all applicable to the resultants of these distributed forces. We shall proceed to discuss distributed forces only when all the elements of the distributed force are parallel to each other. As a very important example of such a distributed force, we may mention the force of gravity which is distributed through the mass of the body on which it acts. Thus, the weight of a body is the resultant of the weights of the separate parts or particles of which it is composed. As another example we have the following: if a straight rod be subjected to a direct pull in the direction of its length, and if it be conceived to be divided into two parts by a plane cross-section, the stress acting at this section is distributed over the surface of the section.

§ 41. **Intensity of a Distributed Force.** — Whenever we have a force uniformly distributed over a certain area, we obtain its *intensity* by dividing its total amount by the area over which it acts, thus obtaining the amount per unit of area.

If the force be not uniformly distributed, or if the intensity vary at different points, we must adopt the following means for finding its intensity. Assume a small area containing the point under consideration, and divide the total amount of force that acts on this small area by the area, thus obtaining the *mean intensity* over this small area: this will be an approximation to the intensity at the given point; and the intensity is the limit of the ratio obtained by making the division, as the area used becomes smaller and smaller.

Thus, also, the *intensity*, at a given point, of a force which is distributed through a certain volume, is the limit of the ratio of the force acting on a small volume containing the given point, to the volume, as the latter becomes smaller and smaller.

§ 42. **Resultant of a Distributed Force.** — 1°. Let the force be distributed over the straight line *AB* (Fig. 22), and let its intensity at the point E where $AE = x$, be represented by $EF = p = \phi(x)$, a function of x; then will the force acting on the portion $Ee = \Delta x$ of the line be $p\Delta x$: and if we denote by R the magnitude of the resultant of the force acting on the entire line AB, and by x_0 the distance of its point of application from A, we shall have

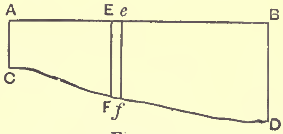


FIG. 22.

$$R = \Sigma p\Delta x \text{ approximately,}$$

or

$$R = \int pdx \text{ exactly;}$$

and, by taking moments about an axis through A perpendicular to the plane of the force, we shall have

$$x_0R = \Sigma x(p\Delta x) \text{ approximately,}$$

or

$$x_0R = \int pxdx \text{ exactly;}$$

whence we have the equations

$$R = \int pdx, \qquad x_0 = \frac{\int pxdx}{\int pdx}.$$

2°. Let the force be distributed over a plane area $EFGH$ (Fig. 23), let this area be referred to a pair of rectangular axes OX and OY, in its own plane, and let the intensity of the force per unit of area at the point P, whose co-ordinates are x and y, be $p = \phi(x, y)$; then will $p\Delta x\Delta y$ be approximately the force acting on the small rectangular area $\Delta x\Delta y$. Then, if we represent by R the magnitude of the resultant of the distributed force, and by x_0, y_0, the co-ordi-

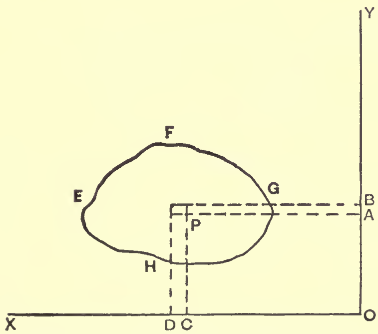


FIG. 23.

nates of the point at which the line of action of the resultant cuts the plane of *EFGH*, we shall have

$$R = \Sigma p \Delta x \Delta y \text{ approximately,}$$
$$x_0 R = \Sigma x (p \Delta x \Delta y) \quad \text{“}$$
$$y_0 \mathrm{R} = \Sigma y (p \Delta x \Delta y) \quad \text{“}$$

or, as exact equations, we shall have

$$R = \int\int p dx dy,$$

$$x_0 = \frac{\int\int p x dx dy}{\int\int p dx dy}, \qquad y_0 = \frac{\int\int p y dx dy}{\int\int p dx dy}.$$

3°. Let the force be distributed through a volume, let this volume be referred to a system of rectangular axes, OX, OY, and OZ, let ΔV represent the elementary volume, whose co-ordinates are x, y, z, and let $p = \phi(x, y, z)$ be the intensity of the force per unit of volume at the point (x, y, z); then, if we represent by R the magnitude of the resultant, and by x_0, y_0, z_0, the co-ordinates of the centre of the distributed force, we shall have, from the principles explained in § 38 and § 39, the approximate equations

$$R = \Sigma p \Delta V, \; x_0 R = \Sigma x (p \Delta V), \; y_0 R = \Sigma y (p \Delta V), \; z_0 R = \Sigma z (p \Delta V);$$

and these give, on passing to the limit, the exact equations

$$R = \int p dV, \quad x_0 = \frac{\int p x dV}{\int p dV}, \quad y_0 = \frac{\int p y dV}{\int p dV}, \quad z_0 = \frac{\int p z dV}{\int p dV}.$$

§ 43. **Centre of Gravity.** — The weight of a body, or system of bodies, is the resultant of the weight of the separate parts or particles into which it may be conceived to be divided; and the *centre of gravity* of the body, or system of bodies, is the centre of the above-stated system of parallel forces, i.e., the point through which the resultant always passes, no matter how the forces are turned. The weight of any one particle is the force which gravity exerts on that particle: hence, if we represent the weight per unit of volume of a body, whether it be the same for all parts or not, by w, we shall have, as an approximation,

$$W = \Sigma w \Delta V, \quad x_0 = \frac{\Sigma wx\Delta V}{\Sigma w\Delta V}, \quad y_0 = \frac{\Sigma wy\Delta V}{\Sigma w\Delta V}, \quad z_0 = \frac{\Sigma wz\Delta V}{\Sigma w\Delta V};$$

and as exact equations,

$$W = \int wdV, \quad x_0 = \frac{\int wxdV}{\int wdV}, \quad y_0 = \frac{\int wydV}{\int wdV}, \quad z_0 = \frac{\int wzdV}{\int wdV}, \qquad (1)$$

where W denotes the entire weight of the body, and x_0, y_0, z_0, the co-ordinates of its centre of gravity.

If, on the other hand, we let $M =$ entire mass of the body, $dM =$ mass of volume dV, and $m =$ mass of unit of volume, we shall have

$$W = Mg, \quad w = mg, \quad wdV = mgdV = gdM.$$

Hence the above equations reduce to

$$M = \int dM, \quad x_0 = \frac{\int xdM}{\int dM}, \quad y_0 = \frac{\int ydM}{\int dM}, \quad z_0 = \frac{\int zdM}{\int dM}. \qquad (2)$$

Equations (1) and (2) are both suitable for determining the centre of gravity; one of the sets being sometimes most convenient, and sometimes the other.

§ 44. **Centre of Gravity of Homogeneous Bodies.** — If the body whose centre of gravity we are seeking is homogeneous, or of the same weight per unit of volume throughout, we shall have, that $w =$ a constant in equations (1); and hence these reduce to

$$W = w\int dV, \quad x_0 = \frac{\int xdV}{\int dV}, \quad y_0 = \frac{\int ydV}{\int dV}, \quad z_0 = \frac{\int zdV}{\int dV}.$$

§ 45. **Effect of a Single Force applied at the Centre of a Straight Rod of Uniform Section and Material.** — If a straight rod of uniform section and material have imparted to it a motion, such that the velocity imparted in a unit of time to each particle of the rod is the same, and if we represent this velocity by f, then if at each point of the rod, we lay off a line xy (Fig. 24) in the direction of the motion, and representing the velocity imparted to that point, the line bounding the other ends of the lines xy will be straight, and parallel to the rod. If we conceive the rod to be divided into any number of small equal parts, and denote the mass of one of these parts by ΔM, then will $f\Delta M$ contain as many units of momentum as there are units of force in the force required to impart to this particle the velocity f in a unit of time; and hence $f\Delta M$ is the measure of this force.

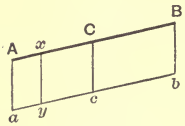


FIG. 24.

Hence the resultant of the forces which impart the velocity f to every particle of the rod will have for its measure

$$fM,$$

where M is the entire mass of the rod; and its point of application will evidently be at the middle of the rod.

It therefore follows that —

The effect of a single force applied at the middle of a straight rod of uniform section and material is to impart to the rod a motion of translation in the direction of the force, all points of the rod acquiring equal velocities in equal times.

§ 46. **Translation and Rotation combined.** — Suppose that we have a straight rod AB (Fig. 25), and suppose that such a force or such forces are applied to it as will impart to the point A in a unit of time the velocity Aa, and to the point B the (different) velocity Bb in a unit of time, both being perpendicular to the length of the rod. It is required to determine the motion of any other point of the rod and that of the entire rod.

Lay off Aa and Bb (Fig. 25), and draw the line ab, and produce it till it meets AB produced in O: then, when these velocities Aa and Bb are imparted to the points A and B, the rod is in the act of rotating around an axis through O perpendicular to the plane of the paper; for when a body is rotating around an axis, the linear velocity of any point of the body is perpendicular to the line joining the point in question with the axis (i.e., the perpendicular dropped from the point in question upon the axis), and proportional to the distance of the point from the axis.

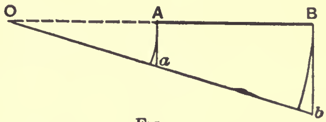


Fig. 25.

Hence: *If the velocities of two of the points in the rod are given, and if these are perpendicular to the rod, the motion of the rod is fixed, and consists of a rotation about some axis at right angles to the rod.*

Another way of considering this motion is as follows: Suppose, as before, the velocities of the points A and B to be represented by Aa and Bb respectively, and hence the velocity of any other point, as x (Fig. 26), to be represented by xy, or the length of the line drawn perpendicular to AB, and limited by AB and ab.

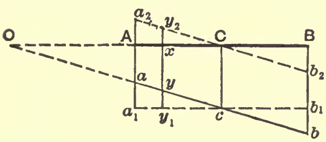


Fig. 26.

Then, if we lay off $Aa_1 = Bb_1 = \frac{1}{2}(Aa + Bb) = Cc$, and draw a_1b_1, and if we also lay off $Aa_2 = a_1a$, and $Bb_2 = b_1b$, we shall have the following relations; viz., —

$$\begin{aligned} Aa &= Aa_1 - Aa_2, \\ Bb &= Bb_1 + Bb_2, \\ xy &= xy_1 - xy_2, \quad \text{etc.,} \end{aligned}$$

or we may say that the actual motion imparted to the rod in a unit of time may be considered to consist of the following two parts: —

1°. A velocity of translation represented by Aa_1, the mean velocity of the rod; all points moving with this velocity.

2°. A varying velocity, different for every different point, and such that its amount is proportional to its distance from C, the centre of the rod, as graphically shown in the triangles Aa_2CBb_2. In other words, the rod has imparted to it two motions: —

1°. A translation with the mean velocity of the rod.

2°. A rotation of the rod about its centre.

§ 47. **Effect of a Force applied to a Straight Rod of Uniform Section and Material, not at its Centre.** — If the force be not at right angles to the rod, resolve it into two components, one acting along the rod, and the other at right angles to it. The first component evidently produces merely a translation of the rod in the direction of its length: hence the second component is the only one whose effect we need to study.

To do this we shall proceed to show, that, when such a rod has imparted to it the motion described in § 46, the single resultant force which is required to impart this motion in a unit of time is a force acting at right angles to the rod, at a point different from its centre; and we shall determine the relation between the force and the motion imparted, so that one may be deduced from the other.

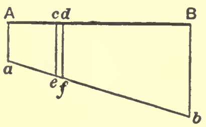


Fig. 27.

Let A be the origin (Fig. 27), and let

$Ac = x$, $cd = dx$.

$AB = l =$ length of the rod.

$ce = f =$ velocity imparted per unit of time at distance x from A.

$Aa = f_1$, $Bb = f_2$.

$w =$ weight per unit of length.

$m =$ mass per unit of length $= \dfrac{w}{g}$.

W = entire weight of rod.

M = entire mass of rod $= \dfrac{W}{g}$.

R = single resultant force acting for a unit of time to produce the motion.

x_0 = distance from A to point of application of R.

Then we shall have,

$$f = f_1 + \frac{f_2 - f_1}{l} x.$$

Hence, from § 42,

$$R = m\int_0^l f dx = m(\text{area } AabB) = \frac{m}{2}(f_1 + f_2)l = \frac{M}{2}(f_1 + f_2). \quad (1)$$

$$x_0 R = m\int_0^l f x dx = \frac{m}{6}(f_1 + 2f_2)l^2 = \frac{M}{6}(f_1 + 2f_2)l. \quad (2)$$

$$\therefore \quad x_0 = \frac{1}{3}\frac{f_1 + 2f_2}{f_1 + f_2} l. \quad (3)$$

We thus have a force R, perpendicular to AB, whose magnitude is given by equation (1), and whose point of application is given by equation (3); the respective velocities imparted by the force being shown graphically in Fig. 27.

EXAMPLES.

1. Given Weight of rod $= W =$ 100 lbs.,
   Length of rod = 3 feet,
   Assume g = 32 feet per second,
   Force applied $= R =$ 5 lbs.,
   Point of application to be 2.5 feet from one end;

determine the motion imparted to the rod by the action of the force for one second.

Solution.

Equation (1) gives us,

$$5 = \left(\frac{100}{32}\right)\frac{f_1 + f_2}{2}, \text{ or } f_1 + f_2 = 3.2.$$

Equation (2) gives,

$$(2.5)(5) = \left(\frac{100}{32}\right)\left(\frac{1}{6}\right)(3)(f_1 + 2f_2), \text{ or } f_1 + 2f_2 = 8$$

$$\therefore \quad f_2 = 4.8, \qquad f_1 = -1.6.$$

Hence the rod at the end nearest the force acquires a velocity of 4.8 feet per second, and at the other end a velocity of -1.6 feet per second. The mean velocity is, therefore, 1.6 feet per second; and we may consider the rod as having a motion of translation in the direction of the force with a velocity of 1.6 feet per second, and a rotation about its centre with such a speed that the extreme end (i.e., a point $\frac{3}{2}$ feet from the centre) moves at a velocity $4.8 - 1.6 = 3.2$ feet per second. Hence angular velocity $= \frac{3.2}{1.5} = 2.14$ per second $= 122°.6$ per second.

2. Given $W = 50$ lbs., $l = 5$ feet. It is desired to impart to it, in one second, a velocity of translation at right angles to its length, of 5 feet per second, together with a rotation of 4 turns per second: find the force required, and its point of application.

3. Assume in example 2 that the velocity of translation is in a direction inclined 45° to the length of the rod, instead of 90°. Solve the problem.

4. Given a force of 3 lbs. acting for one-half a second at a distance of 4 feet from one end of the rod, and inclined at 30° to the rod: determine its motion.

5. Given the same conditions as in example 4, and also a force of 4 lbs., parallel and opposite in direction to the 3-lb. force, and acting also for one-half a second, and applied at 3 feet from the other end: determine the resulting motion.

6. Given two equal and opposite parallel forces, each acting at right angles to the length of the rod, and each equal to 4 lbs., one being applied at 1 foot from one end, and the other at the middle of the rod; find the motion imparted to the rod through the joint action of these forces for one-third of a second.

§ 48. **Moment of the Forces causing Rotation.** — Referring again to Fig. 26, and considering the motion of the rod as a combination of translation and rotation, if we take moments about the centre C, and compare the total moment of the forces causing the rotation alone, whose accelerations are represented by the triangles aa_1cb_1b, with the total moment of the actual forces acting, whose accelerations are represented by the trapezoid $AabB$, we shall find these moments equal to each other; for, as far as the forces represented by the rectangle are concerned, every elementary force $m(xy_1)dx$ on one side of the centre C has its moment $(Cx)\{m(xy_1)dx\}$ equal and opposite to that of the elementary force at the same distance on the other side of C: hence the total moment of the forces represented graphically by the rectangle Aa_1b_1B is zero, and hence —

The moment about C *of those represented by the trapezoid equals the moment of those represented by the triangles.*

Hence, from the preceding, and from what has been previously proved, we may draw the following conclusions: —

1°. If a force be applied at the centre of the rod, it will impart the same velocity to each particle.

2°. If a force be applied at a point different from the centre, and act at right angles to its length, it will cause a translation of the rod, together with a rotation about the centre of the rod.

3°. In this latter case, the moment of the forces imparting the rotation alone is equal to the moment of the single resultant force about the centre of the rod, and the velocity of translation imparted in a unit of time is equal to the number of units of force in the force, divided by the entire mass of the rod.

§ 49. **Effect of a Pair of Equal and Opposite Parallel Forces applied to a Straight Rod of Uniform Section and Material.** — Suppose the rod to be AB (Fig. 28), and let the two equal and opposite parallel forces be Dd and Ee, each equal to F, applied at D and E respectively.

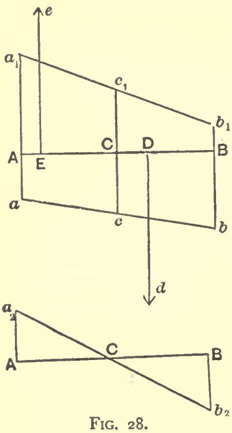


Fig. 28.

The mean velocity imparted in a unit of time by either force will be $\frac{F}{M}$; and, from what we have already seen, the trapezoid $AabB$ will furnish us the means of representing the actual velocity imparted to any point of the rod by the force Dd. The relative magnitudes of Aa and Bb, the accelerations at the ends, will depend, of course, on the position of D; but we shall always have $Cc = \frac{1}{2}(Aa + Bb) = \frac{F}{M}$, a quantity depending only on the *magnitude* of the force. So, likewise, the trapezoid Aa_1b_1B will represent the velocities imparted by the force Ee; and while the relative magnitude of Aa_1 and Bb_1 will depend upon the position of E, we shall always have $Cc_1 = \frac{1}{2}(Aa_1 + Bb_1) = \frac{F}{M}$. Hence, since $Cc = Cc_1$, the centre C of the rod has no motion imparted to it by the given pair of forces, hence the motion of the rod is one of rotation about its centre C.

The resulting velocity of any point of the rod will be the difference between the velocities imparted by the two forces; and if these be laid off to scale, we shall have the second figure. Hence —

A pair of equal and opposite parallel forces, applied to a straight rod of uniform section and material, produce a rotation of the rod about its centre. Also, —

Such a rotation about the centre of the rod cannot be pro-

duced by a single force, but requires a pair of equal and opposite parallel forces.

§ 50. **Statical Couple.** — A pair of equal and opposite parallel forces is called a statical couple.

§ 51. **Effect of a Single Force applied at the Centre of Gravity of a Straight Rod of Non-Uniform Section and Material.** — In the case of a straight rod of non-uniform section and material, we may consider the rod as composed of a set of particles of unequal mass: and if we imagine each particle to have imparted to it the same velocity in a unit of time, then, using the same method of graphical representation as before (Fig. 24), the line ab, bounding the other ends of the lines representing velocities, will be parallel to AB; but if we were to represent by the lines xy, not the velocities imparted, but the forces per unit of length, the line bounding the other ends of these forces would not, in this case, be parallel to AB. Moreover, since these forces are proportional to the masses, and hence to the weights of the several particles, their resultant would act at the centre of gravity of the rod. Hence —

A force applied at the centre of gravity of a straight rod will impart the same velocity to each point of the rod; i.e., will impart to it a motion of translation only.

§ 52. **Effect of a Statical Couple on a Straight Rod of Non-Uniform Section and Material.** — Let such a rod have imparted to it only a motion of rotation about its centre of gravity, and let us adopt the same modes of graphical representation as before.

Let the origin be taken at O (Fig. 29), the centre of gravity of the rod.

Let $Aa = f_1 =$ velocity imparted to A.
$Bb = f_2 =$ velocity imparted to B.
$OA = a$, $OB = b$, $OC = x$.
$CD = f =$ velocity imparted to C.
$dM =$ elementary mass at C.

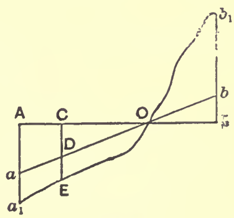


FIG. 29.

Then, from similar triangles, we have

$$f = \frac{f_1}{a}x = \frac{f_2}{b}x,$$

and hence for the force acting on dM we have

$$dF = (CE)dx = \frac{f_1}{a}xdM.$$

Hence the whole force acting on AO, and represented graphically by Aa_1O, is

$$\frac{f_1}{a}\int_{x=o}^{x=a} xdM,$$

and that acting on OB, and represented by BOb_1, is

$$\frac{f_2}{b}\int_{x=-b}^{x=o} xdM = \frac{f_1}{a}\int_{x=-b}^{x=o} xdM.$$

Hence for the resultant, or the algebraic sum, of the two, we have

$$R = \frac{f_1}{a}\int_{x=-b}^{x=a} xdM.$$

But from § 43 we have for the co-ordinate x_0 of the centre of gravity of the rod

$$x_0 = \frac{\int_{x=-b}^{x=a} xdM}{M};$$

but, since the origin is at the centre of gravity, we have

$$x_0 = 0,$$

and hence

$$\int_{x=-b}^{x=a} xdM = 0 \qquad \therefore \quad R = 0.$$

Hence the two forces represented by Aa_1O and Bb_1O are equal in magnitude and opposite in direction: hence the rotation about the centre of gravity is produced by a *Statical Couple.*

Now, a train of reasoning similar to that adopted in the case of a rod of uniform section and material will show that a single force applied at some point which is not the centre of gravity of the rod will produce a motion which consists of two parts; viz., a motion of translation, where all points of the rod have equal velocities, and a motion of rotation around the centre of gravity of the rod.

§ 53. **Moment of a Couple.** — The *moment of a statical couple* is the product of either force by the perpendicular distance between the two forces, this perpendicular distance being called the arm of the couple.

§ 54. **Measure of the Rotatory Effect.** — Before proceeding to examine the effect of a statical couple upon any rigid body whatever, we will seek a means of measuring its effect in the cases already considered.

The measure adopted is the moment of the couple; and, in order to show that it is proper to adopt this measure, it will be necessary to show —

That the moment of the couple is proportional to the angular velocity imparted to the same rod in a unit of time; and from this it will follow —

That two couples in the same plane with equal moments will balance each other if one is right-handed and the other left-handed

If we assume the origin of co-ordinates at C (Fig. 30), the centre of gravity of the rod, and if we denote by α the angular velocity imparted in a unit of time by the forces F and $-F$, and let $CD = x_1$, $CE = x_2$, then we have for the linear velocity of a particle situated at a distance x from C the value

$$\alpha x.$$

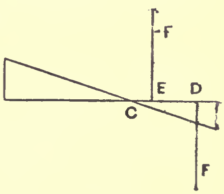


FIG. 30.

The force which will impart this velocity in a unit of time to the mass dM is

$$\alpha x dM.$$

The total resultant force is

$$\alpha \int x dM,$$

which, as we have seen, is equal to zero. The moment of the elementary force about C is

$$x(\alpha x dM) = \alpha x^2 dM,$$

and the sum of the moments for the whole rod is

$$\alpha \int x^2 dM,$$

and this, as is evident if we take moments about C, is equal to

$$Fx_1 - Fx_2 = F(x_1 - x_2) = F(DE).$$

Now, $\int x^2 dM$ is a constant for the same rod: hence any quantity proportional to $F(DE)$ is also proportional to α.

The above proves the proposition.

Moreover, we have

$$F(DE) = \alpha \int x^2 dM$$
$$\therefore \quad \alpha = \frac{F(DE)}{\int x^2 dM},$$

whence it follows, that when the moment of the couple is given, and also the rod, we can find the angular velocity imparted in a unit of time by dividing the former by $\int x^2 dM$.

§ 55. **Effect of a Couple on a Straight Rod when the Forces are inclined to the Rod.** — We shall next show that the effect of such a couple is the same as that of a couple of equal moment whose forces are perpendicular to the rod.

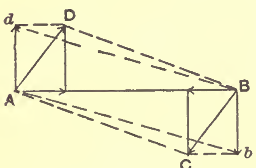


FIG. 31.

In this case let AD and BC be the forces (Fig. 31). The moment of this couple is the product of AD by the perpendicular distance between AD and BC, the graphical representation of this being the area of the parallelogram $ADBC$.

Resolve the two forces into components along and at right angles to the rod. The former have no effect upon the motion of the rod: the latter are the only ones that have any effect upon its motion. The moment of the couple which they form is the product of Ad by AB, graphically represented by parallelogram $AdBb$; and we can readily show that

$$ADBC = AdBb.$$

Hence follows the proposition.

§ 56. **Effect of a Statical Couple on any Rigid Body.**—Refer the body (Fig. 32) to three rectangular axes, OX, OY, and OZ, assuming the origin at the centre of gravity of the body, and OZ as the axis about which the body is rotating. Let the mass of the particle P be ΔM, and its co-ordinates be x, y, z.

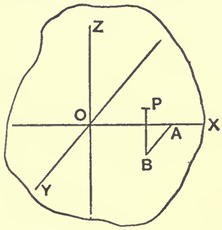


FIG. 32.

Then will the force that would impart to the mass ΔM the angular velocity α in a unit of time be

$$\alpha r \Delta M,$$

where $r =$ perpendicular from P on OZ, or

$$r = \sqrt{x^2 + y^2}.$$

This force may be resolved into two, one parallel to OY and the other to OX; the first component being $\alpha x \Delta M$, and the second $\alpha y \Delta M$.

Proceeding in the same way with each particle, and finding the resultant of each of these two sets of parallel forces, we shall obtain, finally, a single force parallel to OY and equal to

$$\alpha \Sigma x \Delta M,$$

and another parallel to OX, equal to

$$\alpha \Sigma y \Delta M.$$

But, since OZ passes through the centre of gravity of the body, we shall have

$$\Sigma x\Delta M = 0 \quad \text{and} \quad \Sigma y\Delta M = 0.$$

Hence the resultant is in each case, not a single force, but a *statical couple.* Hence, to impart to a body a rotation about an axis passing through its centre of gravity requires the action of a statical couple; and conversely, a statical couple so applied will cause such a rotation as that described.

Further discussion of the motion of rigid bodies resulting from the action of statical couples is unnecessary for our present purpose, hence we shall pass to the deduction of the following propositions, viz.:

PROP. I. Two statical couples in the same plane balance each other when they have equal moments, and tend to produce rotation in opposite directions. Let F_1 at a and $-F_1$ at b represent one couple (left-handed in the figure), and let F_2 at d and $-F_2$ at e represent the other (right-handed in the figure), and let

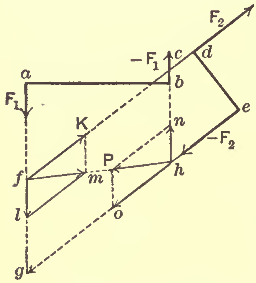


FIG. 33.

$$F_1(ab) = F_2(de);$$

then will these two couples balance each other.

Proof.—The resultant of F_1 at a and F_2 at d will be equal in amount, and directly opposed to the resultant of $-F_1$ at b and $-F_2$ at e and both will act along the diagonal fh of the parallelogram $fchg$. For we have $(fg)(ab) = (fc)(de)$, each being equal to the area of the parallelogram.

$$\therefore \quad \frac{F_1}{fg} = \frac{F_2}{fc}, \qquad \therefore \quad \frac{F_1}{F_2} = \frac{fg}{fc};$$

hence follows the proposition.

Hence follows that for a couple we may substitute another in the same plane, having the same moment, and tending to rotate the body in the same direction.

PROP. II. Two couples in parallel planes balance each other when their moments are equal, and the directions in which they tend to rotate the body are opposite.

Let (Fig. 33 (*a*)) the planes of both couples, be perpendicular to OZ. Reduce them both so as to have their arms equal and transfer them, each in its own plane, till their arms are in the X plane. Let ab be the arm of one couple, and dc that of the other. Then will the two couples form an equilibrate system. For the resultant of the force at a and that at c acts at e, and is twice either one of its components, and hence is equal and directly opposed to the resultant of the force at b and that at d.

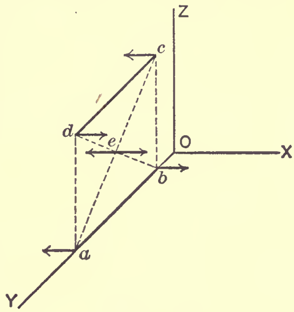


FIG. 33 (*a*).

Hence we may generalize all our propositions in regard to the effect of statical couples and we may conclude that —

In order that two couples may have the same effect, it is necessary —

1°. *That they be in the same or parallel planes.*

2°. *That they have the same moment.*

3°. *That they tend to cause rotation in the same direction* (*i.e., both right-handed or both left-handed when looked at from the same side*).

It also follows, that, for a given statical couple, we may substitute another having the magnitudes of its forces different, provided only the moment of the couple remains the same.

§ 57. **Composition of Couples in the Same or Parallel Planes.** — If the forces of the couples are not the same, reduce them to equivalent couples having the same force, transfer them to the same plane, and turn them so that their arms shall lie in the same straight line, as in Fig. 34; the first couple consisting of the force F at A and $-F$ at B, and the second of F at B and $-F$ at C.

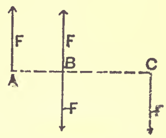


FIG. 34.

The two equal and opposite forces counterbalance each other, and we have left a couple with force F and arm

$$AC = AB + BC$$

$$\therefore \quad \text{Resultant moment} = F.AC = F(AB) + F(BC).$$

Hence: *The moment of the couple which is the resultant of two or more couples in the same or parallel planes is equal to the algebraic sum of the moments of the component couples.*

EXAMPLES.

1. Convert a couple whose force is 5 and arm 6 to an equivalent couple whose arm is 3. Find the resultant of this and another couple in the same plane and sense whose force is 7 and arm 8; also find the force of the resultant couple when the arm is taken as 5.

Solution.

$$\begin{array}{lll}
\text{Moment of first couple} & = 5 \times 6 & = 30 \\
\text{When arm is 3, force} & = \frac{30}{3} & = 10 \\
\text{Moment of second couple} & = 7 \times 8 & = 56 \\
\text{Moment of resultant couple} & = 30 + 56 & = 86 \\
\text{When arm is 5, force} & = \frac{86}{5} & = 17\frac{1}{5}
\end{array}$$

2. Given the following couples in one plane: —

Force.	Arm.		Force.	Arm.
12	17		5	
3	8			6
5	7	Convert to equivalent	8	
6	9	couples having the	6	
12	12	following: —		7
10	9		4	
14	6			20

The first and the last three are right-handed; the second, third, and fourth are left-handed. Find the moment of the resultant couple, and also its force when it has an arm 11.

§ 58. **Representation of a Couple by a Line.** — From the preceding we see that the effect of a couple remains the same as long as —

1°. *Its moment does not change.*

2°. *The direction of its axis* (*i.e.*, *of the line drawn perpendicular to the plane of the couple*) *does not change.*

3°. *The direction in which it tends to make the body turn* (*right-handed or left-handed*) *remains the same.*

Hence a couple may be represented by drawing a line in the direction of its axis (perpendicular to its plane), and laying off on this line a distance containing as many units of length as there are units of moment in the couple, and indicating by a dot, an arrow-head, or some other means, in what direction one must look along the line in order that the rotation may appear right-handed.

This line is called the *Moment Axis* of the couple.

§ 59. **Composition of Couples situated in Planes inclined to Each Other.** — Suppose we have two couples situated neither in the same plane nor in parallel planes, and that we wish to find their resultant couple. We may proceed as follows: Substitute for them equivalent couples with equal arms, then transfer them in their own plane respectively to such positions that their arms shall coincide, and lie in the line of intersection of the two planes.

This having been done, let OO_1 (Fig. 35) be the common arm, F and $-F$ the forces of one couple, F_1 and $-F_1$ those of the other. The forces F and F_1 have for their resultant R, and $-F$ and $-F_1$ have $-R_1$. Moreover, we may readily show that R and $-R_1$ are equal and

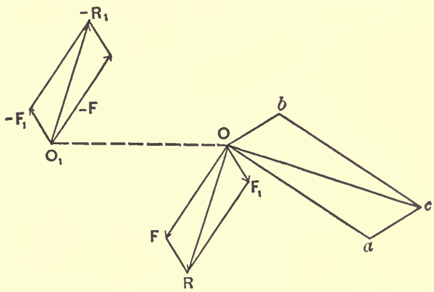


Fig. 35.

parallel, both being perpendicular to OO_1. The resultant of the two couples is, therefore, a couple whose arm is OO_1 and force R, the diagonal of the parallelogram on F and F_1, so that

$$R = \sqrt{F^2 + F_1^2 + 2FF_1 \cos \theta},$$

where θ is the angle between the planes of the couples. Now, if we draw from O the line Oa perpendicular to OO_1 and to F, and hence perpendicular to the plane of the first couple, and if we draw in the same manner Ob perpendicular to the plane of the second couple, so that there shall be in Oa as many units of length as there are units of moment in the first couple, and in Ob as many units of length as there are units of moment in the second couple, we shall have —

1°. The lines Oa and Ob are the moment axes of the two given couples respectively.

2°. The lines Oa and Ob lie in the same plane with F and F_1, this plane being perpendicular to OO_1.

3°. We have the proportion

$$Oa : Ob = F \,.\, OO_1 : F_1 \,.\, OO_1 = F : F_1.$$

4°. If on Oa and Ob as sides we construct a parallelogram, it will be similar to the parallelogram on F and F_1. We shall have the proportion

$$Oc : R = Oa : F = Ob : F_1\,;$$

and since the sides of the two parallelograms are respectively perpendicular to each other, the diagonals are perpendicular to each other; and since we have also

$$Oc = \frac{R \,.\, Oa}{F} \quad \text{and} \quad Oa = F \,.\, OO_1 \quad \therefore \quad Oc = R \,.\, OO_1,$$

it follows that Oc is perpendicular to the plane of the resultant couple, and contains as many units of length as there are units of moment in the moment of the resultant couple; in other words, Oc will represent the *moment axis* of the resultant couple, and we shall have

$$Oc = \sqrt{Oa^2 + Ob^2 + 2\,Oa\,.\,Ob\cos aOb};$$

or, if we let

$$Oa = L, \quad Ob = M, \quad Oc = G, \quad aOb = \theta,$$

$$G = \sqrt{L^2 + M^2 + 2LM\cos\theta}.$$

This determines the moment of the resultant couple; and, for the direction of its moment axis, we have

$$\sin aOc = \frac{M}{G}\sin\theta$$

and

$$\sin bOc = \frac{L}{G}\sin\theta.$$

Hence we can compound and resolve couples just as we do forces, provided we represent the couples by their *moment axes*

EXAMPLES.

1. Given $L = 43$, $M = 15$, $\theta = 65°$; find resultant couple.
2. Given $L = 40$, $M = 30$, $\theta = 30°$; find resultant couple.
3. Given $L = 1$, $M = 5$, $\theta = 45°$; find resultant couple.

§ 60. **Resultant of a Couple and a Single Force in the Same Plane.** — Let M (Fig. 36 or 37) be the moment of the

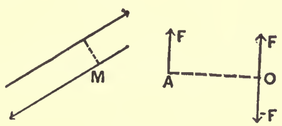


FIG. 36.

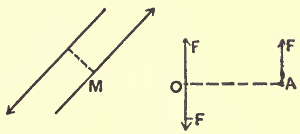


FIG. 37.

given couple, and let $OF = F$ be the single force. For the given couple substitute an equivalent couple, one of whose forces is $-F$ at O, equal and directly opposed to the single force F, these two counterbalancing each other, and leaving only the other force of the couple, which is equal and parallel to the original single force F, and acts along a line whose distance from O is $OA = \frac{M}{F}$. Hence —

The resultant of a single force and a couple in the same plane is a force equal and parallel to the original force, having its line of direction at a perpendicular distance from the original force equal to the moment of the couple divided by the force.

Fig. 36 shows the case when the couple is right-handed, and Fig. 37 when it is left-handed.

§ 61. **Composition of Parallel Forces in General.** — In each case of composition of parallel forces (§§ 34, 35, and 36) it was stated that the method pursued was applicable to all cases except those where

$$\Sigma F = 0.$$

We were obliged, at that time, to reserve this case, because we had not studied the action of a statical couple; but now we will ad pt a method for the composition of parallel forces which will apply in all cases.

(*a*) *When all the forces are in one plane.* Assume, as we did in § 35, the axis OY to be parallel to the forces; assume the forces and the co-ordinates of their lines of direction, as shown in the figure (Fig. 38). Now place at the origin O, along OY, two equal and opposite forces, each equal to F_1; then these three forces, viz., F_1 at D, OA, and OB, produce the same effect as F_1 at D alone; but F_1 at D and OB form a couple (left-handed in the figure) whose moment is $-F_1x_1$. Hence the force F_1 is equivalent to —

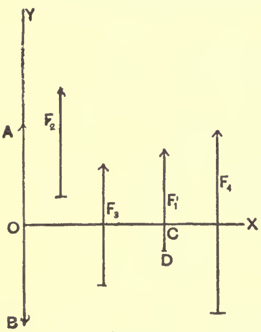


Fig. 38.

1°. An equal and parallel force at the origin, and

2°. A statical couple whose moment is $-F_1x_1$.

Likewise the force F_2 is equivalent to (1°) an equal and parallel force at the origin, and (2°) a couple whose moment is $-F_2x_2$, etc.

Hence we shall have, if we proceed in the same way with all the forces, for resultant of the entire system a single force

$$R = \Sigma F \text{ along } OY,$$

and a single resultant couple

$$M = -\Sigma Fx.$$

(Observe that downward forces and left-handed couples are to be accounted negative.)

Now, there may arise two cases.

1°. When $\Sigma F = 0$, and

2°. When $\Sigma F >< 0$.

Case I. When $\Sigma F = 0$, the resultant force along OY vanishes, and the resultant of the entire system is a statical couple whose moment is

$$M = -\Sigma Fx.$$

Case II. When $\Sigma F >< 0$, we can reduce the resultant to a single force.

Let (Fig. 39) OB represent the resultant force along OY, $R = \Sigma F$. With this, compound the couple whose moment is $M = -\Sigma Fx$, and we obtain as resultant (§ 60) a single force

$$R = \Sigma F,$$

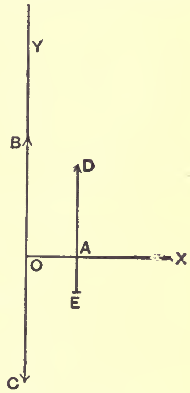


Fig. 39.

whose line of action is at a perpendicular distance from OY equal to

$$AO = x_r = \frac{\Sigma Fx}{\Sigma F}.$$

(*b*) *When the forces are not confined to one plane.* Assume, as before (Fig. 40), OZ parallel to the forces, and let F acting through A be one of the given forces, the co-ordinates of A being x and y. Place at O two equal and opposite forces, each equal to F, and also at B two equal and opposite forces, each equal to F. These five forces produce the same effect as F alone at A, and they may be considered to consist of—

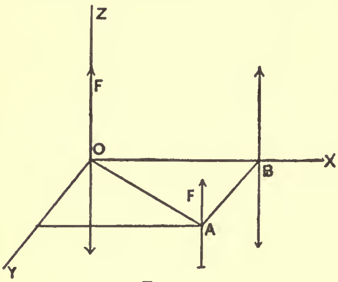


FIG. 40.

1°. A single force F at the origin.

2°. A couple whose forces are F at B and $-F$ at O, and whose moment is $-Fx$ acting in the y plane.

3°. A couple whose forces are F at A and $-F$ at B, and whose moment is Fy acting in the x plane. Treating each of the forces in the same way, we shall have, in place of the entire system of parallel forces, the following forces and couples:—

1°. A single force $R = \Sigma F$ along OZ.

2°. A couple $M_y = -\Sigma Fx$ in the y plane.

3°. A couple $M_x = +\Sigma Fy$ in the x plane.

Now, there may be two cases:—

1°. When $\Sigma F >< 0$.

2°. When $\Sigma F = 0$.

CASE I. When $\Sigma F >< 0$, we can reduce to a single resultant force having a fixed line of direction. Lay off (Fig. 41) along OZ, $OH = \Sigma F$.

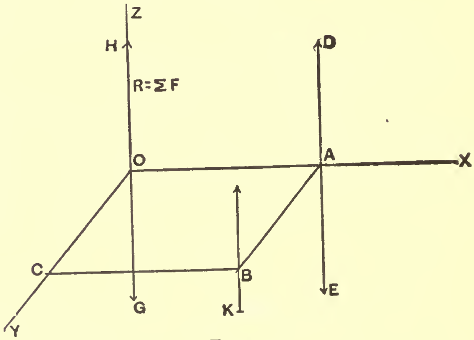


FIG. 41.

Combining this with the first of the above-stated couples, we obtain a force $R = \Sigma F$ at A, where $OA = \dfrac{\Sigma Fx}{\Sigma F} = x_r$. Then combine with this resultant force $R = \Sigma F$ at A, the second couple, and we shall have as single resultant of the entire system a single force

$$R = \Sigma F$$

acting through B, where

$$AB = y_r = \frac{\Sigma Fy}{\Sigma F}.$$

Hence the resultant is a force whose magnitude is

$$R = \Sigma F,$$

the co-ordinates of its line of direction being

$$x_r = \frac{\Sigma Fx}{\Sigma F}, \quad y_r = \frac{\Sigma Fy}{\Sigma F}.$$

CASE II. When $\Sigma F = 0$, there is no single resultant force; but the system reduces to two couples, one in the x plane and one in the y plane, and these two can be reduced to one single resultant couple. (Observe that couples are to be accounted positive when, on being looked at by the observer from the positive part of the axis towards the origin, they are right-handed; otherwise they are negative.)

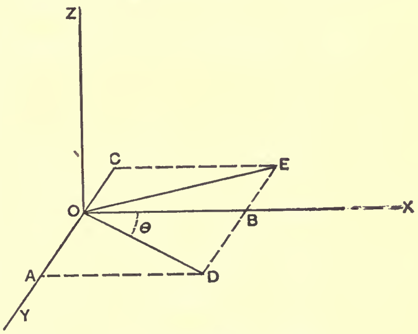


FIG. 42.

The moment axis of the couple in the x plane will be laid off on the axis OX from the origin towards the positive side if the moment is positive, or towards the negative side if it is negative, and likewise for the couple in the y plane.

Hence lay off (Fig. 42) $OB = M_x$, $OA = M_y$, and by completing the rectangle we shall have OD as the moment axis of the resultant couple; hence the resultant couple lies in a plane perpendicular to OD, and its moment bears to OD the same ratio as M_x bears to OB.

Hence we may write

$$OD = M_r = \sqrt{M_x^2 + M_y^2},$$

$$\cos DOX = \frac{M_x}{M_r} = \cos \theta.$$

If M_y had been negative, we should have OE as the moment axis of the resultant couple.

EXAMPLES.

	F.	*x.*	*y.*		*F.*	*x.*	*y.*
1.	5	4	3	2.	5	−4	3
	3	2	1		−2	2	−1
	1	3	5		−3	3	5

Find the resultant in each example.

§ 62. Resultant of any System of Forces acting at Different Points of a Rigid Body, all situated in One Plane. — Let $CF = F$ (Fig. 43) be one of the given forces. Let all the forces be referred to a system of rectangular axes, as in the figure, and let α = angle made by F with OX, etc. Let the co-ordinates of the point of application of F be $AO = x$, $BO = y$.

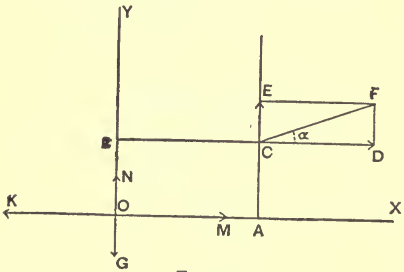


Fig. 43.

We first decompose $CF = F$ into two components, parallel respectively to OX and OY. These components are

$$CD = F\cos\alpha, \quad CE = F\sin\alpha.$$

Apply at O in the line OY two equal and opposite forces, each equal to $F\sin\alpha$, and at O in the line OX two equal and opposite forces, each equal to $F\cos\alpha$. Since these four are mutually balanced, they do not alter the effect of the single force; and hence we have, in place of F at C, the six forces CD, OM, OK, CE, ON, OG. Of these six, CE and OG form a couple whose moment is

$$-(F\sin\alpha)x = -Fx\sin\alpha,$$

CD and OK form a couple whose moment is

$$(F\cos\alpha)y = Fy\cos\alpha.$$

These two couples, being in the same plane, give as resultant moment their algebraic sum, or

$$F(y\cos\alpha - x\sin\alpha).$$

We have, therefore, instead of the single force at C, the following:—

1°. $OM = F\cos\alpha$ along OX.
2°. $ON = F\sin\alpha$ along OY.
3°. The couple $M = F(y\cos\alpha - x\sin\alpha)$ in the given plane.

Decompose in the same way each of the given forces; and we have, on uniting the components along OX, those along OY, and the statical couples respectively, the following:—

1°. A resultant force along OX, $R_x = \Sigma F\cos\alpha$.
2°. A resultant force along OY, $R_y = \Sigma F\sin\alpha$.
3°. A resultant couple in the plane, whose moment is

$$M = \Sigma F(y\cos\alpha - x\sin\alpha).$$

This entire system, on compounding the two forces at O, reduces to

1°. $R = \sqrt{R_x^2 + R_y^2} = \sqrt{(\Sigma F \cos \alpha)^2 + (\Sigma F \sin \alpha)^2}$;

making with OX an angle α_r, where

$$\cos \alpha_r = \frac{\Sigma F \cos \alpha}{R}.$$

2°. A resultant couple in the same plane, whose moment is

$$M = \Sigma F(y \cos \alpha - x \sin \alpha).$$

Now compound this resultant force and couple, and we have, for final resultant, a single force equal and parallel to R, and acting along a line whose perpendicular distance from O is equal to

$$\frac{M}{R}.$$

The equation of this line may be found as follows:

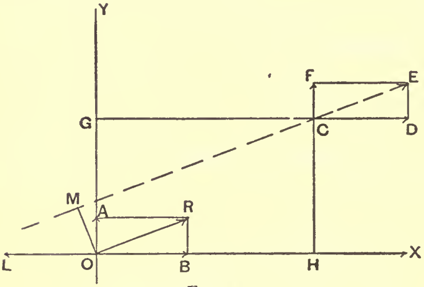


Fig. 44.

Suppose (Fig. 44) the force

$$OB = \Sigma F \cos \alpha,$$

$$OA = \Sigma F \sin \alpha,$$

$$OR = \sqrt{(\Sigma F \cos \alpha)^2 + (\Sigma F \sin \alpha)^2};$$

and let us suppose the resultant couple to be right-handed, and let

$$OM = \frac{M}{R}:$$

then will the line ME parallel to OR be the line of direction of the single resultant force.

Assuming the force R to act at any point $C(x_r, y_r)$ of this line, if we decompose it in the same way as we did the single forces previously, we obtain —

1°. The force $R\cos\alpha_r = \Sigma F\cos\alpha$ along OX.

2°. The force $R\sin\alpha_r = \Sigma F\sin\alpha$ along OY.

3°. The couple $R(y_r\cos\alpha_r - x_r\sin\alpha_r)$.

Hence we must have

$$R(y_r\cos\alpha_r - x_r\sin\alpha_r) = \Sigma F(y\cos\alpha - x\sin\alpha) = M.$$

Hence for the equation of the line of direction we have

$$y_r\cos\alpha_r - x_r\sin\alpha_r = \frac{M}{R}. \qquad (1)$$

Another form for the same equation is

$$y_r(\Sigma F\cos\alpha) - x_r(\Sigma F\sin\alpha) = M. \qquad (2)$$

§ 63. **Conditions of Equilibrium.** — If such a set of forces be in equilibrium, there must evidently be no tendency to translation and none to rotation. Hence we must have

$$R = 0 \quad\text{and}\quad M = 0.$$

Hence the conditions of equilibrium for any system of forces in a plane are three; viz., —

$$\Sigma F\cos\alpha = 0, \quad \Sigma F\sin\alpha = 0, \quad \Sigma F(y\cos\alpha - x\sin\alpha) = 0.$$

Another and a very convenient way to state the conditions of equilibrium for this case is as follows: —

If the forces be resolved into components along two directions at right angles to each other, then the algebraic sum of the components along each of these directions must be zero, and the algebraic sum of the moments of the forces about any axis perpendicular to the plane of the forces must equal zero.

EXAMPLES.

	$F.$	$x.$	$y.$	$\alpha.$	
1. Given	5	3	2	31°	Find the resultant, and the equation of its line of direction.
	10	1	3	40°	
	−7	4	2	54°	

	$F.$	$x.$	$y.$	$\alpha.$	
2. Given	12	27	3	15°	Find the resultant, and the equation of its line of direction.
	4	13	−5	30°	
	8	−5	−4	45°	

§ 64. **Resultant of any System of Forces not confined to One Plane.** — Suppose we have a number of forces applied at different points of a rigid body, and acting in different directions, of which we wish to find the resultant. Refer them all to a system of three rectangular axes, OX, OY, OZ (Fig. 45). Let $PR = F$ be one of the given forces. Resolve it into three components, PK, PH, and PG, parallel respectively to the three axes. Let

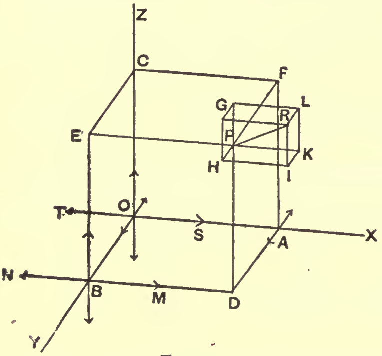


FIG. 45.

$$RPK = \alpha, \quad RPH = \beta, \quad RPG = \gamma.$$

Let $OA = x$, $OB = y$, $OC = z$, be the co-ordinates of the point of application of the force F. Now introduce at B and also at O two forces, opposite in direction, and each equal to PK. We now have, instead of the force PK, the five forces PK, BM, BN, OS, and OT. The two forces PK and BN form a couple in the y plane, whose axis is a line parallel to the axis OY, and whose moment is $(PK)(EB) = (F \cos \alpha)z = Fz \cos \alpha$. The forces BM and OT form a couple in the z plane, whose moment is

$$(BM)(OB) = -Fy\cos\alpha.$$

Now do the same for the other forces PH and PG, and we shall finally have, instead of the force PR, three forces,

$$F\cos\alpha, \quad F\cos\beta, \quad F\cos\gamma,$$

acting at O in the directions OX, OY, and OZ respectively, together with six couples, two of which are in the x plane, two in the y plane, and two in the z plane.

They thus form three couples, whose moments are as follows: —

$$\begin{aligned}
&\text{Around } OX,\ F(y\cos\gamma - z\cos\beta);\\
&\text{Around } OY,\ F(z\cos\alpha - x\cos\gamma);\\
&\text{Around } OZ,\ F(x\cos\beta - y\cos\alpha).
\end{aligned}$$

Treat each of the given forces in the same way, and we shall have, in place of all the forces of the system, three forces,

$$\begin{aligned}
&\Sigma F\cos\alpha \text{ along } OX,\\
&\Sigma F\cos\beta \text{ along } OY,\\
&\Sigma F\cos\gamma \text{ along } OZ;
\end{aligned}$$

and three couples, whose moments are as follows: —

$$\begin{aligned}
&\text{Around } OX,\ M_x = \Sigma F(y\cos\gamma - z\cos\beta);\\
&\text{Around } OY,\ M_y = \Sigma F(z\cos\alpha - x\cos\gamma);\\
&\text{Around } OZ,\ M_z = \Sigma F(x\cos\beta - y\cos\alpha).
\end{aligned}$$

The three forces give a resultant at O equal to

$$R = \sqrt{(\Sigma F\cos\alpha)^2 + (\Sigma F\cos\beta)^2 + (\Sigma F\cos\gamma)^2}, \tag{1}$$

$$\cos\alpha_r = \frac{\Sigma F\cos\alpha}{R}, \quad \cos\beta_r = \frac{\Sigma F\cos\beta}{R}, \quad \cos\gamma_r = \frac{\Sigma F\cos\gamma}{R}. \tag{2}$$

For the three couples we have as resultant

$$M = \sqrt{M_x^2 + M_y^2 + M_z^2}, \tag{3}$$

$$\cos\lambda = \frac{M_x}{M}, \quad \cos\mu = \frac{M_y}{M}, \quad \cos\nu = \frac{M_z}{M}; \tag{4}$$

λ, μ, and ν being the angles made by the moment axis of the resultant couple with OX, OY, and OZ respectively.

Thus far we have reduced the whole system to a single resultant force at the origin, and a couple. Sometimes we can reduce the system still farther, and sometimes not. The following investigation will show when we can do so. Let (Fig. 46) $OP = R$ be the resultant force, and $OC = M$ the moment axis of the resultant couple. Denote the angle between them by θ (a quantity thus far undetermined). Project $OP = R$ on OC. Its projection will be $OD = R\cos\theta$; then project, in its stead, the broken line $OABP$ on OC. By the principles of projections, the projection of this broken line will equal OD.

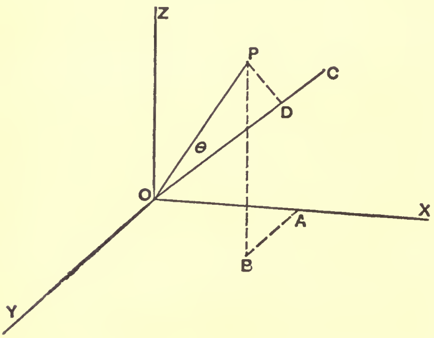


Fig. 46.

Now OA, AB, and BP are the co-ordinates of P, and make with OC the same angle as the axes OX, OY, and OZ; i.e., λ, μ, and ν respectively: hence the length of the projection is

$$OA\cos\lambda + AB\cos\mu + BP\cos\nu.$$

But

$$OA = R\cos\alpha_r, \quad AB = R\cos\beta_r, \quad BP = R\cos\gamma_r.$$

Hence

$$R\cos\theta = R\cos\alpha_r\cos\lambda + R\cos\beta_r\cos\mu + R\cos\gamma_r\cos\nu$$

$$\therefore \quad \cos\theta = \cos\alpha_r\cos\lambda + \cos\beta_r\cos\mu + \cos\gamma_r\cos\nu. \tag{5}$$

This enables us to find the angle between the resultant force and the moment axis of the resultant couple.

The following cases may arise: —

1°. When $\cos\theta = 0$, or $\theta = 90°$, the force lies in the plane of the couple, and we can reduce to a single force acting at a distance from O equal to $\frac{M}{R}$, and parallel to R at O.

2°. When $\cos\theta = 1$, or $\theta = 0$, the moment axis of the couple coincides in direction with the force: hence the plane of the couple is perpendicular to the force, and no farther reduction is possible.

3°. When θ is neither 0° nor 90°, we can resolve the couple M into two component couples, one of which, $M\cos\theta$, acts in a plane perpendicular to the direction of R, and the other, $M\sin\theta$, acts in a plane containing R. The latter, on being combined with the force R at the origin, gives an equal and parallel force whose line of action is at a distance from that of R at O, equal to

$$\frac{M\sin\theta}{R}.$$

4°. When $M = 0$, the resultant is a single force at O.

5°. When $R = 0$, the resultant is a couple.

§ 65. **Conditions of Equilibrium.** — To produce equilibrium, we must have no tendency to translation and none to rotation. Hence we must have

$$R = 0 \quad \text{and} \quad M = 0.$$

Hence we have, in general, six conditions of equilibrium; viz., —

$$\Sigma F\cos\alpha = 0, \quad \Sigma F\cos\beta = 0, \quad \Sigma F\cos\gamma = 0.$$
$$M_x = 0, \quad M_y = 0, \quad M_z = 0.$$

EXAMPLES.

1. Prove that, whenever three forces balance each other, they must lie in one plane.

2. Show how to resolve a given force into two whose sum is given, the direction of one being also given.

3. A straight rod of uniform section and material is suspended by two strings attached to its ends, the strings being of given length, and attached to the same fixed point: find the position of equilibrium of the rod.

4. Two spheres are supported by strings attached to a given point, and rest against each other: find the tensions of the strings.

5. A straight rod of uniform section and material has its ends resting against two inclined planes at right angles to each other, the vertical plane which passes through the rod being at right angles to the line of intersection of the two planes: find the position of equilibrium of the rod, and the pressure on each plane, disregarding friction.

6. A certain body weighs 8 lbs. when placed in one pan of a false balance of equal arms, and 10 lbs. in the other: find the true weight of the body.

7. The points of attachment of the three legs of a three-legged table are the vertices of an isosceles right-angled triangle; a weight of 100 lbs. is supported at the middle of a line joining the vertex of one of the acute angles with the middle of the opposite side: find the pressure upon each leg.

8. A heavy body rests upon an inclined plane without friction: find the horizontal force necessary to apply, to prevent it from falling.

9. A rectangular picture is supported by a string passing over a smooth peg, the string being attached in the usual way at the sides, but one-fourth the distance from the top: find how many and what are the positions of equilibrium, assuming the absence of friction.

10. Two equal and weightless rods are jointed together, and form a right angle; they move freely about their common point: find the ratio of the weights that must be suspended from their extremities, that one of them may be inclined to the horizon at sixty degrees.

11. A weight of 100 lbs. is suspended by two flexible strings, one of which is horizontal, and the other is inclined at an angle of thirty degrees to the vertical: find the tension in each string.

CHAPTER II.

DYNAMICS.

§ 66. **Definitions.** — *Dynamics* is that part of mechanics which discusses the forces acting, when motion is the result.

Velocity, in the case of uniform motion, is the space passed over by the moving body in a unit of time; so that, if s represent the space passed over in time t, and v represent the velocity, then

$$v = \frac{s}{t}.$$

Velocity, in variable motion, is the limit of the ratio of the space (Δs) passed over in a short time (Δt), to the time, as the latter approaches zero: hence

$$v = \frac{ds}{dt}.$$

Acceleration is the limit of the ratio of the velocity (Δv) imparted to the moving body in a short time (Δt), to the time, as the time approaches zero. Hence, if a represent the acceleration,

$$a = \frac{dv}{dt} = \frac{d\left(\frac{ds}{dt}\right)}{dt} = \frac{d^2s}{dt^2}.$$

§ 67. **Uniform Motion.** — In this case the acceleration is zero, and the velocity is constant; and we have the equation

$$s = vt.$$

§ 68. **Uniformly Varying Motion.** — In this case the acceleration is constant: hence a is a constant in the equation

$$\frac{d^2s}{dt^2} = a,$$

and we obtain by one integration

$$v = \frac{ds}{dt} = at + c,$$

where c is an arbitrary constant: to determine it we observe, that, if v_0 represent the value of v when $t = 0$, we shall have

$$v_0 = 0 + c$$

$$\therefore \quad c = v_0$$

$$\therefore \quad v = \frac{ds}{dt} = at + v_0,$$

and by another integration

$$s = \tfrac{1}{2}at^2 + v_0 t,$$

where s is the space passed over in time t; the arbitrary constant vanishing, because, when $t = 0$, s is also zero.

§ 69. **Measure of Force.** — It has already been seen, that, when a body is either at rest or moving uniformly in a straight line, there are either no forces acting upon it, or else the forces acting upon it are balanced. If, on the other hand, the motion of the body is rectilinear, but not uniform, the only unbalanced force acting is in the direction of the motion, and equal in magnitude to the momentum imparted in a unit of time in the direction of the motion, or, in other words, to the limit of the ratio of the momentum imparted in a short time (Δt), to the time, as the latter approaches zero.

Thus, if F denote the force acting in the direction of the motion, m the mass, and a the acceleration, we shall have

$$F = ma = m\frac{dv}{dt} = m\frac{d^2s}{dt^2}. \qquad (1)$$

From (1) we derive

$$mdv = Fdt; \qquad (2)$$

and, if v_0 be the velocity of the moving body at the time when $t = t_0$, and v_1 its velocity when $t = t_1$, we shall have

$$\int_{v_0}^{v_1} mdv = \int_{t_0}^{t_1} Fdt$$

or

$$m(v_1 - v_0) = \int_{t_0}^{t_1} Fdt; \qquad (3)$$

or, in words, the momentum imparted to the body during the time $t = (t_1 - t_0)$ by the force F, will be found by integrating the quantity Fdt between the limits t_1 and t_0.

§ 70. **Mechanical Work.** — Whenever a force is applied to a moving body, the force is either used in overcoming resistances (i.e., opposing forces, such as gravity or friction), and leaving the body free to continue its original motion undisturbed, or else it has its effect in altering the velocity of the body. In either case, the work done by the force is the product of the force, by the space passed through by the body in the direction of the force.

Unit of Work. — The unit of work is that work which is done when a unit of force acts through a unit of distance in the same direction as the force; thus, if one pound and one foot are our units of force and length respectively, the unit of work will be one foot-pound.

If a constant force act upon a moving body in the direction of its motion while the body moves through the space s, the work done by the force is

$$Fs;$$

and this, if the force is unresisted, is the energy, or capacity for performing work, which is imparted to the body upon which the force acts while it moves through the space s.

Thus, if a 10-pound weight fall freely through a height of 5 feet, the energy imparted to it by the force of gravity during this fall is $10 \times 5 = 50$ foot-pounds, and it would be necessary to do upon it 50 foot-pounds of work in order to destroy the velocity acquired by it during its fall. If, on the other hand, the force is a variable, the amount of work done in passing over any finite space in its own direction will be found by integrating, between the proper limits, the expression

$$\int F ds.$$

The power which a machine exerts is the work which it performs in a unit of time.

The unit of power commonly employed is the *horse-power*, which in English units is equal to 33000 foot-pounds per minute, or 550 foot-pounds per second.

§ 71. **Energy.** — The energy of a body is its capacity for performing work.

Kinetic or Actual Energy is the energy which a body possesses in virtue of its velocity; in other words, it is the work necessary to be done upon the body in order to destroy its velocity. This is equal to the work which would have to be done to bring the body from a state of rest to the velocity with which it is moving. Assume a body whose mass is m, and suppose that its velocity has been changed from v_0 to v_1. Then if F be the force acting in the direction of the motion, we shall have, from equation (2), § 69, that

$$Fvdt = mvdv; \qquad (1)$$

but

$$vdt = ds$$

$$\therefore \quad Fds = mvdv. \qquad (2)$$

Hence, by integration,

$$\int_{v_0}^{v_1} mvdv = \int Fds$$

$$\therefore \quad \tfrac{1}{2}m(v_1^2 - v_0^2) = \int Fds; \qquad (3)$$

but $\int Fds$ is the work that has been done on the body by the force, and the result of doing this work has been to increase its velocity from v_0 to v_1. It follows, that, in order to change the velocity from v_0 to v_1, the amount of work necessary to perform upon the body is

$$\tfrac{1}{2}m(v_1^2 - v_0^2) = \tfrac{1}{2}\frac{W}{g}(v_1^2 - v_0^2). \qquad (4)$$

If $v_0 = 0$, this expression becomes

$$\tfrac{1}{2}mv_1^2, \text{ or } \frac{Wv_1^2}{2g}, \qquad (5)$$

which is the expression for the kinetic energy of a body of mass m moving with a velocity v_1.

§ 72. **Atwood's Machine.** — A particular case of uniformly accelerated motion is to be found in Atwood's machine, in which a cord is passed over a pulley, and is loaded with unequal weights on the two sides. Were the weights equal, there would be no unbalanced force acting, and no motion would ensue; but when they are unequal, we obtain as a result a uniformly accelerated motion (if we disregard the action of the pulley), because we have a constant force equal to the difference of the two weights acting on a mass whose weight is the sum of the two weights. Thus, if we have a 10-pound weight on one side and a 5-pound weight on the other, the unbalanced force acting is

$$F = 10 - 5 = 5 \text{ lbs.}$$

The mass moved is $M = \frac{10 + 5}{g}$: hence the resulting acceleration is

$$a = \frac{5}{\left(\frac{15}{g}\right)} = \frac{g}{3}.$$

§ 73. **Normal and Tangential Components of the Forces acting on a Heavy Particle.** — If a body be in motion, either in a straight or in a curved line, and if at a certain instant all forces cease acting on it, the body will continue to move at a uniform rate in a straight line tangent to its path at that point where the body was situated when the forces ceased acting.

If an unresisted force be applied in the direction of the body's motion, the motion will still take place in the same straight line; but the velocity will vary as long as the force acts, and, from what we have seen, the equation

$$F = m\frac{d^2s}{dt^2} \qquad (1)$$

will hold.

If an unresisted force act in a direction inclined to the body's motion, it will cause the body to change its speed, and also its course, and hence to move in a curved line. Indeed, if a force acting on a body which is in motion be resolved into two components, one of which is tangent to its path and the other normal, the tangential component will cause the body to change its speed, and the normal component will cause it to change the direction of its motion.

The measure of the tangential component is, as we have seen,

$$F = m\frac{d^2s}{dt^2};$$

and we will proceed to find an expression for the normal component otherwise known as the *Deviating Force.* For this purpose we may substitute, for a small portion of the curve, a portion of the circle of curvature; hence we will proceed to find an expression for the centrifugal force of a body which moves uniformly with a velocity v in a circle whose radius is r.

CENTRIFUGAL FORCE.

Let AC (Fig. 47) be the space described in the time Δt. Then we have

$$AC = v\Delta t.$$

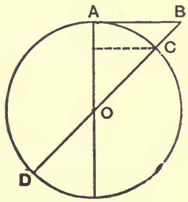


FIG. 47.

The motion AC may be approximately considered as the result of a uniform motion

$$AB = v\Delta t \text{ nearly},$$

and a uniformly accelerated motion

$$BC = \tfrac{1}{2}a(\Delta t)^2 = s,$$

where a = acceleration due to centrifugal force. But

$$(AB)^2 = BC \,.\, BD,$$

or

$$(v\Delta t)^2 = \tfrac{1}{2}a(\Delta t)^2(2r + s),$$

where

$$AO = OC = r$$

$$\therefore \quad v^2 = \tfrac{1}{2}a(2r + s) \text{ approximately}$$

$$\therefore \quad a = \frac{2v^2}{2r + s} \text{ approximately.}$$

For its true value, pass to the limit where $s = 0$.

Hence we have, for the acceleration due to the centrifugal force, the expression

$$\frac{v^2}{r}.$$

Hence the centrifugal force is equal to

$$F = \frac{mv^2}{r} = \frac{Wv^2}{gr}. \qquad (2)$$

DEVIATING FORCE.

If a body is moving in a curved path, whether circular or not, and the unbalanced force acting on it be resolved into tangential and normal components, the tangential component will be, as has already been seen,

$$m\frac{d^2s}{dt^2};$$

and the normal component will be

$$\frac{mv^2}{r} = \frac{m}{r}\left(\frac{ds}{dt}\right)^2,$$

where r is the radius of curvature of the path at the point in question.

RESULTANT FORCE.

Hence it follows that the entire unbalanced force acting on the body will be

$$F = \sqrt{\left(m\frac{d^2s}{dt^2}\right)^2 + \left(\frac{m}{r}\frac{ds^2}{dt^2}\right)^2},$$

or

$$F = m\sqrt{\left(\frac{d^2s}{dt^2}\right)^2 + \frac{1}{r^2}\left(\frac{ds}{dt}\right)^4}. \qquad (3)$$

§ 74. **Components along Three Rectangular Axes of the Velocities of, and of the Forces acting on, a Moving Body.**—If we resolve the velocity $\frac{ds}{dt}$ into three components along OX, OY, and OZ, we shall have, for these components respectively,

$$\frac{dx}{dt}, \quad \frac{dy}{dt}, \quad \text{and} \quad \frac{dz}{dt};$$

this being evident from the fact that dx, dy, and dz are respectively the projections of ds on the axes OX, OY, and OZ; and, from the differential calculus, we have

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}.$$

On the other hand,

$$\frac{dx}{dt}, \quad \frac{dy}{dt}, \quad \text{and} \quad \frac{dz}{dt}$$

are not only the components of the velocity $\frac{ds}{dt}$ in the directions OX, OY, and OZ, but they are also the velocities of the body in these directions respectively.

Now, the case of the accelerations is different; for, while

$$\frac{d^2x}{dt^2}, \quad \frac{d^2y}{dt^2}, \quad \text{and} \quad \frac{d^2z}{dt^2}$$

are the accelerations in the directions OX, OY, and OZ respectively, they are not the components of the acceleration

$$\frac{d^2s}{dt^2}$$

along the three axes.

That they are the former is evident from the fact that $\frac{dx}{dt}$, $\frac{dy}{dt}$, and $\frac{dz}{dt}$ are the velocities in the directions of the axes, and $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2z}{dt^2}$ are their differential co-efficients, and hence represent the accelerations along the three axes. But if we consider the components of the force acting on the body, we shall have for its components along OX, OY, and OZ, if α, β, and γ are the angles made by F with the axes respectively,

$$F\cos\alpha = m\frac{d^2x}{dt^2}, \quad F\cos\beta = m\frac{d^2y}{dt^2}, \quad F\cos\gamma = m\frac{d^2z}{dt^2},$$

$$\therefore \quad F \quad = m\sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2 + \left(\frac{d^2z}{dt^2}\right)^2}; \qquad (1)$$

and we found (§ 73) for F, the value

$$F = m\sqrt{\left(\frac{d^2s}{dt^2}\right)^2 + \frac{1}{r^2}\left(\frac{ds}{dt}\right)^4}. \qquad (2)$$

Hence, equating these values of F, and simplifying, we shall have the equation

$$\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2 + \left(\frac{d^2z}{dt^2}\right)^2 = \left(\frac{d^2s}{dt^2}\right)^2 + \frac{1}{r^2}\left(\frac{ds}{dt}\right)^4$$

Hence it is plain that $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, and $\frac{d^2z}{dt^2}$ can only be the components of the actual acceleration

$$\frac{d^2s}{dt^2}$$

when the last term $\frac{1}{r^2}\left(\frac{ds}{dt}\right)^4$ vanishes, or when $r = \infty$, i.e., when the motion is rectilinear.

Moreover, we have the two expressions (1) and (2) for the force acting upon a moving body.

The truth of the proposition just proved may also be seen from the following considerations : —

If a parallelopiped be constructed with the edges

$$\frac{dx}{dt}, \quad \frac{dy}{dt}, \quad \frac{dz}{dt},$$

the diagonal will be the actual velocity

$$\frac{ds}{dt},$$

and will, of course, coincide in direction with its path.

On the other hand, if a parallelopiped be constructed with the edges

$$\frac{d^2x}{dt^2}, \quad \frac{d^2y}{dt^2}, \quad \frac{d^2z}{dt^2},$$

its diagonal must coincide in direction with the force

$$F = m\sqrt{\left(\frac{d^2s}{dt^2}\right)^2 + \frac{1}{r^2}\left(\frac{ds}{dt}\right)^4},$$

and can coincide in direction with the path, and hence with the actual acceleration

$$\frac{d^2s}{dt^2},$$

only when the force is tangential to the path, and hence when the motion is rectilinear.

§ 75. **Centrifugal Force of a Solid Body.** — When a solid body revolves in a circle, the resultant centrifugal force of the entire body acts in the direction of the perpendicular let fall from the centre of gravity of the body on the axis of rotation, and its magnitude is the same as if its entire weight were concentrated at its centre of gravity.

Proof. — Let (Fig. 48) the angular velocity $= \alpha$, and the total weight $= W$. Assume the axis of rotation perpendicular t the plane of the paper and passing through O; assume, as axis of x, the perpendicular dropped from the centre of gravity upon the axis of rotation. The co-ordinates of the centre of gravity will then be (x_0, y_0), and y_0 will be equal to zero.

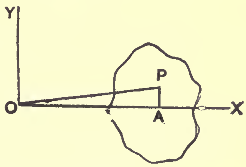


Fig. 48.

If, now, P be any particle of weight w, where $r =$ perpendicular distance from P on axis of rotation, and $x = OA$, $y = AP$, we shall have for the centrifugal force of the particle at P

$$\frac{w}{g}\alpha^2 r;$$

but if we resolve this into two components, parallel respectively to OX and OY, we shall have for these components

$$\left(\frac{w}{g}\alpha^2 r\right)\frac{x}{r} = \frac{\alpha^2}{g}wx \quad \text{and} \quad \left(\frac{w}{g}\alpha^2 r\right)\frac{y}{r} = \frac{\alpha^2}{g}wy,$$

and, for the resultant for the entire body we shall have, parallel to OX,

$$F_x = \frac{\alpha^2}{g}\Sigma wx = \frac{\alpha^2}{g}Wx_0, \tag{1}$$

and

$$F_y = \frac{\alpha^2}{g}\Sigma wy = \frac{\alpha^2}{g}Wy_0 = 0. \tag{2}$$

Hence the centrifugal force of the entire body is

$$F = \frac{\alpha^2}{g}Wx_0; \tag{3}$$

and if we let $v_0 = \alpha x_0 =$ linear velocity of the centre of gravity, we have

$$F = \frac{Wv_0^2}{gx_0},$$

which is the same as though the entire weight of the body were concentrated at its centre of gravity.

EXAMPLES.

1. A 10-pound weight is fastened by a rope 5 feet long to the centre, around which it revolves at the rate of 200 turns per minute; find the pull on the cord.

2. A locomotive weighing 50000 lbs., whose driving-wheels weigh [illegible]00 lbs., is running at 60 miles per hour, the diameter of the drivers

being 6 feet, and the distance from the centre of the wheel to the centre of gravity of the same being 2 inches (the drivers not being properly balanced); find the pressure of the locomotive on the track (a) when the centre of gravity is directly below the centre of the wheel, and (b) when it is directly above.

3. Assume the same conditions, except that the distance between centre of the wheel and its centre of gravity is 5 inches instead of 2.

§ 76. **Uniformly Varying Rectilinear Motion.** — We have already found for this case (§ 68) the equations

$$\frac{d^2s}{dt^2} = a = \text{a constant},$$

$$\frac{ds}{dt} = v = v_0 + at,$$

$$s = v_0 t + \tfrac{1}{2}at^2;$$

and we may write for the force acting, which is, of course, coincident in direction with the motion,

$$F = m\frac{d^2s}{dt^2} = ma = \text{a constant}.$$

§ 77. **Motion of a Body acted on by the Force of Gravity only.** — A useful special case of uniformly varying motion is that of a body moving under the action of gravity only.

The downward acceleration due to gravity is represented by g feet per second, the value of g varying at different points on the surface of the earth according to the following law: —

$$g = g_1(1 - 0.00284 \cos 2\lambda)\left(1 - \frac{2h}{R}\right) \text{ feet per second},$$

where

$g_1 = 32.1695$ feet,
λ = latitude of the place,
h = its elevation above mean sea-level in feet,
$R = 20900000$ feet.

If, now, we represent by h the height fallen through by a descending body in time t, we shall have the equations,

$$v_t = v_0 + gt,$$
$$h = v_0 t + \tfrac{1}{2}gt^2,$$

where v_0 is the initial downward velocity.

If, on the other hand, we represent by v_0 the initial upward velocity, and by h the height to which the body will rise in time t under the action of gravity only, we must write the equations

$$v = v_0 - gt,$$
$$h = v_0 t - \tfrac{1}{2}gt^2.$$

When $v_0 = 0$, the first set of equations gives

$$v = gt,$$
$$h = \tfrac{1}{2}gt^2,$$

which express the law of motion of a body starting from rest and subject to the action of gravity only.

Eliminate t between these equations, and we shall have

$$v^2 = 2gh \qquad \therefore \quad v = \sqrt{2gh},$$

or

$$h = \frac{v^2}{2g}:$$

h is called the *height due to the velocity* v, and represents the height through which a falling body must drop to acquire the velocity v; and

$$v = \sqrt{2gh}$$

is the velocity which a falling body will acquire in falling through the height h. Thus, if a body fall through a height of 50 feet, it will, by that fall, acquire a velocity of about

$$\sqrt{2(32\tfrac{1}{6})(50)} = \sqrt{3216.66} = 56.7 \text{ feet per second.}$$

Again: if a body has a velocity of 40 feet per second, we shall have

$$h = \frac{v^2}{2g} = \frac{1600}{64.3} = 24.8 \text{ feet};$$

and we say that the body has a velocity due to the height 24.8 feet, i. e., a velocity which it would acquire by falling through a height of 24.8 feet.

EXAMPLES.

1. A stone is dropped down a precipice, and is heard to strike the bottom in 4 seconds after it started: how high is the precipice?

2. How long will a stone, dropped down a precipice 500 feet high, take to reach the bottom?

3. What will be its velocity just before striking the ground?

4. A body is thrown vertically upwards with a velocity of 100 feet per second; to what height will it rise?

5. A body is thrown vertically upwards, and rises to a height of 50 feet. With what velocity was it thrown, and how long was it in its ascent?

6. What will be its velocity in its ascent at a point 15 feet above the point from which it started, and what at the same point in its descent?

§ 78. **Unresisted Projectile.** — In the case of an unresisted projectile, we have a body on which is impressed a uniform motion in a certain direction (the direction of its initial motion), and which is acted on by the force of gravity only.

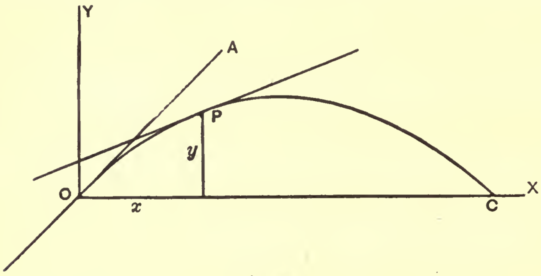


FIG. 49.

Let OPC be the path (Fig. 49), OA the initial direction, and v_0 the initial velocity, and the angle $AOX = \theta$.

Then we shall have, for the horizontal and vertical components of the unbalanced force acting, when the projectile is at P (co-ordinates x and y),

$$m\frac{d^2x}{dt^2} = 0 \text{ along } OX, \text{ and } m\frac{d^2y}{dt^2} = -mg = -W \text{ along } OY.$$

Hence

$$\frac{d^2x}{dt^2} = 0, \qquad (1) \qquad \frac{d^2y}{dt^2} = -g. \qquad (2)$$

Integrating, and observing, that, when $t = 0$, the horizontal and the vertical velocities were respectively $v_0 \cos\theta$ and $v_0 \sin\theta$, we have

$$\frac{dx}{dt} = v_0 \cos\theta, \qquad (3)$$

$$\frac{dy}{dt} = v_0 \sin\theta - gt. \qquad (4)$$

These equations could be derived directly by observing that the horizontal component of the initial velocity is $v_0 \cos\theta$, and that this remains constant, as there is no unbalanced force acting in this direction, also that $v_0 \sin\theta$ is the initial vertical velocity; and, since the body is acted on by gravity only, this velocity will in time t be decreased by gt.

Integrating equations (3) and (4), and observing that for $t = 0$, x and y are both zero, we obtain

$$x = v_0 \cos \theta . t, \tag{5}$$

$$y = v_0 \sin \theta . t - \tfrac{1}{2} g t^2. \tag{6}$$

Eliminate t, and we have

$$y = x \tan \theta - \frac{g x^2}{2 v_0^2 \cos^2 \theta} \tag{7}$$

as the equation of the path, which is consequently a parabola.

Equations (1), (2), (3), (4), (5), (6), and (7) enable us to solve any problem with reference to an unresisted projectile.

Equation (7) may be written

$$\left(y - \frac{v_0^2 \sin^2 \theta}{2g}\right) = - \frac{g}{2 v_0^2 \cos^2 \theta} \left(x - \frac{v_0^2 \sin \theta \cos \theta}{g}\right)^2 \tag{8}$$

which gives for the co-ordinates of the vertex

$$y_1 = \frac{v_0^2 \sin^2 \theta}{2g}, \qquad x_1 = \frac{v_0^2 \sin \theta \cos \theta}{g}.$$

EXAMPLES.

1. An unresisted projectile starts with a velocity of 100 feet per second at an upward angle of 30° to the horizon; what will be its velocity when it has reached a point situated at a horizontal distance of 1000 feet from its starting-point, and how long will be required for it to reach that point?

Solution.

$$v_0 = 100, \qquad \theta = 30^\circ, \qquad v_0 \cos \theta = 86.6, \qquad v_0 \sin \theta = 50,$$
$$g = 32.16.$$

Equation (5) gives us

$$1000 = 86.6\, t$$

$$\therefore \quad t = \frac{1000}{86.6} = 11.55 \text{ seconds.}$$

$$v_0 \sin\theta - gt = 50 - 371.5 = -321.5,$$

$$v = \sqrt{(86.6)^2 + (321.5)^2} = \sqrt{7500 + 103362} = 333.$$

Hence the point in question will be reached in $11\frac{1}{2}$ seconds after starting, and the velocity will then be 333 feet per second.

2. An unresisted projectile is thrown upwards from the surface of the earth at angle of 39° to the horizontal: find the time when it will reach the earth, and the velocity it will have acquired when it reaches the earth, the velocity of throwing being 30 feet per second.

3. A 10-pound weight is dropped from the window of a car when travelling over a bridge at a speed of 25 miles an hour. How long will it take to reach the ground 100 feet below the window, and what will be the kinetic energy when it reaches the ground?

4. With what horizontal velocity, and in what direction, must it be thrown, in order that it may strike the ground 50 feet forward of the point of starting?

5. Suppose the same 10-pound weight to be thrown vertically upwards from the car window with a velocity of 100 feet a minute, how long will it take to reach the ground, and at what point will it strike the ground?

§ 79. **Motion of a Body on an Inclined Plane without Friction.** — If a body move on an inclined plane along the line of steepest descent, subject to the action of gravity only, and if we resolve the force acting on it (i.e., its weight) into two components, along and perpendicular to the plane respectively, the latter component will be entirely balanced by the resistance of the plane, and the former will be the only unbalanced force acting on the body.

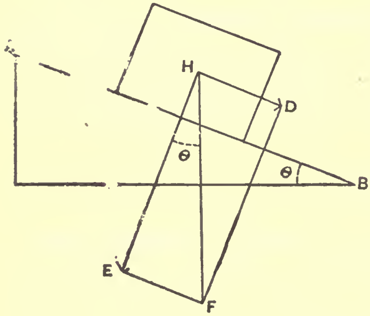


FIG. 50.

Suppose a body whose weight is represented (Fig. 50) by $HF = W$ to move along the inclined path AB under the action of gravity only. Let θ be the inclination of AB to the horizon. Resolve W into two components,

$$HD = W \sin \theta, \quad \text{and} \quad HE = W \cos \theta,$$

respectively parallel and perpendicular to the plane. The former is the only unbalanced force acting on the body, and will cause it to move down the plane with a uniformly accelerated motion; the acceleration being

$$\frac{W \sin \theta}{\left(\frac{W}{g}\right)} = g \sin \theta. \qquad (1)$$

If the body is either at rest or moving downwards at the beginning, it will move downwards; whereas, if it is first moving upwards, it will gradually lose velocity, and move upwards more slowly, until ultimately its upward velocity will be destroyed, and it will begin moving downwards.

The equations for uniformly varying motion are entirely applicable to these cases. Thus, suppose that the body has an initial downward velocity v_0, this velocity will, at the end of the time t, become

$$v = \frac{ds}{dt} = v_0 + (g \sin \theta)t \qquad (2)$$

$$\therefore \quad s = v_0 t + \tfrac{1}{2} g \sin \theta \, . \, t^2, \qquad (3)$$

and, for the unbalanced force acting, we have

$$F = m \frac{d^2 s}{dt^2} = \frac{W}{g}(g \sin \theta) = W \sin \theta. \qquad (4)$$

If, on the other hand, the body's initial velocity is upward, and we denote this upward velocity by v_0, we shall have the equations

$$v = \frac{ds}{dt} = v_0 - (g \sin\theta)t \tag{5}$$

$$s = v_0 t - \tfrac{1}{2}g \sin\theta \, . \, t^2 \tag{6}$$

$$F = -W \sin\theta. \tag{7}$$

Again, if the initial velocity is zero, equations (2) and (3) become

$$v = \frac{ds}{dt} = (g \sin\theta)t, \tag{8}$$

$$s = \tfrac{1}{2}g \sin\theta \, . \, t^2. \tag{9}$$

From these we obtain, for this case,

$$t = \sqrt{\frac{2s}{g \sin\theta}}; \tag{10}$$

and, substituting this value of t in (8), we have

$$v = \sqrt{2g(s \sin\theta)}, \tag{11}$$

or, if we let $s \sin\theta = h =$ the vertical distance through which the body has fallen. we have

$$v = \sqrt{2gh}. \tag{12}$$

Hence, *When a body, starting from rest, falls, under the action of gravity only, through a height* h, *the velocity acquired is* $\sqrt{2gh}$, *whether the path be vertical or inclined.*

EXAMPLES.

1. A body moves from the top to the bottom of a plane inclined to the horizon at 30°, under the action of gravity only: find the time required for the descent, and the velocity at the foot of the plane.

2. In the right-angled triangle shown in the figure (Fig. 51), given $AB = 10$ feet, angle $BAC = 30°$: find the time a body would require, if acted on by gravity only, to fall from rest through each of the sides respectively, AB being vertical.

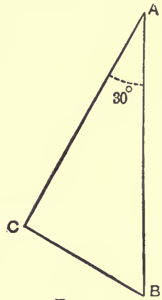


Fig. 51.

3. Given inclination of plane to the horizon $= \theta$, length of plane $= l$: compare the time of falling down the plane with the time of falling down the vertical.

4. A 100-pound weight rests, without friction, on the plane of example 3. What horizontal force is required to keep it from sliding down the plane.

5. Suppose 5 pounds horizontal force to be applied (*a*) so as to oppose the descent, (*b*) so as to aid the descent: find in each case how long it will take the weight to descend from the top to the bottom plane.

§ 80. **Motion along a Curved Line under the Action of Gravity only.** — We shall consider two questions in this regard: (*a*) the velocity at any point of the curve (*b*) the time of descent through any part of the curve.

(*a*) *Velocity at any point.* Let us suppose the body to have started from rest at A, and to have reached the point P in time t, where $AB = x$ (Fig. 52). Then, since the curved line AP may be considered as the limit of a broken line running from A to P, and as it has already been seen that the velocity acquired by falling through a certain height depends only upon the height, and not upon the inclination of the path, we shall have for a curved line also

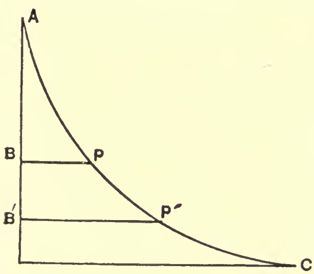


Fig. 52.

$$v = \sqrt{2gAB} = \sqrt{2gx},$$

where v is the velocity at P.

(*b*) *Time down a curve.* Referring to the same figure, let t denote the time required to go from A to P, and Δt the time to go from P to P', where $PP' = \Delta s$, and $BB' = \Delta x$; then, as we have seen that the velocity at P is $\sqrt{2gx}$, we shall have approximately for the space passed over in time Δt, the equation

$$\Delta s = \sqrt{2gx}\Delta t,$$

or, passing to the limit,

$$\frac{ds}{dt} = \sqrt{2gx}. \qquad (1)$$

This equation gives

$$dt = \frac{ds}{\sqrt{2gx}}$$

or

$$t = \int \frac{ds}{\sqrt{2gx}} = \int \frac{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx}{\sqrt{2gx}}, \qquad (2)$$

where, of course, the proper limits of integration must be used.

If t denote the time from A to P, we have

$$t = \int_{x=0}^{x=x} \frac{ds}{\sqrt{2gx}}.$$

EXAMPLE.

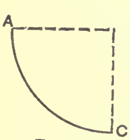


FIG. 53.

A body acted on by gravity only is constrained to move in the arc of a circle from A to C (Fig. 53), radius 10 feet. Find the time of describing the arc (quadrant) and the velocity acquired by the body when it reaches C.

§ 81. **Simple Circular Pendulum.** — To find the time occupied in a vibration of a simple circular pendulum, we take D (Fig. 54) as origin, and DC as axis of x, and the axis of y at right angles to DC. Let $AC = l$ and $BD = h$, we shall have for the time of a single oscillation from A to E

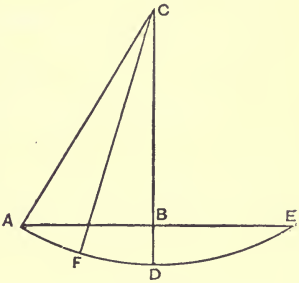


Fig. 54.

$$t = 2 \int_{x=0}^{x=h} \frac{ds}{\sqrt{2g(h-x)}}.$$

Now, from the equation of the circle $AFDE$,

$$y^2 = 2lx - x^2,$$

we have

$$\frac{dy}{dx} = \frac{l-x}{y}$$

$$\therefore \quad \frac{ds}{dx} = \frac{l}{y} = \frac{l}{\sqrt{2lx - x^2}}$$

$$\therefore t = 2\int_0^h \frac{l\,dx}{\sqrt{(2lx - x^2)[2g(h-x)]}} = \frac{2l}{\sqrt{2g}} \int_0^h \frac{dx}{\sqrt{hx - x^2}\sqrt{2l - x}}$$

or

$$t = \sqrt{\frac{l}{g}} \int_0^h \frac{dx}{\sqrt{hx - x^2}} \left(1 - \frac{x}{2l}\right)^{-\frac{1}{2}}.$$

This can only be integrated approximately.

Expanding $\left(1 - \frac{x}{2l}\right)^{-\frac{1}{2}}$ we obtain

$$\left(1 - \frac{x}{2l}\right)^{-\frac{1}{2}} = 1 + \frac{x}{4l} + \frac{3}{32}\frac{x^2}{l^2} + \text{etc.},$$

$$\therefore \quad t = \sqrt{\frac{l}{g}} \int_0^h \left(1 + \frac{x}{4l} + \frac{3}{32}\frac{x^2}{l^2} + \text{etc.}\right) \frac{dx}{\sqrt{hx - x^2}}.$$

The greatest value of x is h; and if h is so small that we may omit $\frac{x}{4l}$, we shall have as our approximate result

$$t = \sqrt{\frac{l}{g}}\int_0^h \frac{dx}{\sqrt{hx - x^2}} = \sqrt{\frac{l}{g}}\left\{\text{versin}^{-1}\frac{2x}{h}\right\}_0^h = \pi\sqrt{\frac{l}{g}}. \qquad (1)$$

If, however, the value of h as compared with l is too large to render it sufficiently accurate to omit $\frac{x}{4l}$, but so small that we can safely omit the higher powers of $\frac{x}{l}$, we shall have

$$t = \sqrt{\frac{l}{g}}\left\{\text{versin}^{-1}\frac{2x}{h} + \frac{1}{4l}\int_0^h \frac{xdx}{\sqrt{hx - x^2}}\right\}_0^h$$

$$= \sqrt{\frac{l}{g}}\left\{\text{versin}^{-1}\frac{2x}{h} + \frac{1}{4l}\left[\frac{h}{2}\text{versin}^{-1}\frac{2x}{h} - \sqrt{hx - x^2}\right]\right\}_0^h$$

or

$$t = \pi\sqrt{\frac{l}{g}}\left(1 + \frac{h}{8l}\right), \qquad (2)$$

a nearer approximation.

The formula

$$t = \pi\sqrt{\frac{l}{g}}$$

is the most used, and is more nearly correct, the smaller the value of h.

EXAMPLES.

1. Find the length of the simple circular pendulum which is to beat seconds at a place where $g = 32\frac{1}{6}$.

Solution.

$$t = \pi\sqrt{\frac{l}{g}} \qquad \therefore \quad l = \frac{t^2g}{\pi^2} = \frac{32\frac{1}{6}}{(3.1416)^2} = 3.259 \text{ feet.}$$

2. What is the time of vibration of a simple circular pendulum 5 feet long?

§ 82. **Simple Cycloidal Pendulum.** — The equation of the cycloid is

$$y = a \operatorname{versin}^{-1}\frac{x}{a} + (2ax - x^2)^{\frac{1}{2}},$$

$$\therefore \quad \frac{dy}{dx} = \sqrt{\frac{2a - x}{x}}$$

$$\therefore \quad \frac{ds}{dx} = \left(\frac{2a}{x}\right)^{\frac{1}{2}}.$$

Hence we shall have, for the time of a single oscillation,

$$t = 2\frac{\sqrt{2a}}{\sqrt{2g}}\int_0^h \frac{dx}{\sqrt{hx - x^2}}$$

or

$$t = 2\left(\frac{a}{g}\right)^{\frac{1}{2}}\left\{\operatorname{versin}^{-1}\frac{2x}{h}\right\}_0^h = 2\pi\sqrt{\frac{a}{g}}.$$

This expression is independent of h, so that the time of vibration is the same whether the arc be large or small.

A body can be made to vibrate in a cycloidal arc by suspending it by a flexible string between two cycloidal cheeks. This is shown from the fact that the evolute of the cycloid is another cycloid (Fig. 55).

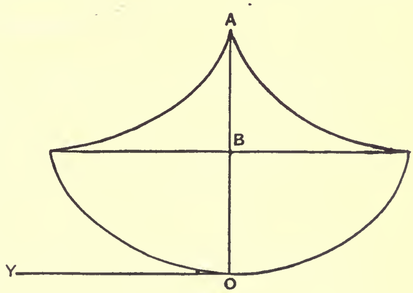


Fig. 55.

To prove this, we have, from the equation of the cycloid,

$$y = a \operatorname{versin}^{-1}\frac{x}{a} + (2ax - x^2)^{\frac{1}{2}},$$

$$\frac{dy}{dx} = \sqrt{\frac{2a - x}{x}}, \quad \frac{ds}{dx} = \sqrt{\frac{2a}{x}},$$

$$\frac{d^2y}{dx^2} = \frac{-a}{x^{\frac{3}{2}}\sqrt{2a - x}}.$$

Hence the radius of curvature is

$$\rho = \frac{\left(\frac{ds}{dx}\right)^3}{\frac{-d^2y}{dx^2}} = 2(2a)^{\frac{1}{2}}\sqrt{2a - x};$$

and since we have for the evolute the relation

$$ds' = d\rho,$$

where ds' is the elementary arc of the evolute,

$$\therefore \quad s' = \int_{x=x}^{x=2a} d\rho;$$

and, observing that when $x = 2a \quad \rho = 0$, we have

$$s' = \rho,$$
$$\therefore \quad s' = 2(2a)^{\frac{1}{2}}\sqrt{2a - x}.$$

If x_1 is the abscissa of the point of the evolute,

$$x_1 = x + \rho\frac{dy}{ds} = 4a - x,$$
$$\therefore \quad s' = 2(2a)^{\frac{1}{2}}\sqrt{x_1 - 2a};$$

and, transforming co-ordinates to B by putting $x_2 + 2a$ for x_1, we obtain

$$s' = 2(2ax_2)^{\frac{1}{2}},$$
$$\therefore \quad s'^2 = 8ax_2,$$

which is the equation of another cycloid just like the first.

The motion along a vertical cycloid may also be obtained by letting a body move along a groove in the form of a cycloid acted on by gravity alone; and in this case the time of descent of the body to the lowest point is precisely the same at whatever point of the curve the body is placed.

§ 83. **Effect of Grade on the Tractive Force of a Railway Train.** — As a useful particular case of motion on an inclined plane, we have the case of a railroad train moving up or down a grade. It is necessary that a certain tractive force be exerted in order to overcome the resistances, and keep the train moving at a uniform rate along a level track. If, on the other hand, the track is not on a level, and if we resolve the weight of the train into components at right angles to and along the plane of the track, we shall have in the latter component a force which must be added to the tractive force above referred to when we wish to know the tractive force required to carry it up grade, and must be subtracted when we wish to know the tractive force required to carry it down grade. The result of this subtraction may give, if the grade is sufficiently steep and the speed sufficiently slow, a negative quantity; and in that case we must apply the brakes, instead of using steam, unless we wish the speed of the train to increase.

EXAMPLES.

1. A railroad train weighing 60000 lbs., and running at 50 miles per hour, requires a tractive force of 618 lbs. on a level; what is the tractive force necessary when it is to ascend a grade of 50 feet per mile? What when it is to descend? Also what is the amount of work per minute in each case?

Solution.

The resolution of the weight will give (Fig. 50, § 79), for the component along the plane,

$$(60000)\frac{50}{5280} = 568.2 \text{ nearly.}$$

Hence

$$\begin{aligned} \text{Tractive force for a level} &= 618.0, \\ \text{Tractive force for ascent} &= 1186.2, \\ \text{Tractive force for descent} &= 49.8. \end{aligned}$$

To ascertain the work done per minute in each case, we have —

(*a*) For a level track, $\frac{618 \times 50 \times 5280}{60} = 2719200$ foot-lbs.

(*b*) Up grade, $2719200 + \frac{60000 \times 50 \times 50}{60} = 5219200$ foot-lbs.

(*c*) Down grade, $2719200 - \frac{60000 \times 50 \times 50}{60} = 219200$ foot-lbs.

2. Suppose the tractive force required for each 2000 lbs. of weight of train to be, on a level track, for velocities of —

5.0 miles per hour,	10.0	20.0	30.0	40.0	50.0	60
6.1 lbs.,	6.6	8.3	11.2	15.3	20.6	27;

find the tractive force required to carry the train of example 1 —

(*a*) Up an incline of 50 feet per mile at 30 miles per hour.

(*b*) Down an incline of 50 feet per mile at 30 miles per hour.

(*c*) Down an incline of 10 feet per mile at 20 miles per hour.

(*d*) What must be the incline down which the train must run to require no tractive force at 40 miles per hour?

3. If in the first example the tractive force remains 618 lbs. while the train is going down grade, what will be its velocity at the end of one minute, the grade being 10 feet per mile?

§ 84. **Harmonic Motion.** — If we imagine a body to be moving in a circle at a uniform rate (Fig. 56), and a second body to oscillate back and forth in the diameter *AB*, both starting from *B*, and if when the first body is at *C* the other is directly under it at *G*, etc., then is the second body said to move in harmonic motion.

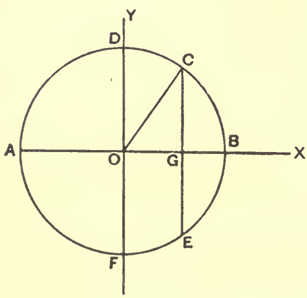


Fig. 56.

A practical case of this kind of motion is the motion of a slotted cross-head of an engine, as shown in the figure (Fig. 57); the crank moving at a uniform rate. In the case of the ordinary crank, and connecting-rod connecting the drive-wheel shaft of a stationary engine with the piston-rod,

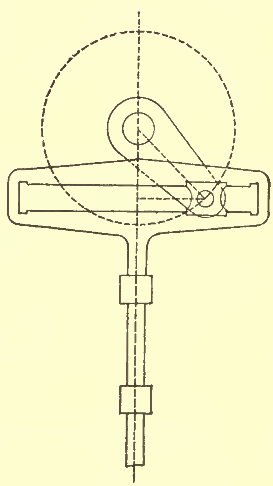


Fig. 57.

we have in the motion of the piston only an approximation to harmonic motion. We will proceed to determine the law of the force acting upon, and the velocity of, a body which is constrained to move in harmonic motion. Let the body itself and the corresponding revolving body be supposed to start from B (Fig. 56), the latter revolving in left-handed rotation with an angular velocity α, and let the time taken by the former in reaching G be t: then will the angle $BOC = \alpha t$; and we shall have, if s denote the space passed over by the body that moves with harmonic motion,

$$s = BG = OB - OC \cos \alpha t,$$

or, if

$$r = OB = OC,$$

$$s = r - r \cos \alpha t, \qquad (1)$$

the velocity at the end of the time t will be

$$v = \frac{ds}{dt} = \alpha r \sin \alpha t, \qquad (2)$$

and the acceleration at the end of time t will be

$$f = \frac{d^2 s}{dt^2} = \alpha^2 r \cos \alpha t. \qquad (3)$$

Hence the force acting upon the body at that instant, in the direction of its motion, is

$$F = m\frac{d^2 s}{dt^2} = m\alpha^2 r \cos \alpha t = m\alpha^2 (OG). \qquad (4)$$

The force, therefore, varies directly as the distance of the body from the centre of its path. It is zero when the body is at the centre of its path, and greatest when it is at the ends of its travel, as its value is then

$$m\alpha^2 r = \frac{W}{g}\alpha^2 r;$$

this being the same in amount as the centrifugal force of the revolving body, provided this latter have the same weight as the oscillating body. On the other hand, the velocity is greatest when $\alpha t = \frac{\pi}{2}$ (i.e., at mid-stroke); and its value is then

$$v = \alpha r,$$

this being also the velocity of the crank-pin at mid-stroke.

EXAMPLE.

Given that the reciprocating parts of an engine weigh 10000 lbs., the length of crank being 1 foot, the crank making 60 revolutions per minute; find the force required to make the cross-head follow the crank, (1) when the crank stands at 30° to the line of dead points, (2) when at 60°, (3) when at the dead point.

§ 85. **Work under Oblique Force.** — If the force act in any other direction than that of the motion, we must resolve it into two components, the component in the direction of the motion being the only one that does work. Thus if the force F is variable, and θ equals the angle it makes with the direction of the motion, we shall have as our expression for the work done

$$\int F \cos \theta ds.$$

Thus if a constant force of 100 lbs. act upon a body in a direction making an angle of 30° with the line of motion, then will the work done by the force during the time in which it moves through a distance of 10 feet be

$$(100)(0.86603)(10) = 866 \text{ foot-lbs.}$$

§ 86. **Rotation of Rigid Bodies.** — Suppose a rigid body (Fig. 58) to revolve about an axis perpendicular to the plane of the paper, and passing through O; imagine a particle whose weight is w to be situated at a perpendicular distance $OA = r$ from the axis of rotation, and let the angular acceleration be α: let it now be required to find the moment of the force or forces required to impart this acceleration; for we know that, if the axis of rotation pass through the centre of gravity of the body, the motion can be imparted only by a statical couple; whereas if it do not pass through the centre of gravity, the motion can be imparted by a single force.

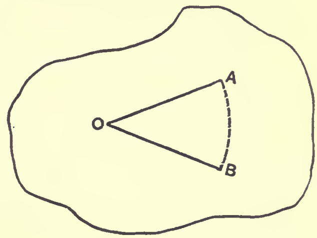


FIG. 58.

We shall have, for the particle situated at A,

Weight $= w$.
Angular acceleration $= \alpha$.
Linear acceleration $= \alpha r$.
Force required to impart this acceleration to this particle

$$= \frac{w}{g}\alpha r.$$

Moment of this force about the axis $= \dfrac{w}{g}\alpha r^2$.

Hence the moment of the force or forces required to impart to the entire body in a unit of time a rotation about the axis through O, with an angular velocity α, is

$$\Sigma\frac{w}{g}\alpha r^2 = \frac{\alpha}{g}\Sigma w r^2 = \frac{\alpha I}{g},$$

where I is used as a symbol to denote the limit of $\Sigma w r^2$, and is called the *Moment of Inertia of the body about the axis through* O.

§ 87. **Angular Momentum.** — This quantity, $\frac{\alpha I}{g}$, which expresses the moment of the force or forces required to impart to the body the angular acceleration α about the axis in question is also called the *Angular Momentum of the body when rotating with the angular velocity α about the given axis.*

§ 88. **Actual Energy of a Rotating Body.**—If it be required to find the actual energy of the body when rotating with the angular velocity ω, we have, for the actual energy of the particle at A,

$$\frac{w}{g}\frac{(\omega r)^2}{2} = \frac{\omega^2}{2g}wr^2,$$

and for that of the entire body

$$\frac{\omega^2}{2g}\Sigma wr^2 = \frac{\omega^2 I}{2g}.$$

This is the amount of mechanical work which would have to be done to bring the body from a state of rest to the velocity ω, or the total amount of work which the body could do in virtue of its velocity against any resistance tending to stop its rotation.

§ 89. **Moment of Inertia.** — The term "moment of inertia" originated in a wrong conception of the properties of matter. The term has, however, been retained as a very convenient one, although the conceptions under which it originated have long ago vanished. The meaning of the term as at present used, in relation to a solid body, is as follows : —

The moment of inertia of a body about a given axis is the limit of the sum of the products of the weight of each of the elementary particles that make up the body, by the squares of their distances from the given axis.

Thus, if w_1, w_2, w_3, etc., are the weights of the particles which are situated at distances r_1, r_2, r_3, etc., respectively from the axis, the moment of inertia of the body about the given axis is

$$I = \text{limit of } \Sigma wr^2.$$

§ 90. **Radius of Gyration.** — The radius of gyration of a body with respect to an axis is the perpendicular distance from the axis to that point at which, if the whole mass of the body were concentrated, the angular momentum, and hence the moment of inertia, of the body, would remain the same as they are in the body itself.

If ρ is the radius of gyration, the moment of inertia would be, when the mass is concentrated,

$$\rho^2 \Sigma w;$$

hence we must have

$$\rho^2 \Sigma w = \Sigma wr^2 = I,$$

whence

$$\rho^2 = \frac{\Sigma wr^2}{\Sigma w} = \frac{I}{W},$$

where $W =$ entire weight of the body.

§ 91. **Moment of Inertia of a Plane Surface.** — The term "moment of inertia," when applied to a plane figure, must, of course, be defined a little differently, as a plane surface has no weight; but, inasmuch as the quantity to which that name is given is necessary for the solution of a great many questions.

The moment of inertia of a plane surface about an axis, either in or not in the plane, is the limit of the sum of the products of the elementary areas into which the surface may be conceived to be divided, by the squares of their distances from the axis in question.

In a similar way, for the radius of gyration ρ of a plane figure whose area is A, we have

$$\rho^2 = \frac{I}{A}.$$

From this definition it will be evident, that, if the surface be referred to a pair of axes in its own plane, the moment of inertia of the surface about OY will be

$$I = \int\int x^2dxdy, \tag{1}$$

and the moment of inertia of the surface about OX will be

$$J = \int\int y^2dxdy. \tag{2}$$

The moment of inertia of the surface about an axis passing through the origin, and perpendicular to the plane XOY, will be

$$\int\int r^2dxdy, \tag{3}$$

where $r =$ distance from O to the point (x, y); hence $r^2 = x^2 + y^2$, and the moment of inertia becomes

$$\int\int(x^2 + y^2)dxdy = \int\int x^2dxdy + \int\int y^2dxdy = I + J. \tag{4}$$

This is called the "polar moment of inertia." If polar co-ordinates be used, this last becomes

$$\int\int\rho^2(\rho d\rho d\theta) = \int\int\rho^3d\rho d\theta. \tag{5}$$

All these quantities are quantities that will arise in the discussion of stresses, and the letters I and J are very commonly used to denote respectively

$$\int\int x^2dxdy \quad \text{and} \quad \int\int y^2dxdy.$$

Another quantity that occurs also, and which will be represented by K, is

$$\int\int xydxdy; \tag{6}$$

and this is called the moment of deviation.

EXAMPLES.

The following examples will illustrate the mode of finding the moment of inertia:—

1. Find the moment of inertia of the rectangle $ABCD$ about OY (Fig. 59).

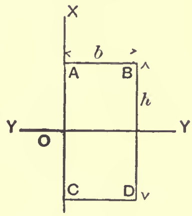


FIG. 59.

Solution.

$$I=\int_{-\frac{h}{2}}^{\frac{h}{2}}\int_{0}^{b}x^2dxdy=b\int_{-\frac{h}{2}}^{\frac{h}{2}}x^2dx=\frac{bh^3}{12}.$$

2. Find the moment of inertia of the entire circle (radius r) about the diameter OY (Fig. 60).

Solution.

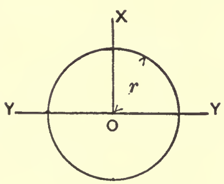


FIG. 60.

$$I=\int_{-r}^{r}\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}}x^2dxdy=2\int_{-r}^{r}x^2\sqrt{r^2-x^2}\,dx$$

$$=2\left\{-\frac{1}{4}x\left(r^2-x^2\right)^{\frac{3}{2}}+\frac{r^2}{4}\int\sqrt{r^2-x^2}\,dx\right\}_{-r}^{r}$$

$$=\frac{\pi r^4}{4}=\frac{\pi d^4}{64}.$$

3. Find the moment of inertia of the circular ring (outside radius r, inside radius r_1) about OY (Fig. 61).

Solution.

$$I=\frac{\pi r^4}{4}-\frac{\pi r_1^4}{4}=\frac{\pi(r^4-r_1^4)}{4}=\frac{\pi(d^4-d_1^4)}{64}.$$

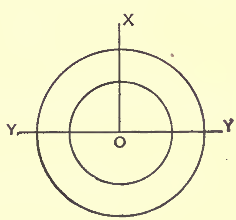


FIG. 61.

4. Find the moment of inertia of an ellipse (semi-axes a and b) about the minor axis OY.

Solution.

Equation of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$$\therefore \quad I_y = \int_{-a}^{a} \int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} x^2 dxdy$$

$$= \frac{2b}{a}\int_{-a}^{a} x^2 \sqrt{a^2 - x^2} dx = \frac{2b}{a}\left(\frac{\pi a^4}{8}\right) = \frac{\pi a^3 b}{4}.$$

On the other hand, $I_x = \frac{\pi a b^3}{4}$.

§ 92. **Moments of Inertia of Plane Figures about Parallel Axes.**

PROPOSITION. — *The moment of inertia of a plane figure about an axis not passing through its centre of gravity is equal to its moment of inertia about a parallel axis passing through its centre of gravity increased by the product obtained by multiplying the area by the square of the distance between the two axes.*

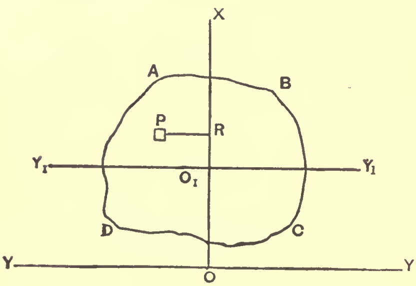


FIG. 62.

PROOF. — Let $ABCD$ (Fig. 62) be the surface; let OY be the axis not passing through the centre of gravity; let P be an elementary area $\Delta x \Delta y$, whose co-ordinates are $OR = x$ and $RP = y$; and let $OO_1 = a =$ a constant $=$ distance between the axes.

Let $O_1R = x_1 =$ abscissa of P with reference to the axis passing through the centre of gravity,

$$\therefore \quad x = a + x_1$$

$$\therefore \quad x^2 = x_1^2 + 2ax_1 + a^2$$

$$\therefore \quad x^2\Delta x\Delta y = x_1^2\Delta x\,\Delta y + 2ax\Delta x\Delta y + a^2\Delta x\Delta y.$$

Hence, summing, and passing to the limit, we have

$$\int\int x^2 dxdy = \int\int x_1^2 dxdy + 2a\int\int x_1 dxdy + a^2\int\int dxdy; \quad (1)$$

but if we were seeking the abscissa of the centre of gravity when the surface is referred to Y_1OY_{11}, and if this abscissa be denoted by x_0, we should have

$$x_0 = \frac{\int\int x_1 dxdy}{\int\int dxdy};$$

and, since $x_0 = 0$, $\therefore \int\int x_1 dxdy = 0$; hence, substituting this value in (1), we obtain

$$\int\int x^2 dxdy = \int\int x_1^2 dxdy + a^2\int\int dxdy. \quad (2)$$

If, now, we call the moment of inertia about OY, I, that about O_1Y_{11}, I_1, and let the area $= A = \int\int dxdy$, we shall have

$$I = I_1 + a^2 A. \quad (3)$$

Q. E. D.

§ 93. **Polar Moment of Inertia of Plane Figures.** — *The moment of inertia of a plane figure about an axis perpendicular to the plane is equal to the sum of its moments of inertia about any pair of rectangular axes in its plane passing through the foot of the perpendicular.*

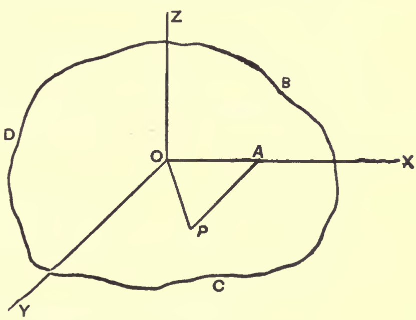


FIG. 63.

PROOF. — Let BCD (Fig. 63) be the surface, and P an elementary area, and let $OA = x$, $AP = y$, $OP = r$; then the moment of inertia of the surface about OZ will be

$$\int\int r^2 dxdy = \int\int (x^2 + y^2) dxdy = \int\int x^2 dxdy + \int\int y^2 dxdy = I + J.$$

Q. E. D.

Hence follows, also, that the sum of the moments of inertia of a plane surface relatively to a pair of rectangular axes in its own plane is isotropic; i.e., the same as for any other pair of rectangular axes meeting at the same point, and lying in its plane.

EXAMPLES.

1. To find the moment of inertia of the rectangle (Fig. 59) about an axis through its centre perpendicular to the plane of the rectangle.

Solution.

$$\text{Moment of inertia about } YY = \frac{bh^3}{12},$$

Moment of inertia about an axis through its

$$\text{centre and perpendicular to } YY = \frac{hb^3}{12};$$

hence

$$\text{Polar moment of inertia} = \frac{bh^3}{12} + \frac{hb^3}{12} = \frac{bh}{12}(h^2 + b^2).$$

2. To find the moment of inertia of a circle about an axis through its centre and perpendicular to its plane (Fig. 60).

Solution.

$$\text{Moment of inertia about } OY = \frac{\pi r^4}{4},$$

$$\text{Moment of inertia about } OX = \frac{\pi r^4}{4};$$

hence

$$\text{Polar moment of inertia} = \frac{\pi r^4}{4} + \frac{\pi r^4}{4} = \frac{\pi r^4}{2}.$$

3. To find the moment of inertia of an ellipse about an axis passing through its centre and perpendicular to its plane.

Solution.

From example 4, § 91, we have

$$I_x = \frac{\pi ab^3}{4} \qquad I_y = \frac{\pi a^3 b}{4}$$

$$\therefore \quad \text{Polar moment of inertia} = \frac{\pi ab}{4}(a^2 + b^2).$$

§ 94. **Moments of Inertia of Plane Figures about Different Axes compared.** — Given the surface KLM (Fig. 64), suppose we have already determined the quantities

$$I = \int\int x^2 dxdy, \; J = \int\int y^2 dxdy, \; K = \int\int xydxdy,$$

it is required to determine, in terms of them, the quantities

$$I_1 = \int\int x_1^2 dx_1 dy_1, \; J_1 = \int\int y_1^2 dx_1 dy_1, \; K_1 = \int\int x_1 y_1 dx_1 dy_1;$$

the angles XOY and X_1OY_1 being both right angles, and $YOY_1 = \alpha$.

We shall have, from the ordinary equations for the transformation of co-ordinates, to be found in any analytic geometry, the equations

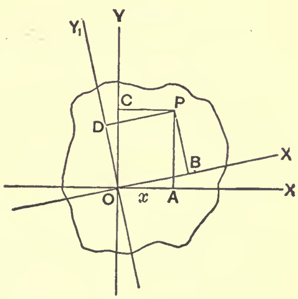


Fig. 64.

$$x_1 = x \cos\alpha + y \sin\alpha,$$

$$y_1 = y \cos\alpha - x \sin\alpha,$$

$$\therefore \quad x_1^2 = x^2 \cos^2\alpha + y^2 \sin^2\alpha + 2xy \cos\alpha \sin\alpha,$$

$$y_1^2 = x^2 \sin^2\alpha + y^2 \cos^2\alpha - 2xy \cos\alpha \sin\alpha,$$

$$x_1 y_1 = xy(\cos^2\alpha - \sin^2\alpha) - (x^2 - y^2) \cos\alpha \sin\alpha.$$

Hence

$$I_1 = \int\int x_1^2 dx_1 dy_1 = \text{limit of } \Sigma x_1^2 \Delta A$$
$$= \cos^2\alpha \text{ limit of } \Sigma x^2 \Delta A + \sin^2\alpha \text{ limit of } \Sigma y^2 \Delta A + 2\cos\alpha\sin\alpha \text{ limit of } \Sigma xy\Delta A$$
$$= (\cos^2\alpha)\int\int x^2 dxdy + (\sin^2\alpha)\int\int y^2 dxdy + 2(\cos\alpha\sin\alpha)\int\int xydxdy.$$
$$J_1 = \int\int y_1^2 dx_1 dy_1 = \text{limit of } \Sigma y_1^2 \Delta A$$
$$= (\sin^2\alpha) \text{ limit of } \Sigma x^2 \Delta A + (\cos^2\alpha) \text{ limit of } \Sigma y^2 \Delta A - 2(\cos\alpha\sin\alpha) \text{ limit of } \Sigma xy\Delta A$$
$$= (\sin^2\alpha)\int\int x^2 dxdy + (\cos^2\alpha)\int\int y^2 dxdy - 2(\cos\alpha\sin\alpha)\int\int xydxdy.$$
$$K_1 = \int\int x_1 y_1 dx_1 dy_1 = \text{limit of } \Sigma x_1 y_1 \Delta A$$
$$= (\cos^2\alpha - \sin^2\alpha) \text{ limit of } \Sigma xy\Delta A - (\cos\alpha\sin\alpha)\{\text{limit of } \Sigma x^2\Delta A - \text{limit of } \Sigma y^2 \Delta A\}$$
$$= (\cos^2\alpha - \sin^2\alpha)\int\int xydxdy - (\cos\alpha\sin\alpha)\left\{\int\int x^2 dxdy - \int\int y^2 dxdy\right\}.$$

Or, introducing the letters I, J, and K, we have

$$I_1 = I\cos^2\alpha + J\sin^2\alpha + 2K\cos\alpha\sin\alpha, \tag{1}$$

$$J_1 = I\sin^2\alpha + J\cos^2\alpha - 2K\cos\alpha\sin\alpha, \tag{2}$$

$$K_1 = (J - I)\cos\alpha\sin\alpha + K(\cos^2\alpha - \sin^2\alpha). \tag{3}$$

The equations (1), (2), and (3) furnish the solution of the problem.

§ 95. **Principal Moments of Inertia in a Plane.**—*In every plane figure, a given point being assumed as origin, there is at least one pair of rectangular axes, about one of which the moment of inertia is a maximum, and a minimum about the other; these moments of inertia being called principal moments of inertia, and the axes about which they are taken being called principal axes of inertia.*

PROOF. — In order that I_1, equation (1), § 94, may be a maximum or a minimum, we must have, as will be seen by differentiating its value, and putting the first differential co-efficient equal to zero,

$$-2I\cos\alpha\sin\alpha + 2J\cos\alpha\sin\alpha + 2K(\cos^2\alpha - \sin^2\alpha) = 0$$

$$\therefore \quad K(\cos^2\alpha - \sin^2\alpha) - (I - J)\cos\alpha\sin\alpha = 0 \qquad (1)$$

$$\therefore \quad \frac{\cos\alpha\sin\alpha}{\cos^2\alpha - \sin^2\alpha} = \frac{K}{I - J} \qquad \therefore \quad \tan 2\alpha = \frac{2K}{I - J}. \qquad (2)$$

Hence, for the value of α given by (2), we have I_1 a maximum or a minimum; and as there are two values of 2α corresponding to the same value of $\tan 2\alpha$, and as these two values differ by 180°, the values of α will differ by 90°, one corresponding to a maximum and the other to a minimum.

Moreover, when the value of α is so chosen, we have

$$K_1 = 0,$$

as is proved by equation (1). Indeed, we might say that the condition for determining the principal axes of inertia is

$$K_1 = 0.$$

§ 96. **Axes of Symmetry of Plane Figures.** — An axis which divides the figure symmetrically is always a principal axis.

PROOF. — Let us assume that the y axis divides the surface symmetrically; then we shall have, with reference to this axis,

$$K = \int\int_{-x}^{x} xydydx = \left\{ \int \frac{x^2}{2} ydy \right\}_{-x}^{x} = 0.$$

And, since K is zero, the axis of y is one principal axis, and of course the axis of x is the other. The same method of reasoning would show $K = 0$ if the x axis were the axis of symmetry.

Hence, whenever a plane figure has an axis of symmetry, this axis is one of the principal axes, and the other is at right angles to it. Thus, for a rectangle, when the axis is to pass through its centre of gravity, the principal axes are parallel to the sides respectively, the moment of inertia being greatest about the shortest axis, and least about the longest. Thus in an ellipse the minor axis is the axis of maximum, and the major that of minimum, moment of inertia, etc. On the other hand, in a circle, or in a square, since the maximum and minimum are equal, it follows that the moments of inertia about all axes passing through the centre are the same.

§ 97. **Conditions for Equal Values of Moment of Inertia.** — When the moments of inertia of a plane figure about three different axes passing through the same point are the same, the moments of inertia about all axes passing through this point are the same.

Proof. — Let I be the moment of inertia about OY, I_1 about OY_1, I_2 about OY_2, and let

$$YOY_1 = \alpha, \qquad YOY_2 = \beta,$$

and let

$$I_1 = I_2 = I.$$

Then, from equation (1), § 94, we have

$$I = I\cos^2\alpha + J\sin^2\alpha + 2K\cos\alpha\sin\alpha,$$

$$I = I\cos^2\beta + J\sin^2\beta + 2K\cos\beta\sin\beta.$$

Hence

$$(I - J)\sin^2\alpha = 2K\cos\alpha\sin\alpha, \tag{1}$$

$$(I - J)\sin^2\beta = 2K\cos\beta\sin\beta. \tag{2}$$

Hence

$$(I - J)\tan\alpha = 2K, \tag{3}$$

$$(I - J)\tan\beta = 2K. \tag{4}$$

And, since $\tan\alpha$ is not equal to $\tan\beta$, we must have

$$I - J = 0 \qquad \text{and} \qquad K = 0.$$

Hence, since $K = 0$ and $I = J$, we shall have, from equation (1), § 94, for the moment of inertia I' about an axis, making any angle θ with OY,

$$I' = I\cos^2\theta + I\sin^2\theta + 0 = I. \tag{5}$$

Hence all the moments of inertia are equal.

§ 98. **Components of Moments of Inertia of Solid Bodies.** — Refer the body to three rectangular axes, OX, OY, and OZ; and let I_x, I_y, and I_z represent its moment of inertia about each axis respectively. Then, if r denote the distance of any particle from OZ, we shall have

$$I_z = \text{limit of } \Sigma wr^2;$$

but

$$r^2 = x^2 + y^2$$

$$\therefore\ I_z = \text{limit of } \Sigma w(x^2 + y^2) = \text{limit of } \Sigma wx^2 + \text{limit of } \Sigma wy^2. \tag{1}$$

In the same way we have

$$I_x = \text{limit of } \Sigma wy^2 + \text{limit of } \Sigma wz^2, \tag{2}$$

$$I_y = \text{limit of } \Sigma wx^2 + \text{limit of } \Sigma wz^2. \tag{3}$$

§ 99. **Moments of Inertia of Solids around Parallel Axes.** — The moment of inertia of a solid body about an axis not passing through its centre of gravity is equal to its moment of inertia about a parallel axis passing through the centre of gravity, increased by the product of the entire weight of the body by the square of the distance between the two axes.

Proof. — Refer the body to a system of three rectangular axes, OX, OY, and OZ, of which OZ is the one about which the moment of inertia is taken. Let the co-ordinates of the centre of gravity of the body with reference to these axes be (x_0, y_0, z_0). Through the centre of gravity of the body draw a system of rectangular axes, parallel respectively to OX, OY, and OZ. Then we shall have for the co-ordinates of any point

$$x = x_0 + x_1,$$
$$y = y_0 + y_1,$$
$$z = z_0 + z_1.$$

Hence

$$\begin{aligned} I_z &= \text{limit of } \Sigma w(x^2 + y^2) = \text{limit of } \Sigma wx^2 + \text{limit of } \Sigma wy^2 \\ &= \text{limit of } \Sigma w(x_0 + x_1)^2 + \text{limit of } \Sigma w(y_0 + y_1)^2 \\ &= x_0^2 \text{ limit of } \Sigma w + y_0^2 \text{ limit of } \Sigma w + 2x_0 \text{ limit of } \Sigma wx_1 \\ &\qquad + 2y_0 \text{ limit of } \Sigma wy_1 + \text{limit of } \Sigma wx_1^2 + \text{limit of } \Sigma wy_1^2 \\ &= (x_0^2 + y_0^2) W + 2x_0 \text{ limit of } \Sigma wx_1 + 2y_0 \text{ limit of } \Sigma wy_1 \\ &\qquad + \text{limit of } \Sigma wr_1^2 \\ &= r_0^2 W + I_z' + 2x_0 \text{ limit of } \Sigma wx_1 + 2y_0 \text{ limit of } \Sigma wy_1. \end{aligned}$$

But, since O_1 is the centre of gravity,

$$\therefore \quad \Sigma wx_1 = 0 \qquad \text{and} \qquad \Sigma wy_1 = 0.$$

Hence

$$I_z = I_z' + Wr_0^2,$$

which proves the proposition.

§ 100. **Examples of Moments of Inertia.**

1. To find the moment of inertia of a sphere whose radius is r and weight per unit of volume w, about the axis OZ drawn through its centre.

Solution.

Divide the sphere into thin slices (Fig. 65) by planes drawn perpendicular to OZ. Let the distance of the slice shown in the figure, above O be z, and its thickness dz: then will its radius be $\sqrt{r^2 - z^2}$; and we can readily see, from example 2, § 93, that its moment of inertia about OZ will be

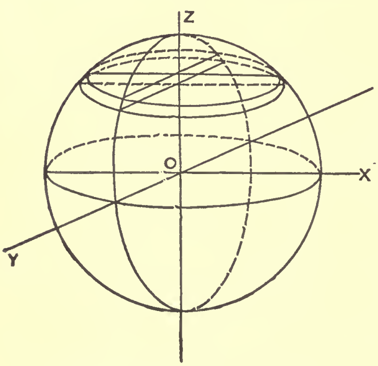


FIG. 65.

$$\frac{w\pi(r^2 - z^2)^2}{2} dz.$$

Hence the moment of inertia of the entire sphere about OZ will be

$$I_z = w\frac{\pi}{2}\int_{-r}^{r}(r^2 - z^2)^2 dz,$$

which easily reduces to

$$I_z = \frac{8}{15} w\pi r^5.$$

2. To find the moment of inertia of an ellipsoid (semi-axes a, b, c) about OZ (Fig. 66).

SOLUTION.—The equation of the ellipsoid is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

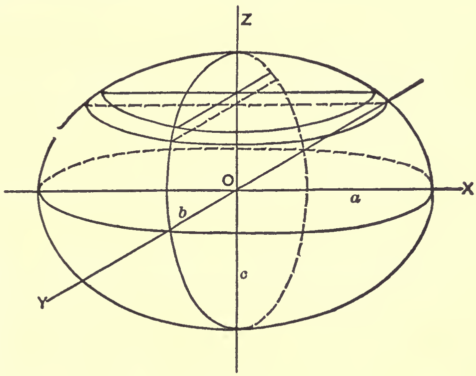


FIG. 66.

Divide it into thin slices perpendicular to OZ, and let the slice shown in the figure be at a distance z from O. Then will this slice be elliptical, and its semi-axes will be

$$\frac{a}{c}\sqrt{c^2 - z^2} \qquad \text{and} \qquad \frac{b}{c}\sqrt{c^2 - z^2};$$

and from example 3, § 93, we readily obtain, for its moment of inertia about OZ,

$$\frac{w\pi}{4}\left[\frac{ab}{c^2}(c^2 - z^2)\right]\left\{\frac{a^2}{c^2}(c^2 - z^2) + \frac{b^2}{c^2}(c^2 - z^2)\right\}dz$$

$$= \frac{w\pi ab(a^2 + b^2)}{4c^4}(c^2 - z^2)^2 dz.$$

Hence, for the moment of inertia of the ellipsoid about OZ, we have

$$I_z = \frac{w\pi ab(a^2 + b^2)}{4c^4}\int_{-c}^{c}(c^2 - z^2)^2 dz = \frac{4}{15} w\pi abc(a^2 + b^2).$$

3. Find the moment of inertia of a right circular cylinder, length a, radius r, about its axis.

Ans. $\dfrac{w\pi r^4 a}{2}$.

4. Find the moment of inertia of the same about an axis perpendicular to, and bisecting its axis. *Ans.* $\dfrac{w\pi ar^2}{4}\left(r^2 + \dfrac{a^2}{3}\right)$.

5. Find the moment of inertia of an elliptic right cylinder, length $2c$, transverse semi-axes a and b, about its longitudinal axis.

Ans. $\dfrac{w\pi abc}{2}(a^2 + b^2)$.

6. Find the moment of inertia of the same about its transverse axis $2b$. *Ans.* $2w\pi abc\left(\dfrac{a^2}{4} + \dfrac{c^2}{3}\right)$.

7. Find the moment of inertia of a rectangular prism, sides $2a$, $2b$, $2c$, about central axis $2c$. *Ans.* $\frac{8}{3}wabc(a^2 + b^2)$.

§ 101. **Centre of Percussion.** — Suppose we have a body revolving, with an angular velocity α, about an axis perpendicular to the plane of the paper, and passing through O. Join O with the centre of gravity, G, and take OG as axis of x; the axis of y passing through O, and lying in the plane of the paper. If, with a radius $OA = r$, we describe an arc CA (Fig. 67), all particles situated in this arc have a linear velocity αr. The force which would impart this velocity to any one of them, as that at A, in a unit of time, is

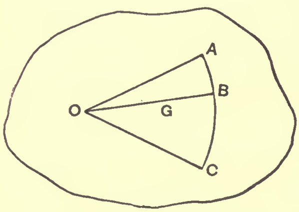


Fig. 67.

$$\frac{w}{g}\alpha r,$$

and this may be resolved into two,

$$\frac{w}{g}\alpha x \quad \text{and} \quad \frac{w}{g}\alpha y,$$

respectively perpendicular and parallel to OG. The moment of this force about the axis is

$$\frac{w}{g}\alpha r^2;$$

hence the total moment of the forces which would impart to the body in a unit of time the angular velocity α, is, as has been shown already,

$$\frac{\alpha I}{g} = \frac{\alpha}{g}\Sigma wr^2.$$

The resultant of the forces acting on the body is

$$\frac{\alpha}{g}\Sigma wx,$$

since, the centre of gravity being on *OB*, it follows that $\Sigma wy = 0$; and hence

$$\frac{\alpha}{g}\Sigma wy = 0.$$

Hence the perpendicular distance from *O* to the line of direction of the resultant force is measured along *OG*, and is

$$l = \frac{\frac{\alpha}{g}I}{\frac{\alpha}{g}\Sigma wx} = \frac{I}{\Sigma wx}; \qquad (1)$$

and the point of application of the resultant force may be conceived to be at a point on *OG* at a distance l from *O;* and this point of application of the resultant of the forces which produce the rotation is called the *Centre of Percussion.*

If ρ = radius of gyration about the axis through *O*, and if x_0 = distance from *O* to the centre of gravity, we have

$$x_0\Sigma w = \Sigma wx.$$

Hence

$$l = \frac{\Sigma wr^2}{\Sigma wx} = \frac{I}{x_0\Sigma w} = \frac{1}{x_0}\left(\frac{I}{W}\right) = \frac{\rho^2}{x_0} \qquad \therefore \quad \rho^2 = x_0 l;$$

or, in words, —

The radius of gyration is a mean proportional between the distance l, *and the distance* x_0, *between the axis of oscillation and the centre of gravity.*

The centre of percussion with respect to a given axis of oscillation O *has been defined as the point of application of the* *resultant of the forces which cause the body to rotate around the point* O.

Another definition often given is, that it is *the point at which, if a force be applied, there will be no shock on the axis of oscillation;* and these two definitions are equivalent to each other.

Let the particles of the body under consideration be conceived, for the sake of simplicity, to be distributed along a single line AB, and suppose a force F applied at D (Fig. 68). Conceive two equal and opposite forces, each equal to F, applied at C, the centre of gravity of the body.

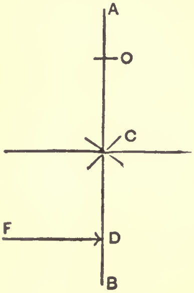


FIG. 68.

Then these three forces are equivalent to a single force F applied at the centre of gravity C, which produces translation of the whole body; and, secondly, a couple whose moment is $F(CD)$, whose effect is to produce rotation around an axis passing through the centre of gravity C. Under this condition of things it is evident that the centre of gravity C will have imparted to it in a unit of time a forward velocity equal to $\frac{F}{M}$, where M is the entire mass of the body; the point D will have imparted to it a greater forward velocity; while those points on the upper side of C will have imparted to them a less and less velocity as they recede from C, until, if the rod is sufficiently long, the particle at A will acquire a backward velocity.

Hence there must be some point which for the instant in question is at rest; i.e., where the velocity due to rotation is just equal and opposite to that due to the translation, or about which, for the instant, the body is rotating: and if this point were fixed by a pivot, there would be no stress on the pivot caused by the force applied at D.

An axis through this point is called the *Instantaneous Axis.*

§ 102. **Interchangeability of the Centre of Percussion and Axis of Oscillation.**—If we take, as axis of oscillation, a line perpendicular to the plane of the paper, and passing through D, then will O be the new centre of percussion.

PROOF. — We have seen (§ 101) that

$$l = \frac{\rho^2}{x_0},$$

where $l = OD$, $x_0 = OC$, and ρ = radius of gyration about an axis through O perpendicular to the plane of the paper.

Moreover, if ρ_0 represent the radius of gyration about an axis through C perpendicular to the plane of the paper, we shall have

$$\rho^2 = \rho_0^2 + x_0^2$$

$$\therefore \qquad l = \frac{\rho_0^2}{x_0} + x_0$$

$$\therefore \quad l - x_0 = \frac{\rho_0^2}{x_0} = CD.$$

Now if D is taken as axis of oscillation, we shall have for the distance l_1 to the corresponding centre of percussion,

$$l_1 = \frac{\rho_1^2}{CD} = \frac{\rho_1^2}{l - x_0},$$

where ρ_1 = radius of gyration about the axis of oscillation through D.

$$\therefore \quad l_1 = \frac{\rho_1^2}{CD} = \frac{\rho_0^2 + CD^2}{CD} = \frac{\rho_0^2}{CD} + CD = x_0 + (l - x_0) = l.$$

Hence the new centre of percussion is at O. Q. E. D.

§ 103. **Impact or Collision.** — Impact or collision is a pressure of inappreciably short duration between two bodies.

The direction of the force of impact is along the straight line drawn normal to the surfaces of the colliding bodies at their point of contact, and we may call this line the line of impact.

The action that occurs in the case of collision may be described as follows: at first the bodies undergo compression; the mutual pressure between them constantly increasing, until, when it has reached its maximum, the elasticity of the materials begins to overpower the compressive force, and restore the bodies wholly or partially to their original shape and dimensions.

Central impact occurs when the line joining the centres of gravity of the bodies coincides with the line of impact.

Eccentric impact occurs when these lines do not coincide.

Direct impact occurs when the line along which the relative motion of the bodies takes place, coincides with the line of impact.

Oblique impact occurs when these lines do not coincide.

CENTRAL IMPACT.

§ 104. **Equality of Action and Re-action.** — One fundamental principle that holds in all cases of central impact is the equality of action and re-action; in other words, we must have, that, at every instant of the time during which the impact is taking place, the pressure that one body exerts upon the other is equal and opposite to that exerted by the second upon the first.

The direct consequence of this principle is, that the algebraic sum of the momenta of the two bodies before impact remains unaltered by the impact, and hence that this sum is just the same at every instant of, and after, the impact.

If we let

m_1, m_2, be the respective masses,

c_1, c_2, their respective velocities before impact,

v_1, v_2, their respective velocities after impact,

v', v'', their respective velocities at any given instant during the time while impact is taking place, then we must have the following two equations true; viz., —

$$m_1 v_1 + m_2 v_2 = m_1 c_1 + m_2 c_2, \qquad (1)$$

$$m_1 v' + m_2 v'' = m_1 c_1 + m_2 c_2. \qquad (2)$$

§ 105. **Velocity at Time of Greatest Compression.** — At the instant when the compression is greatest — i.e., at the instant when the elasticity of the bodies begins to overcome the deformation due to the impact, and to tend to restore them to their original forms — the values of v' and v'' must be equal to each other; in other words, the colliding bodies must be moving with a common velocity

$$v = v' = v''. \qquad (1)$$

To determine this velocity, we have, from equation (2), § 104, combined with (1),

$$v = \frac{m_1 c_1 + m_2 c_2}{m_1 + m_2}. \qquad (2)$$

§ 106. **Co-efficient of Restitution.** — In order to determine the values v_1, v_2, of the velocities after impact, we need two equations, and hence two conditions. One of them is furnished by equation (1), § 104. The second depends upon the nature of the material of the colliding bodies, and we may distinguish three cases: —

1°. *Inelastic Impact.* — In this case the velocity lost up to the time of greatest compression is not regained at all, and the velocity after impact is the common velocity v at the instant of greatest compression. In this case the whole of the work used up in compressing the bodies is lost, as none of it is restored by the elasticity of the material.

2°. *Elastic Impact.* — In this case the velocity regained after the greatest compression, is equal and opposite to that lost up to the time of greatest compression; therefore

$$v - v_1 = c_1 - v. \quad (1) \qquad v_2 - v = v - c_2. \quad (2)$$

We may also define this case as that in which the work lost in compressing the bodies is entirely restored by the elasticity of the material, so that

$$\frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2} = \frac{m_1 c_1^2}{2} + \frac{m_2 c_2^2}{2}. \qquad (3)$$

Either condition will lead to the same result.

3°. *Imperfectly Elastic Impact.* — In this case a part only of the velocity lost up to the time of greatest compression is regained after that time.

If, when the two bodies are of the same material, we call e the co-efficient of restitution, then we shall so define it that

$$\frac{v - v_1}{c_1 - v} = \frac{v_2 - v}{v - c_2} = e;$$

or, in words, the co-efficient of restitution is the ratio of the velocity regained after compression to that lost previous to that time.

In this case only a part of the work done in producing the compression is regained, hence there is loss of energy. Its amount will be determined later.

Strictly speaking, all bodies belong to the third class; the value of e being always a proper fraction, and never reaching unity, the value corresponding to perfect elasticity; nor zero, the value corresponding to entire lack of elasticity.

§ 107. **Inelastic Impact.** — In this case the velocity after impact is the common velocity at the time of greatest compression; hence

$$v = v_1 = v_2 \qquad (1)$$

$$\therefore \quad v = \frac{m_1 c_1 + m_2 c_2}{m_1 + m_2}. \qquad (2)$$

And for the loss of energy due to impact we have

$$\frac{m_1 c_1^2}{2} + \frac{m_2 c_2^2}{2} - (m_1 + m_2)\frac{v^2}{2},$$

which, on substituting the value of v, reduces to

$$\frac{m_1 m_2}{2(m_1 + m_2)}(c_1 - c_2)^2. \tag{3}$$

§ 108. **Elastic Impact.** — In this case we have, of course, the condition, equation (1), § 104,

$$m_1 v_1 + m_2 v_2 = m_1 c_1 + m_2 c_2,$$

and, for second equation, we may use equation (3), § 106; viz.,

$$\frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2} = \frac{m_1 c_1^2}{2} + \frac{m_2 c_2^2}{2}.$$

Combining these two equations, we shall obtain

$$v_1 = c_1 - \frac{2m_2(c_1 - c_2)}{m_1 + m_2}, \tag{1}$$

$$v_2 = c_2 + \frac{2m_1(c_1 - c_2)}{m_1 + m_2}. \tag{2}$$

We can obtain the same result without having to solve an equation of the second degree, by using instead the equations (1) and (2) of § 106, together with (1) of § 104; i.e., —

$$m_1 v_1 + m_2 v_2 = m_1 c_1 + m_2 c_2;$$

$$v - v_1 = c_1 - v,$$

or

$$v_2 - v = v - c_2,$$

and (§ 105)

$$v = \frac{m_1 c_1 + m_2 c_2}{m_1 + m_2}.$$

As the result of combining these equations, and eliminating v, we should obtain equations (1) and (2), as above, for the values of v_1 and v_2. In this case the energy lost by the collision is zero.

§ 109. **Special Cases of Inelastic Impact.** — (*a*) Let the mass m_2 be at rest. Then $c_2 = 0$,

$$\therefore \quad v = \frac{m_1 c_1}{m_1 + m_2} \tag{1}$$

$$\therefore \quad \text{Loss of energy} = \frac{m_1 m_2}{m_1 + m_2} \frac{c_1^2}{2}. \tag{2}$$

(*b*) Let m_2 be at rest, and let $m_2 = \infty$; i.e., let the mass m_1 strike against another which is at rest, and whose mass is infinite. We have

$$m_2 = \infty, \qquad c_2 = 0,$$

$$\therefore \quad v = \frac{m_1 c_1}{m_1 + m_2} = 0, \tag{3}$$

$$\text{Loss of energy} = \frac{m_1}{\dfrac{m_1}{m_2} + 1} \frac{c_1^2}{2} = \frac{m_1 c_1^2}{2}, \tag{4}$$

or the moving body is reduced to rest by the collision, and all its energy is expended in compression.

(*c*) Let $m_1 c_1 = -m_2 c_2$; i.e., let the two bodies move towards each other with equal momenta:

$$\therefore \quad v = \frac{m_1 c_1 + m_2 c_2}{m_1 + m_2} = 0, \tag{5}$$

$$\text{and the loss of energy} = \frac{m_1 c_1^2}{2} + \frac{m_2 c_2^2}{2}, \tag{6}$$

the entire energy being lost.

§ 110. **Special Cases of Elastic Impact.** — (*a*) Let the mass m_2 be at rest. Then $c_2 = 0$,

$$v_1 = c_1 - \frac{2 m_2 c_1}{m_1 + m_2} \tag{1}$$

$$\therefore \quad v_2 = \frac{2 m_1 c_1}{m_1 + m_2}. \tag{2}$$

(*b*) Let m_2 be at rest, and let $m_2 = \infty$. Then we have $c_2 = 0$,

$$\therefore \quad v_1 = c_1 - \frac{2c_1}{\frac{m_1}{m_2} + 1} = c_1 - 2c_1 = -c_1, \qquad (3)$$

$$v_2 = 0. \qquad (4)$$

Hence the moving body retraces its path in the opposite direction with the same velocity.

(*c*) Let $m_1c_1 = -m_2c_2$. Then our equations of condition become

$$m_1v_1 + m_2v_2 = 0,$$

$$\frac{m_1v_1^2}{2} + \frac{m_2v_2^2}{2} = \frac{m_1c_1^2}{2} + \frac{m_2c_2^2}{2};$$

and from these we readily obtain

$$v_1 = -c_1,$$

$$v_2 = -c_2;$$

i.e., both bodies return on their path with the same velocity with which they approached each other.

§ 111. **Examples of Elastic and of Inelastic Impact.**

1. With what velocity must a body weighing 8 pounds strike one weighing 25 pounds in order to communicate to it a velocity of 2 feet per second, (*a*) when the bodies are perfectly elastic, (*b*) when wholly inelastic.

2. Suppose sixteen impacts per minute take place between two bodies whose weights are respectively 1000 and 1200 pounds, their initial velocities being 5 and 2 feet per second respectively: find the loss of energy, the bodies being inelastic.

§ 112. **Imperfect Elasticity.** — In this case we have the relations (see § 106)

$$\frac{v - v_1}{c_1 - v} = \frac{v_2 - v}{v - c_2} = e,$$

where

$$v = \frac{m_1 c_1 + m_2 c_2}{m_1 + m_2};$$

and we have also

$$m_1 v_1 + m_2 v_2 = m_1 c_1 + m_2 c_2.$$

Determining from them the values of v_1 and v_2, we obtain

$$v_1 = v(1 + e) - ec_1, \tag{1}$$

$$v_2 = v(1 + e) - ec_2, \tag{2}$$

or, by substituting for v its value,

$$v_1 = \frac{m_1 c_1 + m_2 c_2}{m_1 + m_2}(1 + e) - ec_1, \tag{3}$$

$$v_2 = \frac{m_1 c_1 + m_2 c_2}{m_1 + m_2}(1 + e) - ec_2. \tag{4}$$

These may otherwise be put in the form

$$v_1 = c_1 - (1 + e)\frac{m_2}{m_1 + m_2}(c_1 - c_2), \tag{5}$$

$$v_2 = c_2 + (1 + e)\frac{m_1}{m_1 + m_2}(c_1 - c_2). \tag{6}$$

Moreover, we have for the loss of energy due to impact

$$E = \frac{m_1}{2}(c_1^2 - v_1^2) + \frac{m_2}{2}(c_2^2 - v_2^2)$$

or

$$E = \tfrac{1}{2}\{m_1(c_1 - v_1)(c_1 + v_1) + m_2(c_2 - v_2)(c_2 + v_2)\};$$

but, from (5) and (6) respectively,

$$c_1 - v_1 = \frac{(1 + e)m_2(c_1 - c_2)}{m_1 + m_2}$$

$$c_2 - v_2 = -\frac{(1 + e)m_1(c_1 - c_2)}{m_1 + m_2}$$

$$\therefore \quad E = \frac{(1 + e)(c_1 - c_2)}{2(m_1 + m_2)} \{m_1m_2(c_1 + v_1) - m_1m_2(c_2 + v_2)\}$$

$$\therefore \quad E = \frac{m_1m_2}{2(m_1 + m_2)}(c_1 - c_2)(1 + e)(c_1 - c_2 + v_1 - v_2).$$

But, from (1) and (2),

$$v_1 - v_2 = -e(c_1 - c_2)$$

$$\therefore \quad E = \frac{m_1m_2}{2(m_1 + m_2)}(c_1 - c_2)(1 + e)(c_1 - c_2)(1 - e)$$

or

$$E = (1 - e^2)\frac{m_1m_2}{2(m_1 + m_2)}(c_1 - c_2)^2. \tag{7}$$

When $e = 1$, or the elasticity is perfect, this loss of energy becomes zero.

When $e = 0$, or the bodies are totally inelastic, then the loss of energy becomes

$$\frac{m_1m_2}{2(m_1 + m_2)}(c_1 - c_2)^2, \tag{8}$$

as has been already shown in § 107.

An interesting fact in this connection is, that since (8) is the work expended in producing compression, and (7) is the work lost in all, therefore the work restored by the elasticity of the body is

$$\frac{e^2}{2}\left\{\frac{m_1m_2}{m_1 + m_2}(c_1 - c_2)^2\right\}; \tag{9}$$

so that e^2, or the square of the co-efficient of restitution, is the ratio of the work restored by the elasticity of the bodies, to the work expended in compressing the bodies up to the time of greatest compression.

§ 113. **Special Cases.** — (*a*) Let m_2 be at rest, therefore $c_2 = 0$. Then we shall have

$$v_1 = c_1 \left\{ 1 - \frac{m_2(1 + e)}{m_1 + m_2} \right\} = c_1 \frac{m_1 - em_2}{m_1 + m_2}, \qquad (1)$$

$$v_2 = (1 + e)c_1 \frac{m_1}{m_1 + m_2}, \qquad (2)$$

and for loss of energy

$$E = (1 - e^2) \frac{m_1 m_2}{2(m_1 + m_2)} c_1^2. \qquad (3)$$

(*b*) When $m_2 = \infty$, and $c_2 = 0$, we have

$$v_1 = -ec_1, \qquad (4)$$

$$v_2 = 0,$$

$$E = (1 - e^2) \frac{m_1 c_1^2}{2}. \qquad (5)$$

(*c*) When $m_1 c_1 = -m_2 c_2$, then

$$v_1 = -ec_1,$$

$$v_2 = -ec_2,$$

$$E = \frac{(1 - e^2) m_1 c_1 (c_1 - c_2)}{2} = \frac{(1 - e^2) m_2 c_2 (c_2 - c_1)}{2}$$

$$= (1 - e^2) \frac{m_1(m_1 + m_2)}{2m_2} c_1^2. \qquad (6)$$

§ 114. **Values of *e* as Determined by Experiment.** — Since we have

$$e = \frac{v - v_1}{c_1 - v},$$

we shall have, when

$$m_2 = \infty \quad \text{and} \quad c_2 = 0,$$

$$v = \frac{m_1 c_1 + m_2 c_2}{m_1 + m_2} = 0.$$

Hence

$$e = -\frac{v_1}{c_1}.$$

Now, if we let a round ball fall vertically upon a horizontal slab from the height H, we shall have for the velocity of approach

$$c_1 = \sqrt{2gH};$$

and if we measure the height h to which it rises on its rebound, we shall have

$$-v_1 = \sqrt{2gh}.$$

Hence

$$e = -\frac{v_1}{c_1} = \sqrt{\frac{h}{H}}.$$

In this way the value of e can be determined experimentally for different substances.

Newton found for values of e: for glass, $\frac{15}{16}$; for steel, $\frac{5}{9}$; and Coriolis gives for ivory from 0.5 to 0.6.

On the other hand, if we desired to adopt as our constant the ratio of the work restored, to the work spent in compression, we should have for our constant e^2, and hence the squares of the preceding numbers.

EXAMPLES.

1. If two trains of cars, weighing 120000 and 160000 lbs., come into collision when they are moving in opposite directions with velocities 20 and 15 feet per second respectively, what is the loss of mechanical effect expended in destroying the locomotives and cars?

2. Two perfectly inelastic balls approach each other with equal velocities, and are reduced to rest by the collision; what must be the ratio of their weights?

3. Two steel balls, weighing 10 lbs. each, are moving with velocities 5 and 10 feet per second respectively, and in the same direction: find their velocities after impact, the fastest ball being in the rear, and overtaking the other; also the loss of mechanical effect due to the impact, assuming $e = 0.55$.

§ 115. Oblique Impact.

Let m_1, m_2, be the masses of the colliding bodies;

c_1, c_2, their respective velocities before impact;

α_1, α_2, the angles made by c_1, c_2, with the line of centres;

v_1, v_2, the components of the velocities after impact;

$c_1 \cos \alpha_1$, $c_2 \cos \alpha_2$, the components of c_1, c_2, along the line of centres;

$c_1 \sin \alpha_1$, $c_2 \sin \alpha_2$, the components of c_1, c_2, at right angles to the line of centres;

v the common component of the velocity at the instant of greatest compression along line of centres;

v', v'', actual velocities after impact;

α', α'', angles they make with line of centres;

v_c', v_c'', actual velocities when compression is greatest;

α_c', α_c'', angles they make with line of centres.

Then we shall have, by proceeding in the same way as was done in § 112,

$$v_1 = c_1 \cos \alpha_1 - (1 + e)\frac{m_2}{m_1 + m_2}(c_1 \cos \alpha_1 - c_2 \cos \alpha_2), \quad (1)$$

$$v_2 = c_2 \cos \alpha_2 + (1 + e)\frac{m_1}{m_1 + m_2}(c_1 \cos \alpha_1 - c_2 \cos \alpha_2), \quad (2)$$

$$v' = \sqrt{v_1^2 + c_1^2 \sin^2 \alpha_1}, \tag{3}$$

$$v'' = \sqrt{v_2^2 + c_2^2 \sin^2 \alpha_2}, \tag{4}$$

$$\cos \alpha' = \frac{v_1}{v'}, \tag{5}$$

$$\cos \alpha'' = \frac{v_2}{v''}, \tag{6}$$

$$v = \frac{m_1 c_1 \cos \alpha_1 + m_2 c_2 \cos \alpha_2}{m_1 + m_2}, \tag{7}$$

$$v_c' = \sqrt{v^2 + c_1^2 \sin^2 \alpha_1}, \tag{8}$$

$$v_c'' = \sqrt{v^2 + c_2^2 \sin^2 \alpha_2}, \tag{9}$$

$$\cos \alpha_c' = \frac{v}{v_c'}, \tag{10}$$

$$\cos \alpha_c'' = \frac{v}{v_c''}. \tag{11}$$

And for the energy lost in impact, we have

$$E = (1 - e^2)\frac{m_1 m_2}{2(m_1 + m_2)}(c_1 \cos \alpha_1 - c_2 \cos \alpha_2)^2. \tag{12}$$

When the bodies are perfectly elastic,

$$e = 1,$$

and equations (1), (2), and (12) become respectively

$$v_1 = c_1 \cos \alpha_1 - \frac{2m_2}{m_1 + m_2}(c_1 \cos \alpha_1 - c_2 \cos \alpha_2),$$

$$v_2 = c_2 \cos \alpha_2 + \frac{2m_1}{m_1 + m_2}(c_1 \cos \alpha_1 - c_2 \cos \alpha_2),$$

$$E = 0.$$

The rest remain the same in form.

When the bodies are totally inelastic,

$$e = 0,$$

and equations (1), (2), and (12) become respectively

$$v_1 = c_1 \cos \alpha_1 - \frac{m_2}{m_1 + m_2}(c_1 \cos \alpha_1 - c_2 \cos \alpha_2),$$

$$v_2 = c_2 \cos \alpha_2 + \frac{m_1}{m_1 + m_2}(c_1 \cos \alpha_1 - c_2 \cos \alpha_2),$$

$$E = \frac{m_1 m_2}{2(m_1 + m_2)}(c_1 \cos \alpha_1 - c_2 \cos \alpha_2)^2.$$

The rest remain the same in form.

§ 116. **Impact of Revolving Bodies.** — Let the bodies A and B revolve about parallel axes, and impinge upon each other.

Draw a common normal at the point of contact. This common normal will be the line of impact.

Let ϵ_1 = angular velocity of A before impact,
ϵ_2 = angular velocity of B before impact,
ω_1 = angular velocity of A after impact,
ω_2 = angular velocity of B after impact,
a_1 = perpendicular from axis of A on line of impact,
a_2 = perpendicular from axis of B on line of impact,
I_1 = moment of inertia of A about its axis,
I_2 = moment of inertia of B about its axis.

Then we shall have

$a_1\epsilon_1 = c_1$ = linear velocity of A at point of contact before impact;
$a_2\epsilon_2 = c_2$ = linear velocity of B at point of contact before impact;
$a_1\omega_1 = v_1$ = linear velocity of A at point of contact after impact;
$a_2\omega_2 = v_2$ = linear velocity of B at point of contact after impact;

$\frac{I_1\epsilon_1^2}{2g} = \left(\frac{I_1}{a_1^2}\right)\frac{c_1^2}{2g}$ = actual energy of A before impact;

$\frac{I_2\epsilon_2^2}{2g} = \left(\frac{I_2}{a_2^2}\right)\frac{c_2^2}{2g}$ = actual energy of B before impact;

$\frac{I_1\omega_1^2}{2g} = \left(\frac{I_1}{a_1^2}\right)\frac{v_1^2}{2g}$ = actual energy of A after impact;

$\frac{I_2\omega_2^2}{2g} = \left(\frac{I_2}{a_2^2}\right)\frac{v_2^2}{2g}$ = actual energy of B after impact;

Hence it follows that we have the case explained in § 112 for imperfectly elastic impact, provided only we write

$$\frac{I_1}{a_1^2} \text{ instead of } m_1 g \quad \text{and} \quad \frac{I_2}{a_2^2} \text{ instead of } m_2 g.$$

Hence we shall have

$$\omega_1 = \epsilon_1 - a_1(a_1\epsilon_1 - a_2\epsilon_2)\frac{I_2}{I_1 a_2^2 + I_2 a_1^2}(1 + e), \qquad (1)$$

$$\omega_2 = \epsilon_2 + a_2(a_1\epsilon_1 - a_2\epsilon_2)\frac{I_1}{I_1 a_2^2 + I_1 a_2^2}(1 + e), \qquad (2)$$

The case of perfect elasticity is obtained by making $e = 1$.

The case of total lack of elasticity is obtained by making $e = 0$.

In the latter case the loss of energy is

$$\frac{(a_1\epsilon_1 - a_2\epsilon_2)^2}{2g} \frac{I_1 I_2}{I_1 a_2^2 + I_2 a_1^2}, \qquad (3)$$

as can be seen by substituting the proper values in equation (8), § 112.

CHAPTER III.

ROOF-TRUSSES.

§ 117. **Definitions and Remarks.** — *The term "truss" may be applied to any framed structure intended to support a load.*

In the case of any truss, the external loads may be applied only at the joints, or some of the truss members may support loads at points other than the joints.

In the latter case those members are subjected, not merely to direct tension or compression, but also to a bending-action, the determination of which we shall defer until we have studied the mode of ascertaining the stresses in a loaded beam; and we shall at present confine ourselves to the consideration of the direct stresses of tension and compression.

For this purpose any loads applied between two adjacent joints must be resolved into two parallel components acting at those joints, and the truss is then to be considered as loaded at the joints. By this means we shall obtain the entire stresses in the members whenever the loads are concentrated at the joints; and, when certain members are loaded at other points, our results will be the direct tensions and compressions of these members, leaving the stresses due to bending yet to be determined.

A tie is a member suited to bear only tension.

A strut is a member suited to bear compression.

§ 118. **Frames of Two Bars.** — Frames of two bars may consist, (1) of two ties (Fig. 69), (2) of two struts (Fig. 70), (3) of a strut and a tie (Fig. 71).

CASE I. *Two Ties* (Fig. 69). — Let the load be represented graphically by $CF = W$. Then if we resolve it into two components, CD and CE, acting along CB and CA respectively, CD will represent graphically the pull or tension in the tie CB, and CE that in the tie CA.

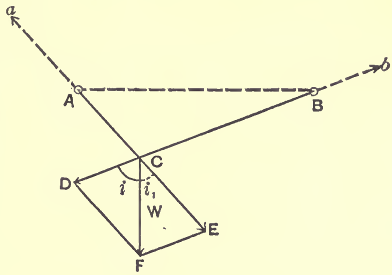


FIG. 69.

The force acting on CB at B is equal and opposite to CD, while that acting on CA at A is equal and opposite to CE.

To compute these stresses analytically, we have

$$CE = CF\frac{\sin CFE}{\sin CEF} = W\frac{\sin i}{\sin(i + i_1)},$$

$$CD = CF\frac{\sin CFD}{\sin CDF} = W\frac{\sin i_1}{\sin(i + i_1)}.$$

CASE II. *Two Struts* (Fig. 70). — Let the load be represented graphically by $CF = W$. Then will the components CD and CE represent the thrusts in the struts CB and CA respectively, and the re-actions of the supports at B and A will be equal and opposite to them. For analytical solution, we derive from the figure

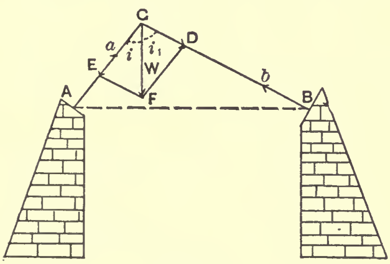


FIG. 70.

$$CE = W\frac{\sin i_1}{\sin(i + i_1)}, \qquad CD = W\frac{\sin i}{\sin(i + i_1)}.$$

CASE III. *A Strut and a Tie* (Fig. 71). — Let the load be represented graphically by $CF = W$. Resolve it, as before, into components along the members of the truss. Then will

CE represent the tension in the tie AC, and CD will represent the thrust in the strut BC; and we may deduce the analytical formulæ as before.

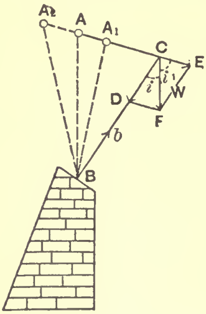


Fig. 71.

§ 119. **Stability for Lateral Deviations.** — In Case I, if the joint C be moved a little out of the plane of the paper, the load at C has such a direction that it will cause the truss to rotate around AB so as to return to its former position; hence such a frame is stable as regards lateral deviations.

In Case II the effect of the load, if C were moved a little out of the plane of the paper, would be to cause rotation in such a way as to overturn the truss; hence such a frame is unstable as regards lateral deviations.

In Case III the stability for lateral deviations will depend upon whether the load $CF = W$ is parallel to AB, is directed away from it or towards it. If the first is the case (i.e., if A is the point of suspension of the tie), the frame is neutral, as the load has no effect, either to restore the truss to its former position, or to overturn it; if the second is the case (i.e., if A_1 is the point of suspension of the tie), the truss is stable; and, if the third is the case (i.e., if A_2 is the point of suspension of the tie), it is unstable as regards lateral deviations.

§ 120. **General Methods for Determining the Stresses in Trusses.** — In the determination of the stresses as above, it would have been sufficient to construct only the triangle CFD by laying off $CF = W$ to scale, and then drawing CD parallel to CB, and FD parallel to CA, and the triangle CFD would have given us the complete solution of the problem. Moreover, the determination of the supporting forces of any truss, and of the stresses in the several members, is a question of equilibrium.

Adopting the following as definitions, viz., —

External forces are the loads and supporting forces, *Internal forces* are the stresses in the members: we must have

1°. The external forces must form a balanced system; i.e., the supporting forces must balance the loads.

2°. The forces (external and internal) acting at each joint of the truss must form a balanced system; i.e., the external forces (if any) at the joint must be balanced by the stresses in the members which meet at that joint.

3°. If any section be made, dividing the truss into two parts, the external forces which act upon that part which lies on one side of the section, must be balanced by the forces (internal) exerted by that part of the truss which lies on the other side of the section, upon the first part.

The above three principles, the triangle, and polygon of forces, and the conditions of equilibrium for forces in a plane, enable us to determine the stresses in the different members of roof and bridge trusses.

§ 121. **Triangular Frame.** — Given the triangular frame ABC (Fig. 72), and given the load W at C in magnitude and direction, given also the direction of the supporting force at B, to find the magnitude of this supporting force, the magnitude and direction of the other supporting force, and the stresses in the members.

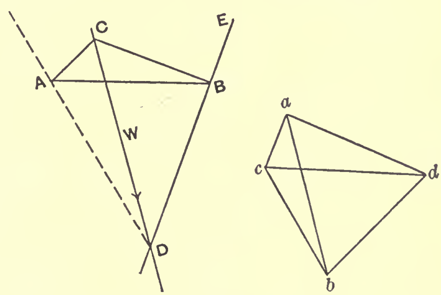


Fig. 72.

Solution. — Join A with D, the point of intersection of the line of direction of the load and the line BE. Then will DA be the direction of the other supporting force; for the three external forces, in order to form a balanced system, must meet in a point, except when they are parallel. Then draw ab to scale, parallel to CD and equal to W. From *a* draw *ac* parallel to *BD*, and from *b* draw *bc* parallel to *AD;* then will the triangle *abca* be the triangle of external forces, the sides *ab*, *bc*, and *ca*, taken in order, representing respectively the load *W*, the supporting force at *A*, and the supporting force at *B*.

Then from *a* draw *ad* parallel to *BC*, and from *c* draw *cd* parallel to *AB;* then will the triangle *acd* be the triangle of forces for the joint *B*, and the sides *ca*, *ad*, and *dc*, taken in order, will represent respectively the supporting force at *B*, the force exerted by the bar *BC* at the point *B*, and the force exerted by the bar *AB* at the point *B*.

Since, therefore, the force *ad* exerted by the bar *CB* at *B* is directed *away from* the bar, it follows that *CB* is in compression; and, since the force *dc* exerted by the bar *AB* at *B* is directed *towards* the bar, it follows that *AB* is in tension.

In the same way *bdc* is the triangle of forces for the point *A;* the sides *bc*, *cd*, and *db* representing respectively the supporting force at *A*, the force exerted by the bar *AB* at *A*, and the force exerted by the bar *AC* at *A*.

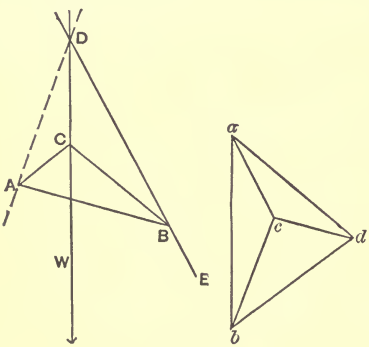


Fig. 73.

The bar *AB* is again seen to be in tension, as the force *cd* exerted by the bar *AB* at *A* is directed towards the bar.

So likewise the triangle *abd* is the triangle of forces for the point *C*.

Fig. 73 shows the case when the supporting forces meet the load-line above, instead of below, the truss.

§ 122. **Triangular Frame with Load and Supporting Forces Vertical.** — Fig. 74 shows the construction when the load and also the supporting forces are vertical. In this case the diagram becomes very much simplified, the triangle of external forces *abd* becoming a straight line. The diagram is otherwise constructed just like the last one.

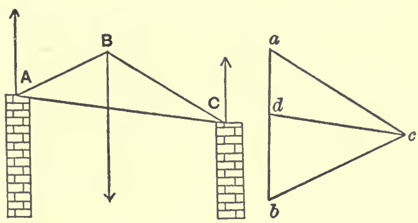


FIG. 74.

§ 123. **Bow's Notation.** — The notation devised by Robert H. Bow very much simplifies the construction of the stress diagrams of roof-trusses.

This notation is as follows: Let the radiating lines (Fig. 75) represent the lines of action of a system of forces in equilibrium, and let the polygon *abcdefa* be the polygon representing these forces in magnitude and direction; then denote the sides of the polygon in the ordinary way, by placing small letters at the vertices, but denote the radiating lines by capital letters placed in the angles. Thus the line *AB* is the line of direction of the force *ab*, etc. In applying the notation to roof-trusses, we letter the truss with capital letters in the spaces, and the stress diagram with small letters at the vertices. If, then, in drawing the polygon of equilibrium for any one joint of the truss, we take the forces always in the same order, proceeding always in right-handed or always in left-handed rotation, we shall be led to the simplest diagrams. Hereafter this notation will be used exclusively in determining the stresses in roof-trusses.

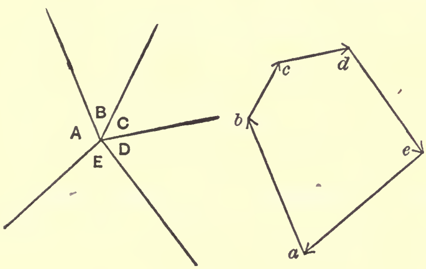


FIG. 75.

§ 124. **Isosceles Triangular Frame: Concentrated Load** (Fig. 76.) — Let the load W act at the apex, the supporting forces being vertical; each will be equal to $\frac{1}{2}W$: hence the polygon of external forces will be the triangle *abc*, the sides of which, *ab*, *bc*, and *ca*, all lie in one straight line. Then begin at the left-hand support, and proceed again in right-handed rotation, and we have as the triangle of forces at this joint *cad*, the forces *ca*, *ad*, and *dc*, these being respectively the supporting force, the stress in AD, and that in DC; the directions of these forces being indicated by the order in which the letters follow each other: thus, *ca* is an upward force, *ad* is a downward force; and, this being the force exerted by the bar AD at the left-hand support, we conclude that the bar AD is in compression. Again: *dc* is directed towards the right, or towards the bar itself, and hence the bar DC is in tension. The triangle of forces for the other support is *bcd*, and that for the apex *abd*.

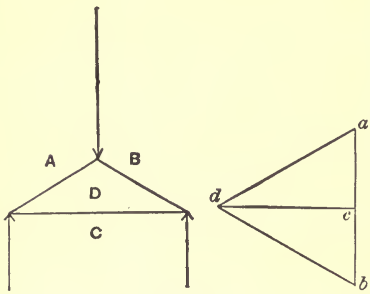


FIG. 76.

§ 125. **Isosceles Triangular Frame: Distributed Load.** — Let the load W be uniformly distributed over the two rafters AF and FB (Fig. 77); then will these two rafters be subjected to a direct stress, and also to a bending action: and if we resolve the load on each rafter into two components at the ends of the rafter, then, considering these components as the loads at the joints, we shall determine correctly by our diagram the direct stresses in all the bars of the truss.

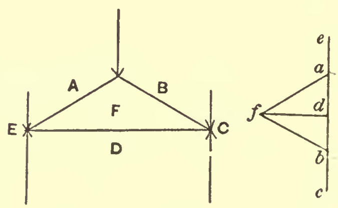


FIG. 77.

The load distributed over AF is $\frac{W}{2}$; and of this, one-half is the component at the support, and one-half at the apex, and similarly for the other rafter. This gives as our loads, $\frac{W}{4}$ at each support, and $\frac{W}{2}$ at the apex. The polygon of external forces is *eabcde*, where the sides are as follows: —

$$ea = \frac{W}{4}, \quad ab = \frac{W}{2}, \quad bc = \frac{W}{4}, \quad cd = \frac{W}{2}, \quad de = \frac{W}{2}.$$

Then, beginning at the left-hand support, we shall have for the polygon of forces the quadrilateral *deafd*, where $de = \frac{W}{2} =$ supporting force, $ea = \frac{W}{4} =$ downward load at support, $af =$ stress in AF (compression), $fd =$ stress in FD (tension). The polygon for the apex is *abf*, and that for the right-hand support *cdfbc*.

§ 126. **Polygonal Frame.** — Given a polygonal frame (Fig. 78) formed of bars jointed together at the vertices of the angles, and free to turn on these joints, it is evident, that, in order that the frame may retain its form, it is necessary that the directions of, and the proportions between, the loads at the different joints, should be specially adapted to the given form: otherwise the frame will change its form. We will proceed to solve the following problem: Given the form of the frame, the magnitude of one load as AB, and the direction of all the external forces (loads and supporting forces) except one, we shall have sufficient data to determine the magnitudes of all,

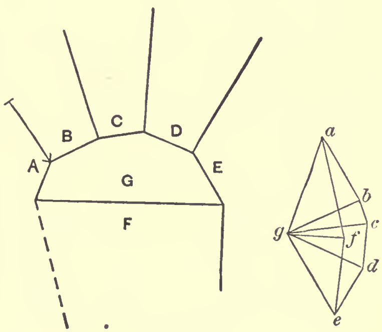


FIG. 78.

and the direction of the remaining external forces, and also the stresses in the bars

Let the direction of all the loads be given, and also that of the supporting force *EF*, that of the supporting force *AF* being thus far unknown; and let the magnitude of *AB* be given. Then, beginning at the joint *ABG*, we have for triangle of forces *abg* formed by drawing *ab* ∥ and = *AB*, then drawing *ga* ∥ *AG*, and *bg* ∥ *BG*; *ga* and *bg* both being thrusts. Then, passing to the joint *BCG*, we have the thrust in *BG* already determined, and it will in this case be represented by *gb*. If, now, we draw *bc* ∥ *BC*, and *gc* ∥ *GC*, we shall have determined the load *BC* as *bc*, and we shall have *cg* and *gb* as the thrusts in *CG* and *GB* respectively. Continuing in the same way, we obtain the triangles *gcd*, *gde*, and *gfe*, thus determining the magnitudes of the loads *cd*, *de*, and of the supporting force *ef*; and then the triangle *gaf*, formed by joining *a* and *f*, gives us *af* for the magnitude and direction of the left-hand support. The polygon *abcdefa* of external forces is called the *Force Polygon*, while the frame itself is called the *Equilibrium Polygon*.

§ 127. **Polygonal Frame with Loads and Supporting Forces Vertical.** — In this case (Fig. 79) we may give the form of the frame and the magnitude of one of the loads, to determine the other loads and the supporting forces, and also the stresses in the bars; or we may give the form of the frame and the magnitude of the resultant of the loads, to find the loads and supporting forces. In the former case let the load *AB* be given. Then, proceeding in the same way as before, we find the diagram of Fig. 79; the polygon of external forces *abcdefa* falling all in one straight line.

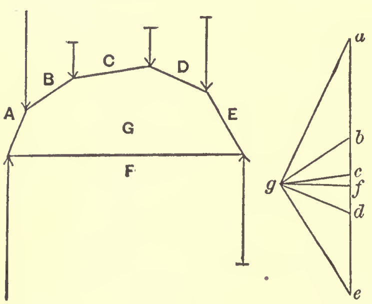


Fig. 79.

If, on the other hand, the whole load ae be given, we observe that this is borne by the stresses in the extreme bars AG and GE; hence, drawing $ag \parallel AG$, and $eg \parallel EG$, we find eg and ga as the stresses in EG and GA respectively. Then, proceeding to the joint ABG, we find, since ga is the force exerted by GA at this point, that, drawing $gb \parallel GB$, we shall have ab as the part of the load acting at the joint ABG, etc.

§ 128. **Funicular Polygon.** — If the frame of Fig. 79 be inverted, we shall have the case of Fig. 80, where all the bars, except FG, are subjected to tension; FG itself being subjected to compression. The construction of the diagram of stresses being entirely similar to that already explained for Fig. 79, the explanation will not be repeated here. If the compression piece be omitted, the case becomes that of a chain hung at the upper joints (the supporting forces then becoming identical with the tensions in the two extreme bars), the line gf would then be omitted from the diagram, and the polygon of external forces would become $abcdega$.

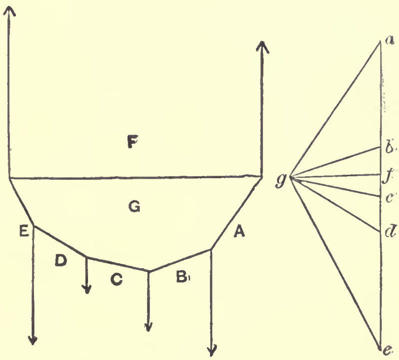


FIG. 80.

§ 129. **Triangular Truss: Wind Pressure.** — Inasmuch as the pressure of the wind on a roof has been shown by experiment to be normal to the roof on the side from which it blows, we will next consider the case of a triangular truss with the load distributed over one rafter only, and normal to the rafter.

There may be three cases: —

1°. When there is a roller under one end, and the wind blows from the other side.

2°. When there is a roller under one end, and the wind blows from the side of the roller.

3°. When there is no roller under either end.

The last arrangement should always be avoided except in small and unimportant constructions; for the presence of a roller under one end is necessary to allow the truss to change its length with the changes of temperature, and to prevent the stresses that would occur if it were confined.

CASE I. — Using Bow's notation, we have (Fig. 81) the whole load represented in the diagram by *db*. Its resultant acts at the middle of the rafter *AE*, whereas the supporting force at the right-hand end is (in consequence of the presence of the roller) vertical. Hence, to find the line of action of the other supporting force, produce the line of action of the load till it meets a vertical line drawn through the roller, and join their point of intersection with the support where there is no roller. We thus obtain *CD* as the line of action of the left-hand support.

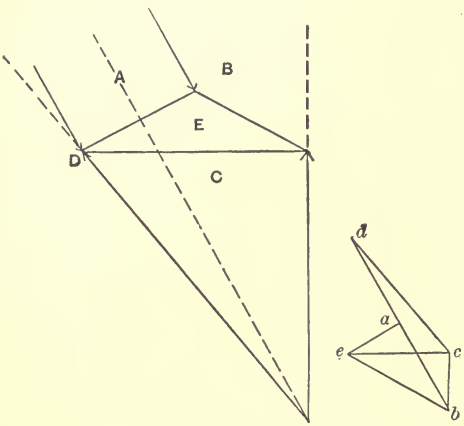


FIG. 81.

We can now determine the magnitude of the supporting forces *bc* and *cd* by constructing the triangle *bcdb* of external forces.

Now resolve the normal distributed force *db* into two single forces (equal to each other in this case), *da* and *ab* respectively, acting at the left-hand support and at the apex.

Now proceed to the left-hand support. We find four forces in equilibrium, of which two are entirely known; viz., *cd* and *da:* hence, constructing the quadrilateral *cdaec*, we have *ae* as the thrust in *AE*, and *ec* as the tension in *EC*.

Next proceed to the apex, and construct the triangle of equilibrium *abea*, and we obtain *be* as the thrust in *BE*.

The triangle *bceb* is then the triangle of equilibrium for the right-hand support.

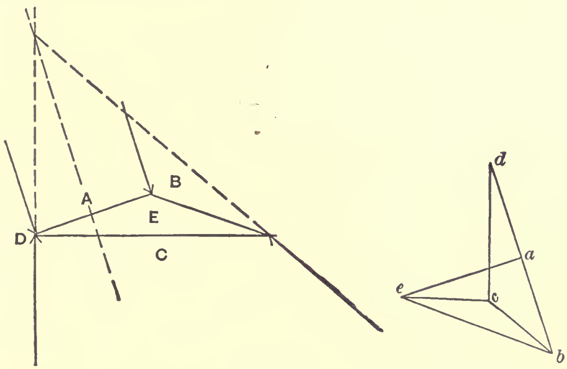


FIG. 82.

CASE II. — In this case (Fig. 82) we follow the same method of procedure, only the point of intersection of the load and supporting forces is above, instead of below, the truss. The figure explains itself so fully that it is unnecessary to explain it here.

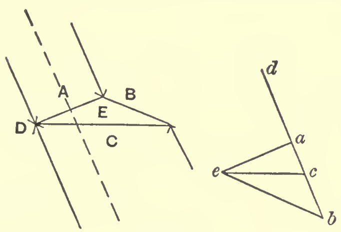


FIG. 83.

CASE III. — In this case the supports are capable of exerting resistance in any direction whatever; so that, if any circumstance should determine the direction of one of them, that of the other would be determined also. When there is no such circumstance, it is customary to assume them parallel to the load (Fig. 83). Making this assumption, we begin by dividing the line *db*, which represents the load, into two parts, inversely proportional to the two segments into which the line of action of the resultant of the load (the dotted line in the figure) divides the line *EC*. We thus obtain the supporting forces *bc* and *cd*, and *bcdb* is the triangle of external forces. We then follow the same method as in the preceding cases.

§ 130. **General Determination of the Stresses in Roof-Trusses.** — In order to compute the stresses in the different members of a roof-truss, it is necessary first to know the amount and distribution of the load.

This consists generally of —

1°. The weight of the truss itself.

2°. The weight of the purlins, jack-rafters, and superincumbent roofing, as the planks, slate, shingles, felt, etc.

3°. The weight of the snow.

4°. The weight of the ceiling of the room immediately below if this is hung from the truss, or the weight of the floor of the loft, and its load, if it be used as a room.

5°. The pressure of the wind; and this may blow from either side.

6°. Any accidental load depending on the purposes for which the building is used. As an instance, we might have the case where a system of pulleys, by means of which heavy weights are lifted, is attached to the roof.

In regard to the first two items, and the fourth, whenever the construction is of importance, the actual weights should be determined and used. In so doing, we can first make an approximate computation of the weight of the truss, and use it in the computation of the stresses; the weights of the ceiling or of the floor below being accurately determined. After the stresses in the different members have been ascertained by the use of these loads, and the necessary dimensions of the members determined, we should compute the actual weight of the truss; and if our approximate value is sufficiently different from the true value to warrant it, we should compute again the stresses. This second computation will, however, seldom be necessary.

In making these computations, the weights of a cubic foot of the materials used will be needed; and average values are given in the following table with sufficient accuracy for the purpose.

Weight of some Building Materials per Cubic Foot.	Pounds.
Timber.	
Chestnut	41
Hemlock	25
Maple	41
Oak, live	59
Oak, white	49
Pine, white	25 to 30
Pine, yellow, Southern	45
Spruce	25 to 30
Iron.	
Iron, cast	450
Iron, wrought	480
Steel	490
Other Substances.	
Asphaltum	80 to 90
Mortar, hardened	103
Snow, freshly fallen	5 to 12
Snow, compacted by rain	15 to 50
Slate	140 to 180

Weight of Slating per Square Foot. According to Trautwine.	Pounds.
$\frac{1}{8}$ inch thick on laths	4.75
$\frac{1}{8}$ " " " 1-inch boards	6.75
$\frac{1}{8}$ " " " $1\frac{1}{4}$ " "	7.30
$\frac{3}{16}$ " " " laths	7.00
$\frac{3}{16}$ " " " 1-inch boards,	9.00
$\frac{3}{16}$ " " " $1\frac{1}{4}$ " "	9.55
$\frac{1}{4}$ " " " laths	9.25
$\frac{1}{4}$ " " " 1-inch boards,	11.25
$\frac{1}{4}$ " " " $1\frac{1}{4}$ " "	11.80
With slating-felt add	$\frac{1}{4}$ lb.
With $\frac{1}{4}$-inch mortar add	3 lbs.

Number of Nails in One Pound.	No.
3-penny	450
4 "	340
6 "	150
8 "	100
10 "	60
12 "	40
20 "	25

As to the weight of the snow upon the roof, Stoney recommends the use of 20 pounds per square foot in moderate climates; and this would seem to the writer to be borne out by the experiments of Trautwine as recorded in his handbook, although Trautwine himself considers 12 pounds per square foot as sufficient.

§ 131. **Wind Pressure.**—While a great deal of work has been done to ascertain the direction and the greatest intensity of the pressure of the wind upon exposed surfaces, as those of roofs and bridges, nevertheless the amount of information on the subject is very small, inasmuch as but few experiments have been under the conditions of practice. Before giving a summary of what has been done the following statements will be made:

1°. The pressure of the wind upon a roof, or other surface, is assumed to be normal to the surface upon which it blows; and what little experimenting has been done upon the subject tends to confirm this view.

2°. Inasmuch as more attempts have been made to determine experimentally the velocity of the wind than its pressure, hence there have been a good many experiments to determine the relation between the velocity and the pressure upon a surface to which the direction of the wind is normal.

3°. A few experimenters have tried to determine the relation between the intensity of the pressure on a surface normal to the direction of the wind and one inclined to its direction.

4°. While the above have been the investigations most commonly pursued, other subjects of experiment have been—

(*a*) The variation of pressure with density; (*b*) with temperature; (*c*) with humidity; (*d*) with the size of surface pressed upon; (*e*) with the shape of surface pressed upon; (*f*) whether the pressure corresponding to a certain velocity is the same whether the air moves against a body at rest, or whether the body moves in quiet air.

By way of references to the literature of the subject may be given the following, as most of the work that has been done is included in them or in other references which they contain:

1°. Proceedings of the British Institution of Civil Engineers, vol. lxix., year 1882, pages 80 to 218 inclusive.
2°. A. R. Wolff : Treatise on Windmills.
3°. C. Shaler Smith : Proceedings American Society of Civil Engineers, vol. x., page 139.
4°. A. L. Rotch : Report of Work of the Blue Hill Meteorological Observatory, 1887.
5°. Engineering, Feb. 28th, 1890 : Experiments of Baker.
6°. Engineering, May 30, June 6, June 13, 1890: Experiments of O. T. Crosby.

The first gives an account of a very full discussion of the subject, by a large number of Engineers. The second contains a recommendation that the temperature of the air be considered in estimating the pressure. The fifth gives an account of Baker's experiments on wind pressure in connection with the building of the Forth Bridge.

Before an account is given of the experimental work that has been done, the following statements will be made of what are some of the methods in most common use :

1°. A great many engineers very commonly call from 40 to 55 pounds per square foot the maximum pressure on a vertical surface at right angles to the direction of the wind. One rather common practice, in the case of bridges, is to estimate 30 pounds per square foot on the loaded, or 50 pounds per square foot on the unloaded structure. Nevertheless pressures of 80 and 90 pounds per square foot have been registered and recorded by the use of small pressure-plates, and by computation from anemometer records.

2°. By way of determining the intensity of the pressure on an inclined surface in terms of that on a surface normal to the direction of the wind, four methods more or less used will be enumerated here :

(*a*) Duchemin's formula, which Professor W. C. Unwin recommends, is as follows, viz. :

$$p = p_1 \frac{2 \sin \theta}{1 + \sin^2 \theta},$$

where p = intensity of normal pressure on roof, p_1 = intensity of pressure on a plane normal to the direction of the wind.

(*b*) Hutton's formula,

$$p = p_1 (\sin \theta)^{1.84 \cos \theta - 1}.$$

Unwin claims that this and Duchemin's formula give nearly the same results for all angles of inclination greater than 15°.

The following table gives the results obtained by the use of each, on the assumption that $p_1 = 40$:

θ	Duchemin.	Hutton.	θ	Duchemin.	Hutton.
5°	6.89	5.10	50°	38.64	38.10
10°	13.59	9.60	55°	39.21	39.40
15°	19.32	14.20	60°	39.74	40.00
20°	24.24	18.40	65°	39.82	40.00
25°	28.77	22.60	70°	39.91	40.00
30°	32.00	26.50	75°	39.96	40.00
35°	34.52	30.10	80°	40.00	40.00
40°	36.40	33.30	85°	40.00	40.00
45°	37.73	36.00	90°	40.00	40.00

(*c*) A formula very commonly favored, but which does not agree with any experiments that have been made, is

$$p = p_1 \sin^2 \theta.$$

It gives much lower results, as a rule, than either of the others, but it is favored by many because, if we assume the wind to blow in parallel lines till it strikes the surface, and then to get suddenly out of the way, forming no eddies on the back side of the surface and meeting no lateral resistance on the front side, all of which are conditions that do not exist, we could then deduce it as follows:

Assume a unit surface making an angle θ with the direction of the wind, the total pressure on this surface in the direction of the wind would be $p_1 \sin \theta$; and by resolving this into normal and tangential components we should have, for the former,

$$p = p_1 \sin^2 \theta.$$

(*d*) Another rule which is sometimes used, but which has nothing to recommend it, is to consider the normal intensity of the wind pressure per square foot of roof surface as equal to the number of degrees of inclination of the roof to the horizontal. The wind pressure allowed for by this rule is very excessive, as it would be 90 pounds per square foot for a vertical surface.

Taking up, now, the experimental work that has been done, we will begin with the attempts to determine velocities and pressures, and the relation between them.

1°. In regard to velocities, these are determined by using some kind of an anemometer, and in all these cases there are several difficulties and sources of error, as follows:

(*a*) In many cases the anemometers have not even been graduated experimentally, but it has been assumed outright that the velocity of the air is just three times the linear velocity of the cups of a cup anemometer.

(*b*) When they have been graduated, it has generally been done by attaching them to the end of the arm of a whirling machine, which, when the arm is long, and the velocity moderate, will do very well, but is the more inaccurate the shorter the arm and the higher the velocity of motion.

(*c*) The wind always comes in gusts, and hence the anemometer does not register the average velocity of the wind at any one moment, but that of the particular portion that comes in contact with it, and this is always a small portion, on account of the small size of the anemometer.

(*d*) In order to get an indication which is not affected by the cross-currents reflected from the surrounding buildings and chimneys, it is necessary to put the anemometer very high up, and then, of course, we obtain the indications corresponding to that height, which is greater than that of the buildings, and it is well known that the velocity of the wind increases very considerably with the height.

Next, as to the direct determination of pressure, this has usually been done by means of some kind of pressure-plate, either round or square, but of small size, thus allowing the eddies formed on the back side of the plate to have a considerable effect. The results obtained by the use of different sizes and different shapes of plates have therefore differed very considerably; and while some have claimed that the pressure per square foot increases with the size of the surface pressed upon, it has been very thoroughly proved by the more modern investigations that the opposite is true, and that the pressure decreases with the size.

While the records from small pressure-plates have frequently shown very high pressures per square foot, as 80, 90, or even over 100 pounds per square foot, it has become very generally recognized by engineers that by far the greater part of existing buildings and bridges would be overturned by winds of such force, or anywhere near such force, and it has not been customary among them to make use of such high figures for wind pressure on bridges and roofs in computing the stability of structures. While some of the figures in general use have already been given, nevertheless the tendency of modern investigation seems to be to obtain rather lower figures. In this connection it is well to refer to the work done by Baker in connection with the construction of the Forth Bridge. The following description is taken from "Engineering" of Feb. 28th, 1890:

"The wind pressure to be provided for in the calculations for bridges in exposed positions is 56 lbs. per square foot, according to the Board of Trade regulations, and this twice over the whole area of the girder surface exposed, the resistance to such pressure to be by dead-weight in the structure alone.

"The most violent gales which have occurred during the construction of the Forth Bridge are given, with the pressures recorded on the wind gauges, in the annexed table:

Year.	Month and Day.	Pressure in pounds per square foot.					Direction of Wind.
		Revolving Gauge.	Small fixed Gauge.	Large fixed Gauge.	In centre of large Gauge.	Right-hand top of large Gauge.	
1883	Dec. 11,	33	39	22			S. W.*
1884	Jan. 26,	65	41	35			S. W.*
1884	Oct. 27,	29	23	18			S. W.
1884	Oct. 28,	26	29	19			S. W.
1885	Mar. 20,	30	25	17			W.
1885	Dec. 4,	25	27	19			W.
1886	Mar. 31,	26	31	19			S. W.
1887	Feb. 4,	26	41	15			S. W.
1888	Jan. 5,	27	16	7			S. E.
1888	Nov. 17,	35	41	27			W.
1889	" 2,	27	34	12			S. W.
1890	Jan. 19,	27	28	16			S. W.
1890	" 21,	26	38	15			W.
1890	" 22,	27	24	18	23½	22	S. W. by W.

* These data are unreliable, owing to faulty registration by the indicator-needle, as will be presently explained. They were altered after this date. The barometer fell to 27.5 inches on that occasion, over three quarters of an inch within an hour.

"The pressure-gauges, which were put up in the summer of 1882 on the top of the old castle of Inchgarvie, and from which daily records have been taken throughout, were of very simple construction. The maximum pressures only were taken. The most unfavorable direction from which the wind pressure can strike the bridge is nearly due east and west, and two out of the three gauges were fixed to face these directions, while a third was so arranged as to register for any direction of wind.

"The principal gauge is a large board 20 feet long by 15 feet high, or 300 square feet area, set vertically with its faces east and west. The weight of this board is carried by two rods suspended from a framework surrounding the board, and so arranged as to offer as little resistance as possible to the passage of the wind, in order not to create eddies near the edge of the board. In the horizontal central axis of the board there are fixed two pins which fit into the lower eyes of the suspension-rods, the object being to balance the board as nearly as possible. Each of the four corners of the board is held between two spiral springs, all carefully and easily adjusted so that any pressure exerted on either face will push it evenly in the opposite direction, but upon such pressure being removed the compressed springs will force the board back to its normal position. To the four corners four irons are attached, uniting in a pyramidal formation in one point, whence a single wire passes over a pulley to the registering apparatus below. In order to ascertain to some extent how far great gusts of wind are quite local in their action, and exert great pressure only upon a very limited area, two circular spaces, one in the exact centre and one in the right-hand top corner, about 18 inches in diameter, were cut out of the board and circular plates inserted, which could register independently the force of the wind upon them.

"By the side of the large square board, at a distance of about 8 feet, another gauge, a circular plate of 1.5 square feet area, facing east and west, was fixed up with separate registration. This was intended as a check upon the indications given by the large board.

"Another gauge of the same dimensions as the last, but with the disc attached to the short arm of a double vane, so that it would face the wind from whatever direction it might come, was set up.

"On one occasion the small fixed board appeared to register 65 pounds to the square foot—a registration which caused no little alarm and anxiety. Mr. Baker found, upon investigation, that the registering apparatus was not in good order, and after adjusting it the highest pressure recorded was 41 pounds.

"In order to determine the effect of the wind upon surfaces like that of the exposed surface of the bridge, he devised an apparatus which consisted of a light wooden rod suspended in the middle, so as to balance correctly, by a string from the ceiling. At one end was attached a cardboard model of the surface, the resistance of which was to be tested, and at the opposite end was placed a sheet of cardboard facing the same way as the model, so arranged that by means of another and adjustable sheet, which would slide in and out of the first, the surface at that end could be increased or decreased at the will of the operator. The mode of working is for a person to pull it from its perpendicular position towards himself, and then gently release it, being careful to allow both ends to go together. If this is properly done, it is evident that the rod will in swinging retain a position parallel to its original position, supposing that the model at one end and the cardboard frame at the other are balanced as to weight, and that the two surfaces exposed to the air pressure coming against it in swinging are exactly alike. Should one area be greater than the other, the model or cardboard sheet, whichever it may be, will be lagging behind, and twist the string."

The experiments carried on in various ways by different people and at different times are generally in agreement with each other and with the results of more elaborate processes. The information specially desired was in regard to the wind pressure upon surfaces more or less sheltered by those immediately in front of them. In this regard Mr. Baker satisfied himself that, while the results differed very considerably according to the distance apart of the surfaces, in no case was the area affected by the wind, in any girder which had two or more surfaces exposed, more than 1.8 times the area of the surface directly fronting the wind, and, as the calculations had been made for twice this surface, the stresses which the structure will receive from this cause will be less than those provided for.

Next, as to the relation between velocity and pressure, a great many formulæ have been devised, to agree with the results of different experimenters. Most all of them make the pressure proportional to the square of the velocity; while some add a term proportional to the velocity itself, and when higher velocities are reached, as those usual in gunnery, terms have been introduced with powers of the velocity higher than the second. It is hardly worth while to consider these different formulæ, as it is rather the pressure than the velocity that the engineer is interested in, and correct information in this regard is to be obtained rather from pressure-boards than from anemometers. Nevertheless, it may be stated that one of the most usual formulæ is that of Smeaton, and is

$$P = \frac{V^2}{200},$$

where $P =$ pressure in pounds per square foot, and $V =$ velocity in miles per hour. This formula agrees very well with a number of experiments that have been made where anemometers have been used to determine the velocity, and small pressure-plates (say one square foot) to determine the pressure; thus this formula satisfies very well the experiments made at the Blue Hill Meteorological Observatory, near Boston, Mass., U. S. A.

It was originally deduced from some very old experiments of Rouse; and it agrees with a good many, but disagrees with other experiments. It is probably the formula that has been more quoted than any other.

A little ought also to be said in regard to the pressure of the wind on very high structures, as on the piers of high viaducts and on tall chimneys. In this regard it is to be observed:

1°. The pressure, as well as the velocity of the wind, becomes greater the higher up from the ground the surface exposed is situated.

2°. From calculations on chimneys that have stood for a long time, Rankine deduced, as the greatest average wind pressure that could be realized in the case of tall chimneys, 55 pounds per square foot.

3°. In making the piers of high viaducts, it would seem desirable not to make them solid, but to use only four uprights at the corners connected by lattice work, in order to expose a smaller surface to the wind. Nevertheless, as was explained, it will not do to separate the structure into its component parts, and to estimate the pressure on each part separately and then add the results together to get the total effect; but we really need some such experiments as those of Baker.

4°. Some old experiments of Borda bear out the common practice of assuming the wind pressure on the surface of a circular cylinder one half that which would exist on its projection on a plane normal to the direction of the wind.

There remains now only to refer to a serial article by O. T. Crosby, in "Engineering" of May 30, June 6th, and June 13th, 1890, containining some experiments made by him on wind pressure near Baltimore, Md. The first two numbers contain rather a summary of what has been done by others, and it is in the copy of June 13th that is to be found the account of his own work, which was done in order to determine the resistances of the air to fast-moving trains.

He used a whirling arrangement turning about a vertical axis, to the end of which was attached a car, the circumference through which the car moved being 36 feet.

In order to determine whether the circular motion produced any disturbing effect, he ran a car having a cross-section of 5.1 square feet on a circular track about two miles in circumference, the speed of the car being about 50 miles per hour, and the results obtained in this way agreed very nearly with those obtained from his whirling table. The special peculiarity of his results is that he obtained, by plotting them, the law that the pressure varies directly as the first power of the velocity, and not as the square or some higher power; also, his pressures, after the velocity had passed 25 or 30 miles per hour, are much lower than those given by Smeaton and others, the pressure on a normal plane surface moving at 115 miles per hour being about 27 pounds per square foot.

The cars used were generally about 3 feet long without the front. The fronts attached were: 1°. Normal plane surface; 2°. Wedge, base 1, height 1; 3°. Pyramid, base 1, height 2; 4°. Wedge and cyma, base 1, height 2; 5°. Parabolic wedge, base 1, height 2.

His experiments covered a range of velocities from 30 to 130 miles per hour.

The law of the first powers of the velocities seems peculiar, and certainly ought not to be accepted without further corroborative evidence; but the low values of the pressures agree with Baker's results and with the tendency of the more modern investigations.

§ 132. **Approximate Estimation of the Load.** — In all important constructions, the estimates of the loads should be made as above described. For smaller constructions, and for the purposes of a preliminary computation in all cases, we only estimate the roof-weight roughly; and some authors even assume the wind pressure as a vertical force.

Trautwine recommends the use of the following figures for the total load per square foot, including wind and snow, when the span is 75 feet or less: —

Roof covered with corrugated iron, unboarded . . .	28 lbs.
Roof plastered below the rafters	38 "
Roof, corrugated iron on boards	31 "
Roof plastered below the rafters	41 "
Roof, slate, unboarded or on laths	33 "
Roof, slate, on boards $1\frac{1}{4}$ inches thick	35 "
Roof, slate, if plastered below the rafters	46 "
Roof, shingles on laths	30 "
Roof plastered below rafters or below tie-beam . . .	40 "

From 75 to 100 feet, add 4 lbs. to each.

§ 133. **Distribution of the Loads.** — The methods for determining the stresses, which will be used here, give correct results only when the loads are concentrated at joints, and are not distributed over any members of the truss.

In constructions of importance, this concentration of the loads at the joints should always be effected if possible; because, when this is the case, the stresses in the members of the truss can be, if properly fitted, obliged to resist only stresses of direct tension, or of direct compression; but, when there is a load distributed over any member of the truss, that member, in addition to the direct compression or direct tension, is subjected to a bending-stress. The effect of this bending cannot be discussed until we have studied the theory of beams. Nevertheless, it is a fact that we have no experimental knowledge of the behavior of a piece under combined compression and bending; and we know that when certain pieces are to resist bending, in addition to tension, they must be made much larger in proportion than is necessary when they bear tension only.

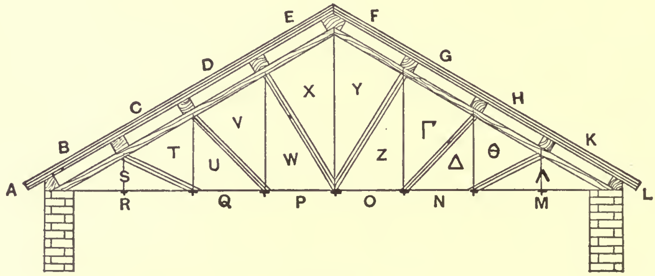


FIG. 84.

The manner in which this concentration of the loads is effected, is shown in Fig. 84, which is intended to represent one of a series of trusses that supports a roof, the rafters being the two lower ones in the figure. Resting on two consecutive trusses, and extending from one to the other, are beams called purlins, which should be placed only above the joints of the truss, and which are shown in cross-section in the figure. On these purlins are supported the jack-rafters parallel to the rafters, and at sufficiently frequent intervals to support suitably the plank and superincumbent roofing-materials.

By this means we secure that the entire bending-stress comes upon the jack-rafters and purlins, and that the rafters proper are subjected only to a direct compression. When, however, the load is distributed, i.e., when the roofing rests directly on the rafters, or when the purlins are placed at points other than the joints, the bending-stress should be taken into account; and while the methods to be developed here will give the stresses in all the members that are not subjected to bending, the bending-stress may be largely in excess of the direct stress in those pieces that are subjected to bending. How to take this into account will be explained later.

Another important item to consider is, that, in constructions of importance, a roller should be placed under one end of the truss to allow it to move with the change of temperature; as otherwise some of the members will be either bent, or at least subjected to initial stresses. The presence of a roller obliges the supporting force at that point to be vertical, whether the load be vertical or inclined.

It is customary, and does not entail any appreciable error, to consider the weight of the truss itself, as well as that of the superincumbent load, as concentrated at the upper joints; i.e., those on the rafters.

In the case of a ceiling on the room below, or of a loft whose floor rests on the lower joints, we must, of course, account the proper load as resting on the lower joints.

§ 134. **Direct Determination of the Stresses.** — This, as we have seen, is merely a question of equilibrium of forces in a plane, where certain forces acting are known, and others are to be determined.

As to the methods of solution, we might adopt —

1°. A graphical solution, laying off the loads to scale, and constructing the diagram by the use of the propositions of the polygon, and the triangle of forces, and then determining the results by measuring the lines representing the stresses to the same scale.

2°. An analytical solution, imposing the analytical conditions of equilibrium, as given under the "Composition of Forces," between the known and unknown forces.

3°. A third method is to construct the diagram as in the graphical solution, but then, instead of determining the results by measuring the resulting lines to scale, to compute the unknown from the known lines of the diagram by the ordinary methods of trigonometry.

The first, or purely graphical, method, is one which has received a very large amount of attention of late years, and in which a great deal of progress has been made. It is, doubtless, very convenient for a skilled draughtsman, and especially convenient for one who, though skilled in draughting, is not free with trigonometric work; but it seems to me, that, when the results are determined by computation from the diagram, there is less chance of a slight error in some unfavorable triangle vitiating all the results. I am therefore disposed to recommend for roof-trusses the third method.

In the case of bridge-trusses, on the other hand, I believe the graphical not to be as convenient as a purely analytic method.

§ 135. **Roof-Trusses.** — In what follows, the graphical solutions will be explained with very little reference to the trigonometric work, as that varies in each special case, and to one who has a reasonable familiarity with the solution of plane triangles, it will present no difficulty; whereas to deduce the formulæ for each case would complicate matters very much. Proceeding to special examples, let us take, first, the truss shown in Fig. 85, and let the vertical load upon it be W uniformly distributed over the top of the roof, the purlins being at the joints on the rafters.

The loads at the several joints will then be as follows, viz. (Fig. 85a), —

$$ab = kl = \frac{W}{16}, \quad bc = cd = de = ef = fg = gh = hk = \frac{W}{8}.$$

Then the supporting forces will be

$$lm = ma = \frac{W}{2}.$$

We thus have, as polygon of external forces, $abcdefghklma$.

Now proceed to either support, say, the left-hand one; and there we have the two forces *ab* and *ma* known, while *by* and *ym* are unknown. We then construct the quadrilateral *maby* in the figure, and thus determine *by* and *ym*. As to whether these represent thrust or tension, we need only remember that they are the forces exerted by the respective bars at the joints: and, since *by* is directed away from the bar *BY*, this bar is in compression; whereas, *ym* being directed towards the bar *YM*, that bar is in tension.

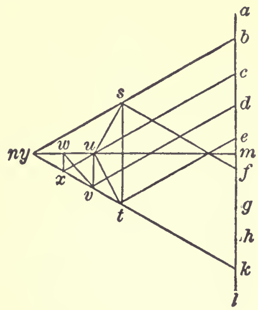


Fig. 85*a*.

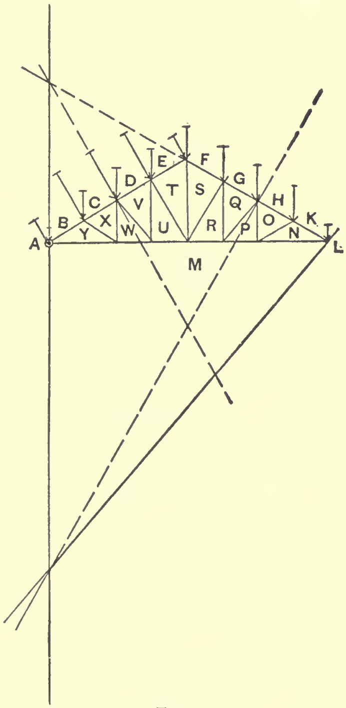


Fig. 85.

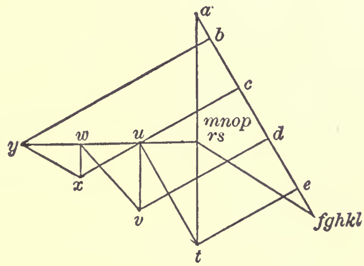


Fig. 85*b*.

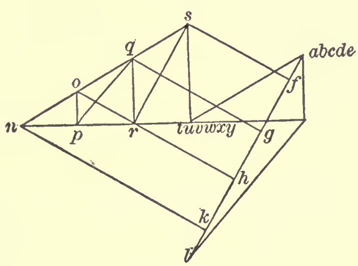


Fig. 85*c*.

Having determined these two stresses, we next proceed to another joint, where we have only two unknown forces. Take the joint at which the load bc acts, and we have as known quantities the load bc, and also the force exerted by the bar YB, which is in compression. This force is now directed away from the bar, and hence is represented by yb. The unknown forces are the stresses in CX and XY. Hence we construct the quadrilateral $cxybc$; and we thus determine the stresses in CX and XY as cx and xy, both being thrusts.

Next proceed to the joint YXW, and construct the quadrilateral $myxwm$, and thus determine the tension xw and the tension wm.

Next proceed to the joint where cd acts, and so on. We thus obtain the diagram (Fig. 85a) giving all the stresses.

The truss in the figure was constructed with an angle of 30° at the base, and hence gives special values in accordance with that angle.

In order to show how we may compute the stresses from the diagram, the computation will be given.

From triangle bmy, we have $bm = \frac{7}{16} W$

$$\therefore \quad ym = \frac{7}{16} W \cot 30° = \frac{7\sqrt{3}}{16} W$$

$$by = \frac{7}{16} W \operatorname{cosec} 30° = \frac{7}{8} W = ky.$$

From the triangle umc, we have $cm = \frac{5}{16} W$,

$$um = \frac{5\sqrt{3}}{16} W,$$

$$yw = \frac{1}{2} yu = \frac{1}{2}\left(\frac{2}{16} W\sqrt{3}\right) = \frac{\sqrt{3}}{16} W,$$

$$yx = yw \sec 30^\circ = \left(\frac{\sqrt{3}}{16} W\right)\frac{2}{\sqrt{3}} = \frac{W}{8} = xv = vt,$$

$$xw = \tfrac{1}{2} yx = \frac{W}{16}, \qquad uv = \frac{W}{8}, \qquad st = 2\left(\frac{3}{16} W\right) = \frac{3}{8} W,$$

$$wv = \sqrt{(wu)^2 + (uv)^2} = W\sqrt{\frac{3}{256} + \frac{4}{256}} = \frac{\sqrt{7}}{16} W,$$

$$ut = \sqrt{(wu)^2 + (\tfrac{1}{2}st)^2} = W\sqrt{\frac{3}{256} + \frac{9}{256}} = \frac{\sqrt{3}}{8} W,$$

$$cx = wm \sec 30^\circ = \left(\frac{3\sqrt{3}}{8} W\right)\frac{2}{\sqrt{3}} = \frac{3}{4} W,$$

$$vd = um \sec 30^\circ = \left(\frac{5\sqrt{3}}{16} W\right)\frac{2}{\sqrt{3}} = \frac{5}{8} W,$$

$$et = \left(\frac{\sqrt{3}}{4} W\right)\frac{2}{\sqrt{3}} = \frac{1}{2} W.$$

Hence we shall have for the stresses, —

RAFTERS (compression).

$by = kn$	$=$	$\frac{7}{8} W.$
$cx = ho$	$=$	$\frac{3}{4} W.$
$dv = gq$	$=$	$\frac{5}{8} W.$
$ct = fs$	$=$	$\frac{1}{2} W.$

HORIZONTAL TIES (tension).

$my = mn$	$=$	$\frac{7\sqrt{3}}{16} W.$
$mw = mp$	$=$	$\frac{3\sqrt{3}}{8} W.$
$mu = mr$	$=$	$\frac{5\sqrt{3}}{16} W.$

VERTICALS (tension).

$xw = op$	$=$	$\frac{W}{16}.$
$vu = qr$	$=$	$\frac{W}{8}.$
$ts =$		$\frac{3}{8} W.$

DIAGONAL BRACES (compression).

$xy = on$	$=$	$\frac{W}{8}.$
$wv = qp$	$=$	$\frac{\sqrt{7}}{16} W.$
$tu = sr$	$=$	$\frac{\sqrt{3}}{8} W.$

Next, as to the stresses due to wind pressure, we will suppose that there is a roller under the left-hand end of the truss, and none under the right-hand end; and we will proceed to determine the stresses due to wind pressure.

First, suppose the wind to blow from the left-hand side of the truss, and let the total wind pressure be (Fig. 85*b*) $af = W_1$. The resultant, of course, acts along the dotted line drawn perpendicular to the left-hand rafter at its middle point, as shown in Fig. 85.

The left-hand supporting force will be vertical: hence, producing the above-described dotted line, and a vertical through the roller to their intersection, and joining this point with the right-hand end of the truss, we have the direction of the right-hand supporting force. In this case, since the angle of the truss is 30°, the line of action of the right-hand supporting force coincides in direction with the right-hand rafter. We now construct the triangle of external forces *afm*, and we obtain the supporting forces *fm* and *ma*. We then have, as the loads at the joints,

$$ab = \frac{W_1}{8} = ef,$$

$$bc = \frac{W_1}{4} = cd = de.$$

Then proceed as before to the left-hand joint; and we find that two of the four forces acting there are known, viz., *ma* and *ab*, and two are unknown, viz., the stresses in *BY* and *YM*. Then construct the quadrilateral *mabym*, and we have the stresses *by* and *ym*; the first being compression and the second tension, as shown by reasoning similar to that previously adopted.

Then pass to the next joint on the rafter, and construct the quadrilateral *ybcxy*, where *yb* and *bc* are already known, and we obtain *cx* and *xy*; and so proceed as before from joint to joint, remembering, that, in order to be able to construct the polygon of forces in each case, it is necessary that only two of the forces acting should be unknown.

When the wind blows from the other side, we shall obtain the diagram shown in Fig. 85*c*.

After having determined the stresses from the vertical load diagram and those from the two wind diagrams, we should, in order to obtain the greatest stress that can come on any one member of the truss, add to the stress due to the vertical load the greater of the stresses due to the wind pressure.

§ 136. **Roof-Truss with Loads at Lower Joints.** — In Fig. 86 is drawn a stress diagram for the truss shown in Fig. 84 on the supposition that there is also a load on the lower joints. In this case let W be the whole load of the truss, except the ceiling, and W_1 the weight of the ceiling below; the latter being supported at the lower joints and on the two extreme vertical suspension rods. Then will the loads at the joints be as follows; viz., —

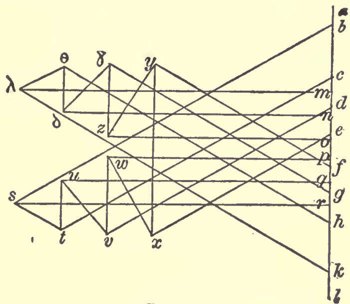


FIG. 86.

$$
\begin{aligned}
ab &= \tfrac{1}{16}(W + W_1) &&= kl, \\
bc &= \tfrac{1}{8}(W + W_1) &&= hk, \\
cd &= \tfrac{1}{8}W &&= gh = de = fg = ef, \\
mn &= \tfrac{1}{8}W_1 = rq &&= on = qp = op.
\end{aligned}
$$

Observe that there is no joint at the lower end of either of the end suspension rods, but that whatever load is supported by these is hung directly from the upper joints, where *bc* and *hk* act.

We have also for each of the supporting forces *lm* and *ra*

$$\tfrac{1}{2}(W + W_1).$$

Hence we have, for the polygon of external forces,

$$abcdefghklmnopqra,$$

which is all in one straight line, and which laps over on itself.

In constructing the diagram, we then proceed in the same way as heretofore.

§ 137. **General Remarks.** — As to the course to be pursued in general, we may lay down the following directions: —

1°. *Determine all the external forces; in other words, the loads being known, determine the supporting forces.*

2°. *Construct the polygon of forces for each joint of the truss, beginning at some joint where only two of the forces acting at that joint are unknown.* This is usually the case at the support. Then proceed from joint to joint, bearing in mind that we can only determine the polygon of forces when the magnitudes of all but two sides are known.

3°. *Adopt a certain direction of rotation, and adhere to it throughout; i.e., if we proceed in right-handed rotation at one joint, we must do the same at all, and we shall thus obtain neat and convenient figures.*

4°. *Observe that the stresses obtained are the forces exerted by the bars under consideration, and that these are thrusts when they act away from the bars, and tensions when they are directed towards the bars.*

We will next take some examples of roof-trusses, and construct the diagrams of some of them, calling attention only to special peculiarities in those cases where they exist.

It will be assumed that the student can make the trigonometric computations from the diagram.

The scale of load and wind diagram will not always be the same; and the stress diagrams will in general be smaller than is advisable in using them, and very much too small if the results were to be obtained by a purely graphical process without any computation.

The loads will in all cases be assumed to be distributed uniformly over the jack-rafters, or, in other words, concentrated at the joints.

Those cases where no stress diagram is drawn may be considered as problems to be solved.

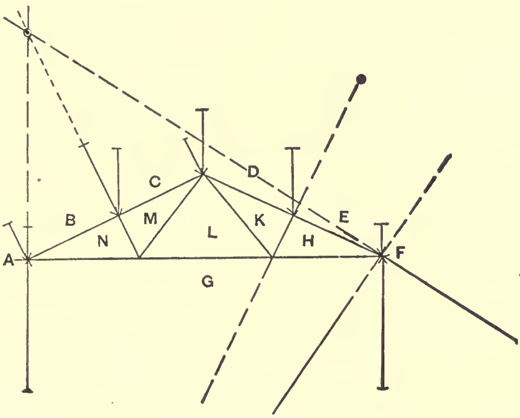


FIG. 87.

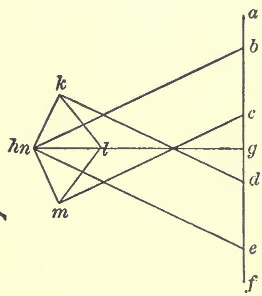


FIG. 87*a*.

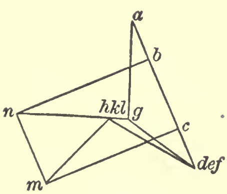


FIG. 87*b*.

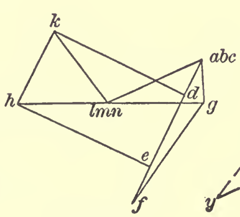


FIG. 87*c*.

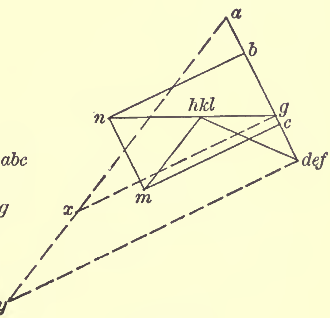


FIG. 87*d*.

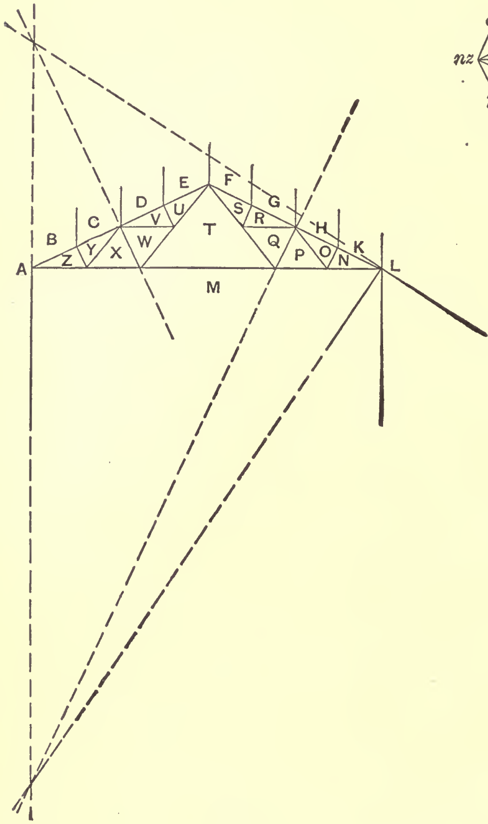


FIG. 88.

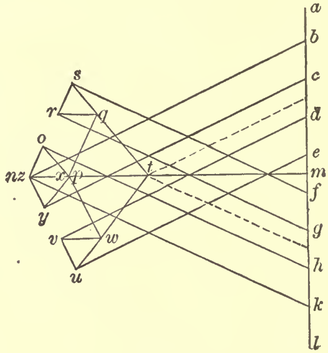


FIG. 88*a*.

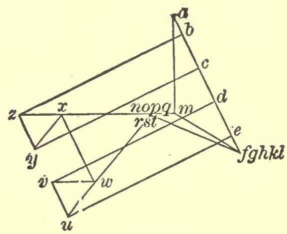


FIG. 88*b*.

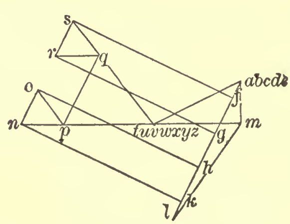


FIG. 88*c*.

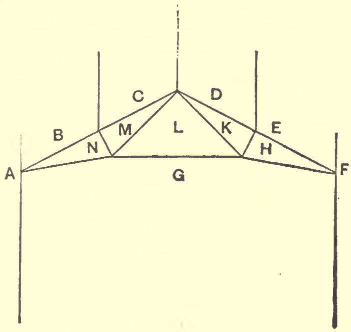


FIG. 89.

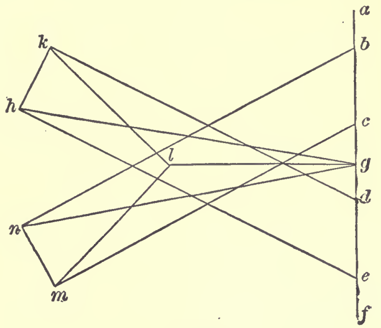


FIG. 89*a*.

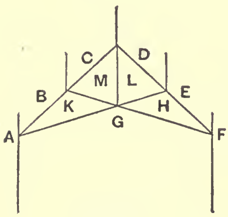


FIG. 90.

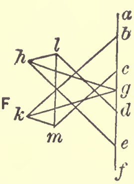


FIG. 90*a*.

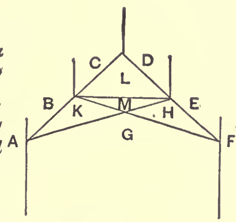


FIG. 91.

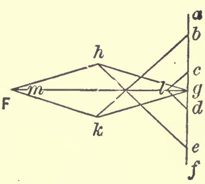


FIG. 91*a*.

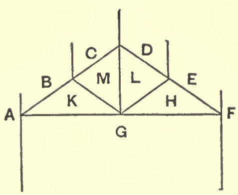


FIG. 92.

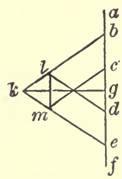


FIG. 92*a*.

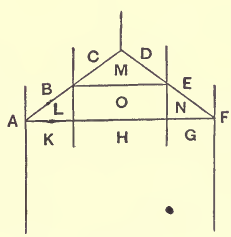


FIG. 93.

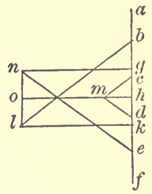


FIG. 93*a*.

§ 138. **Hammer-Beam Truss** (Fig. 94). — This form of truss is frequently used in constructions where architectural effect is the principal consideration rather than strength. It is not an advantageous form from the point of view of strength,

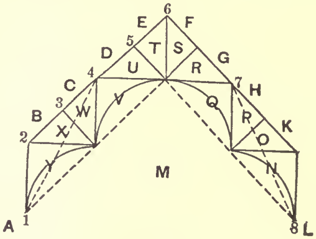


FIG. 94.

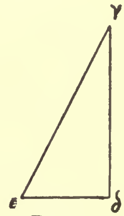


FIG. 94*a*.

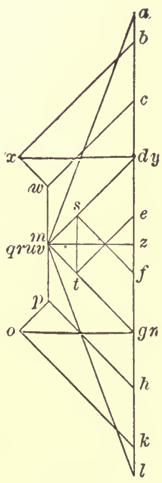


FIG. 94*b*.

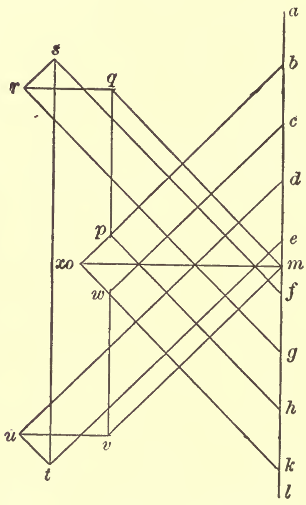


FIG. 94*c*.

for the absence of a tie-rod joining the two lower joints causes a tendency to spread out at the base, which tendency is usually counteracted by the horizontal thrust furnished by the buttresses against which it is supported.

When such a thrust is furnished (or were there a tie-rod), and the load is symmetrical and vertical, the bars are not all needed, and some of them are left without any stress. In the case in hand, it will be found that UV, VM, MQ, and QR are not needed. We must also observe that the effect of the curved members MY, MV, MQ, and MN on the other parts of the truss is just the same as though they were straight, as shown in the dotted lines. The curved piece, of course, has to be subjected to a bending-stress in order to resist the stress acting upon it. If, as is generally the case, the abutments are capable of furnishing all the horizontal thrust needed, it will first be necessary to ascertain how much they will be called upon to furnish. To do this, observe that we have really a truss similar to that shown in Fig. 92, supported on two inclined framed struts, of which the lines of resistance are the dotted lines (Fig. 94) 1 4 and 7 8, and that, under a symmetrical load, this polygonal frame will be in equilibrium, and, moreover, the curved pieces MV and MQ will be without stress, these only being of use to resist unsymmetrical loads, as the snow or wind.

Let the whole load, concentrated by means of the purlins at the joints of the rafters, be W. Then will the truss 4 6 7 have to bear $\frac{1}{2}W$, and this will give $\frac{W}{4}$ to be supported at each of the points 4 and 7. Moreover, on the space 2 4 is distributed $\frac{W}{4}$, which has, as far as overturning the strut is concerned, the same effect as $\frac{W}{8}$ at 2, and $\frac{W}{8}$ at 4. Hence the load to be supported at 4 by the inclined strut is a vertical load equal to $(\frac{1}{4} + \frac{1}{8})W = \frac{3}{8}W$. We may then find the force that must be furnished by the abutment, or by the tie-rod, in either of the two following ways: —

1°. By constructing the triangle $\gamma\delta\epsilon$ (Fig. 94*a*), with $\gamma\delta = \frac{3}{8}W$, $\gamma\epsilon \parallel 14$, and $\epsilon\delta$ parallel to the horizontal thrust of the abutment; then will $\gamma\delta\epsilon$ be the triangle of forces at 1, and $\epsilon\delta$ will be the thrust at 1.

2°. Multiply $\frac{3}{8}W$ by the perpendicular distance from 4 to 1 2, and divide by the height of 4 above 1 8 for the thrust of the abutment; in other words, take moments about the point 1.

Now, to construct the diagram of stresses, let, in Fig. 94*b*, the loads be

$$ab,\ bc,\ cd,\ de,\ ef,\ fg,\ gh,\ hk, \text{ and } kl,$$

and let

$$lz = za = \tfrac{1}{2}W$$

be the vertical component of the supporting force; let zm be the thrust of the abutment: then will lm and ma be the real supporting forces; and we shall have, for polygon of external forces,

$$abcdefghklma.$$

Then, proceeding to the joint 1, we obtain, for polygon of forces,

$$maym;$$

and, proceeding from joint to joint, we obtain the stresses in all the members of the truss, as shown in Fig. 94*b*.

It will be noticed that UV and RQ are also free from stress.

If we had no horizontal thrust from the abutment, and the supporting forces were vertical, the members MV and MQ would be called into action, and MY and MN would be inactive. To exhibit this case, I have drawn diagram 94*c*, which shows the stresses that would then be developed. AY and NL would become merely part of the supports.

In this latter case the stresses are generally much greater than in the former, and a stress is developed in UV.

§ 139. **Hammer-Beam Truss: Wind Pressure.** — Fig. 95 shows the stress diagram of the hammer-beam truss for wind pressure when there is no roller under either end, and when the wind blows from the left. A similar diagram would give the stresses when it blows from the right.

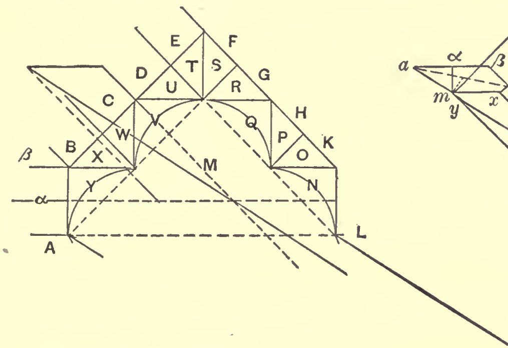


FIG. 95.

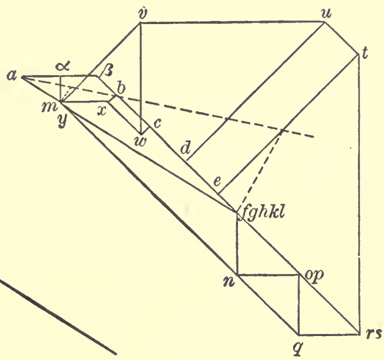


FIG. 95*a*.

The cases when there is a roller are not drawn: the student may construct them for himself.

§ 140. **Scissor-Beam Truss.** — We have already discussed two forms of scissor-beam truss in Figs. 90 and 91. These trusses having the right number of parts, their diagrams present no difficulty. Another form of the scissor-beam truss is shown in Fig. 96, and its diagram presents no difficulty.

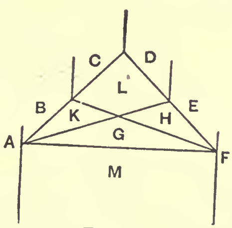


FIG. 96.

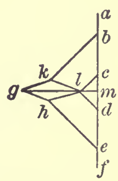


FIG. 96*a*.

The only peculiarity to be noticed is, that, after having constructed the polygon of external forces,

abcdefma,

we cannot proceed to construct the polygon of equilibrium for one of the supports, because there are three unknown forces there. We therefore begin at the apex *CD*, and construct the triangle of forces *cdl* for this point; then proceed to joint *CB*, and construct the quadrilateral

bclkb;

then proceed to the left-hand support, and obtain

mabkgm;

and so continue.

§ 141. **Scissor-Beam Truss without Horizontal Tie.** — Very often the scissor-beam truss is constructed without any horizontal tie, in which case it has the appearance of Fig. 97, where there is sometimes a pin at *GKLH* and sometimes not.

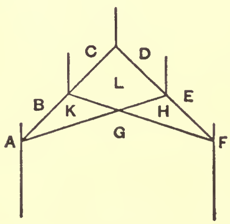


FIG. 97.

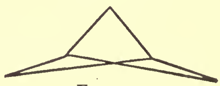


FIG. 97*a*.

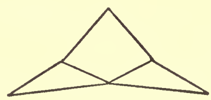


FIG. 97*b*.

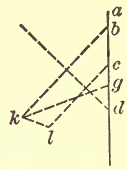


FIG. 97*c*.

In this case, if the abutments are capable of furnishing horizontal thrust to take the place of the horizontal tie of Fig. 96, we are reduced back to that case. If the abutments are not capable of furnishing horizontal thrust, we are then relying on the stiffness of the rafters to prevent the deformation of the truss; for, were the points *BC* and *DE* really joints, with pins, the deformation would take place, as shown in Fig. 97*a* or Fig. 97*b*, according as the two inclined ties were each made in one piece or in two (i.e., according as they are not pinned together at *KH*, or as they are pinned). This necessity of depending on the stiffness of the rafters, and the liability to deformation if they had joints at their middle points, become apparent as soon as we attempt to draw the diagram. Such an attempt is made in Fig. 97*c*, where *abcdefga* is the polygon of external forces, *gabkg* the polygon of stresses for the left-hand support, *kbclk* that for joint *BC*. Then, on proceeding to draw the triangle of stresses for the vertex, we find that the line joining *d* and *l* is not parallel to *DL*, and hence that the truss is not stable. We ought, however, in this latter case, when the supporting forces are vertical, and when we rely upon the stiffness of the rafters to prevent deformation, to be able to determine the direct stresses in the bars; and for this we will employ an analytical instead of a graphical method, as being the most convenient in this case.

Let us assume that there is no pin at the intersection of the two ties, and that the two rafters are inclined at an angle of 45° to the horizon.

We then have, if $W =$ the entire load, and $\alpha =$ angle between BK and KG,

$$ab = ef = \frac{W}{8}, \quad bc = cd = de = \frac{W}{4},$$

$$\tan\alpha = \tfrac{1}{2}, \quad \sin\alpha = \frac{1}{\sqrt{5}}, \quad \cos\alpha = \frac{2}{\sqrt{5}},$$

Let x be the stress in each tie, and let $y = cl = dl =$ thrust in each upper half of the rafters.

Then we must observe that the rafter has, in addition to its direct stresses, a tendency to bend, due to a normal load at the middle, this normal load being equal to the sum of the normal components of bc and of x, when these are resolved along and normal to the rafter. Hence

$$\text{normal load} = x\cos\alpha + \frac{W}{4}\sin 45^\circ.$$

This, resolved into components acting at each end of the rafter, gives a normal downward force at each end equal to

$$\tfrac{1}{2}x\cos\alpha + \tfrac{1}{8}W\sin 45^\circ.$$

Hence, resolving all the forces acting at the left-hand support into components along and at right angles to the rafter, and imposing the condition of equilibrium that the algebraic sum of their normal components shall equal zero, we have, if we call upward forces positive,

$$\tfrac{3}{8}W\sin 45^\circ - (\tfrac{1}{2}x\cos\alpha + \tfrac{1}{8}W\sin 45^\circ) - x\sin\alpha = 0; \quad (1)$$

but, since

$$x\cos\alpha = 2x\sin\alpha,$$

we have from (1)

$$2x\sin\alpha = \frac{W}{4}\sin 45^\circ$$

$$\therefore \quad x\sin\alpha = \frac{W}{8}\sin 45^\circ$$

$$\therefore \quad x = \tfrac{1}{8}W\frac{\sin 45^\circ}{\sin\alpha}. \quad (2)$$

Then, proceeding to the apex of the roof, we have that the load

$$cd = \frac{W}{4}$$

gives, when resolved along the two rafters, a stress in each equal to

$$\frac{W}{4}\sin 45^\circ.$$

Hence the load to be supported in a direction normal to the rafter at the apex is

$$\frac{W}{4}\sin 45^\circ + (\tfrac{1}{2}x\cos\alpha + \frac{W}{8}\sin 45^\circ).$$

Hence, substituting for x its value, we have

$$y = cl = dl = \frac{W}{2}\sin 45^\circ. \quad (3)$$

Then, proceeding to the left-hand support, and equating to zero the algebraic sum of the components along the rafter, we have

$$\begin{aligned} bk &= (ga - ab)\cos 45^\circ + x\cos\alpha \\ &= \tfrac{3}{8}W\sin 45^\circ + \tfrac{1}{4}W\sin 45^\circ = \tfrac{5}{8}W\sin 45^\circ. \quad (4) \end{aligned}$$

We have thus determined in (2), (3), and (4) the values of x, y, and $bk = eh$.

By way of verification, proceed to the middle of the left-hand rafter, and we find the algebraic sum of the components of bc and x along the rafter to be

$$\tfrac{1}{4} W \cos 45^\circ - x \sin \alpha = \tfrac{1}{8} W \sin 45^\circ ;$$

and this is the difference between bk and cl, as it should be.

We have thus obtained the direct stresses; and we have, in addition, that the rafter itself is also subjected to a bending-moment from a normal load at the centre, this load being equal to

$$x \cos \alpha + \frac{W}{4} \sin 45^\circ = \frac{W}{2} \sin 45^\circ .$$

How to take this into account will be explained under the "Theory of Beams."

§ 142. **Examples.** — The following figures of roof-trusses may be considered as a set of examples, for which the stress diagrams are to be worked out.

Observe, that, wherever there is a joint, the truss is to be supposed perfectly flexible, i.e., free to turn around a pin.

FIG. 98.

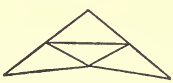


FIG. 99.

FIG. 100.

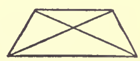


FIG. 101.

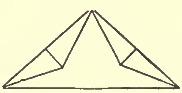


FIG. 102.

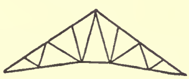


FIG. 103.

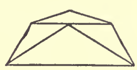


FIG. 104.

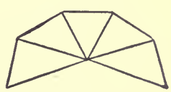


FIG. 105.

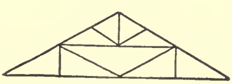


FIG. 106.

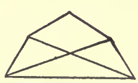


FIG. 107.

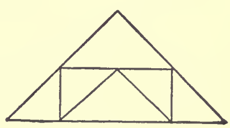


FIG. 108

CHAPTER IV.

BRIDGE-TRUSSES.

§ 143. **Method of Sections.** — It is perfectly possible to determine the stresses in the members of a bridge-truss graphically, or by any methods that are used for roof-trusses.

In this work an analytical method will be used; i.e., a method of sections. This method involves the use of the analytical conditions of equilibrium for forces in a plane explained in § 63. These are as follows; viz., —

If a set of forces in a plane, which are in equilibrium, be resolved into components in two directions at right angles to each other, then —

1°. The algebraic sum of the components in one of these directions must be zero.

2°. The algebraic sum of the components in the other of these directions must be zero.

3°. The algebraic sum of the moments of the forces about any axis perpendicular to the plane of the forces must be zero.

Assume, now, a bridge-truss (Figs. 109, 110, 111, 112, pages 186 and 187) loaded at a part or all of the joints. Conceive a vertical section *ab* cutting the horizontal members 6–8 and 7–9 and the diagonal 7–8, and dividing the truss into two parts. Then the forces acting on either part must be in equilibrium, in other words, the external forces, loads, and supporting forces, acting on one part, must be balanced by the stresses in the members cut by the section; i.e., by the forces exerted by the other part of the truss on the part under consideration. Hence we must have the three following conditions; viz., —

1°. The algebraic sum of the vertical components of the above-mentioned forces must be zero.

2°. The algebraic sum of the horizontal components of these forces must be zero.

3°. The algebraic sum of the moments of these forces about any axis perpendicular to the plane of the truss must be zero.

§ 144. **Shearing-Force and Bending-Moment.** — Assuming all the loads and supporting forces to be vertical, we shall have the following as definitions.

The *Shearing-Force* at any section is the force with which the part of the girder on one side of the section tends to slide by the part on the other side.

In a girder free at one end, it is equal to the sum of the loads between the section and the free end.

In a girder supported at both ends, it is equal in magnitude to the difference between the supporting force at either end, and the sum of the loads between the section and that supporting force.

The *Bending-Moment* at any section is the resultant moment of the external forces acting on the part of the girder to one side of the section, tending to rotate that part of the girder around a horizontal axis lying in the plane of the section.

In a girder free at one end, it is equal to the sum of the moments of the loads between the section and the free end, about a horizontal axis in the section.

In a girder supported at both ends, it is the difference between the moment of either supporting force, and the sum of the moments of the loads between the section and that support; all the moments being taken about a horizontal axis in the section.

§ 145. **Use of Shearing-Force and Bending-Moment.** — The three conditions stated in § 143 may be expressed as follows: —

1°. The algebraic sum of the horizontal components of the stresses in the members cut by the section must be zero.

2°. The algebraic sum of the vertical components of the stresses in the members cut by the section must balance the shearing-force.

3°. The algebraic sum of the moments of the stresses in the members cut by the section, about any axis perpendicular to the plane of the truss, and lying in the plane of the section, must balance the bending-moment at the section.

As the conditions of equilibrium are three in number, they will enable us to determine the stresses in the members, provided the section does not cut more than three; and this determination will require the solution of three simultaneous equations of the first degree with three unknown quantities (the stresses in the three members).

By a little care, however, in choosing the section, we can very much simplify the operations, and reduce our work to the solution of one equation with only one unknown quantity; the proper choice of the section taking the place of the elimination.

§ 146. **Examples of Bridge-Trusses.** — Figs. 109–112 represent two common kinds of bridge-trusses: in the first two the braces are all diagonal, in the last two they are partly vertical and partly diagonal.

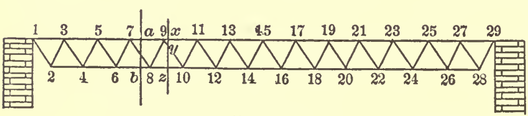


Fig. 109.

The first two are called Warren girders, or half-lattice girders; since there is only one system of bracing, as in the figures. When, on the other hand, there are more than one system, so that the diagonals cross each other, they are called lattice girders.

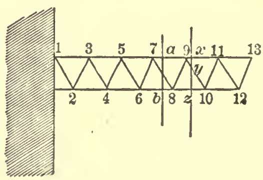


Fig. 110.

§ 147. **General Outline of the Steps to be taken in determining the Stresses in a Bridge-Truss under a Fixed Load.**

1°. If the truss is supported at both ends, find the supporting forces.

2°. Assume, in all cases, a section, in such a manner as not to cut more than three members if possible, or, rather, three of those that are brought into action by the loads on the truss; and it will save labor if we assume the section so as to cut two of the three very near their point of intersection.

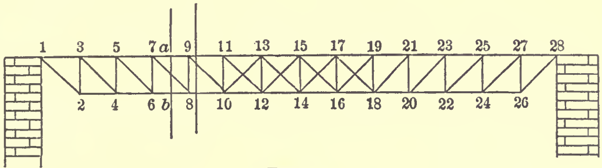


FIG. 111.

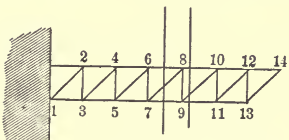


FIG. 112.

3°. Find the shearing-force at the section.

4°. Find the bending-moment at the section.

5°. Impose the analytical conditions of equilibrium on *all* the forces acting on the part of the girder to one side of the section, — the part between the section and the free end when the girder is free at one end, or either part when it is supported at both ends.

In the cases shown in Figs. 109 and 110, we may describe the process as follows; viz., —

(*a*) Find the stress in the diagonal from the fact, that (since the stress in the diagonal is the only one that has a vertical component at the section) the vertical component of the stress in the diagonal must balance the shearing-force.

(*b*) Take moments about the point of intersection of the diagonal and horizontal chord near which the section is taken; then the stresses in those members will have no moment, so that the moment of the stress in the other horizontal must balance the bending-moment at the section. Hence the stress in the horizontal will be found by dividing the bending-moment at the section by the height of the girder.

The above will be best illustrated by some examples.

EXAMPLE I. — Given the semi-girder shown in Fig. 110, loaded at joint 13 with 4000 pounds, and at each of the joints 1, 3, 5, 7, 9, and 11 with 8000 pounds. Suppose the length of each chord and each diagonal to be 5 feet. Required the stress in each member.

Solution. — For the purpose of explaining the method of procedure, we will suppose that we desire to find first the stresses in 8–10 and 9–10.

Assume a vertical section very near the joint 9, but to the right of it, so that it shall cut both 8–10 and 9–10.

If, now, the truss were actually separated into two parts at this section, the right-hand part would, in consequence of the loads acting on it, separate from the other part. This tendency to separate is counteracted by the following three forces: —

1°. The pull exerted by the part 9–x of the bar 9–11 on the part x–11 of the same bar.

2°. The thrust exerted by the part 8–z of the bar 8–10 on the part z–10 of the same bar.

3°. The pull exerted by the part 9–y of the bar 9–10 on the part y–10 of the same bar.

The shearing-force at this section is

$$8000 + 4000 = 12000 \text{ lbs.},$$

and this is equal to the vertical component of the stress in the diagonal. Hence

$$\text{Stress in } 9\text{–}10 = \frac{12000}{\sin 60^\circ} = 12000(1.1547) = 13856 \text{ lbs.}$$

This stress is a pull, as may be seen from the fact, that, in order to prevent the part of the girder to the right of the section from sliding downwards under the action of the load, the part 9–y of the diagonal 9–10 must pull the part y–10 of the same diagonal.

Next take moments about 9: and, since the moment of the stresses in 9–11 and 9–10 about 9 is zero, we must have that the moment of the stress in 8–10; i.e., the product of this stress by the height of the girder, must equal the bending-moment.

The bending-moment about 9 is

$$8000 \times 5 + 4000 \times 10 = 80000 \text{ foot-lbs.}$$

Hence

$$\text{Stress in 8–10} = \frac{80000}{4.33} = 80000(0.23094) = 18475 \text{ lbs.}$$

Proceed in a similar way for all the other members. The work may be arranged as in the following table; the diagonal stresses being deduced from the shearing-forces by multiplying by 1.1547, and the chord stresses from the bending-moments by multiplying by 0.23094.

Section just to the right of	Shearing-Force in lbs.	Stresses in Diagonals cut by Section, in lbs.		Bending-Moment, in foot-lbs.	Stresses in Chords opposite the respective Joints.	
		Tension.	Compression.		Tension.	Compression.
1	44000	50806		720000		166277
2	44000		50806	610000	140873	
3	36000	41569		500000		115470
4	36000		41569	410000	94685	
5	28000	32331		320000		73901
6	28000		32331	250000	57735	
7	20000	23094		180000		41569
8	20000		23094	130000	30022	
9	12000	13856		80000		18475
10	12000		13856	50000	11547	
11	4000	4619		20000		4618
12	4000		4619	10000	2309	

EXAMPLE II. — Given the truss (Fig. 109) loaded at each of the lower joints with 10000 lbs.: find the stresses in the members. The length of chord is equal to the length of diagonal = 10 ft.

Throughout this chapter, tensions will be written with the minus, and compressions with the plus sign.

Solution.—Total load = 14(10000) = 140000 lbs.

Each supporting force = 70000 "

The entire work is shown in the following tables:—

Section taken just to the Right of	Section taken just to the Left of	Shearing-Force at Section, in lbs.	Bending-Moment about Joint, in foot-lbs.
1	29	70000 = 70000	0
2	28	70000 − 10000 = 60000	70000 × 5 = 350000
3	27	70000 − 10000 = 60000	70000 × 10 − 10000 × 5 = 650000
4	26	70000 − 20000 = 50000	70000 × 15 − 10000 × 10 = 950000
5	25	70000 − 20000 = 50000	70000 × 20 − 10000 (5 + 15) = 1200000
6	24	70000 − 30000 = 40000	70000 × 25 − 10000 (10 + 20) = 1450000
7	23	70000 − 30000 = 40000	70000 × 30 − 10000 (5 + 15 + 25) = 1650000
8	22	70000 − 40000 = 30000	70000 × 35 − 10000 (10 + 20 + 30) = 1850000
9	21	70000 − 40000 = 30000	70000 × 40 − 10000 (5 + 15 + 25 + 35) = 2000000
10	20	70000 − 50000 = 20000	70000 × 45 − 10000 (10 + 20 + 30 + 40) = 2150000
11	19	70000 − 50000 = 20000	70000 × 50 − 10000 (5 + 15 + 25 + 35 + 45) = 2250000
12	18	70000 − 60000 = 10000	70000 × 55 − 10000 (10 + 20 + 30 + 40 + 50) = 2350000
13	17	70000 − 60000 = 10000	70000 × 60 − 10000 (5 + 15 + 25 + 35 + 45 + 55) = 2400000
14	16	70000 − 70000 = 0	70000 × 65 − 10000 (10 + 20 + 30 + 40 + 50 + 60) = 2450000
15	-	70000 − 70000 = 0	70000 × 70 − 10000 (5 + 15 + 25 + 35 + 45 + 55 + 65) = 2450000

Numbers of Diagonals.		Stresses in Diagonals, in lbs.
1- 2	28-29	$-70000 \times 1.1547 = -80829$
2- 3	27-28	$+60000 \times 1.1547 = +69282$
3- 4	26-27	$-60000 \times 1.1547 = -69282$
4- 5	25-26	$+50000 \times 1.1547 = +57735$
5- 6	24-25	$-50000 \times 1.1547 = -57735$
6- 7	23-24	$+40000 \times 1.1547 = +46188$
7- 8	22-23	$-40000 \times 1.1547 = -46188$
8- 9	21-22	$+30000 \times 1.1547 = +34641$
9-10	20-21	$-30000 \times 1.1547 = -34641$
10-11	19-20	$+20000 \times 1.1547 = +23094$
11-12	18-19	$-20000 \times 1.1547 = -23094$
12-13	17-18	$+10000 \times 1.1547 = +11547$
13-14	16-17	$-10000 \times 1.1547 = -11547$
14-15	15-16	$+0$ 0

LOWER CHORDS.

Numbers of Chords.		Stresses in Chords, in lbs.
2- 4	26-28	$-\ 650000 \times 0.11547 = -\ 75056$
4- 6	24-26	$-\ 1200000 \times 0.11547 = -138564$
6- 8	22-24	$-\ 1650000 \times 0.11547 = -190526$
8-10	20-22	$-\ 2000000 \times 0.11547 = -230940$
10-12	18-20	$-\ 2250000 \times 0.11547 = -259808$
12-14	16-18	$-\ 2450000 \times 0.11547 = -277128$
14-16		$-\ 2450000 \times 0.11547 = -282902$

UPPER CHORDS.

Numbers of Chords.		Stresses in Chords, in lbs.
1– 3	27–29	$350000 \times 0.11547 = +40415$
3– 5	25–27	$950000 \times 0.11547 = +109697$
5– 7	23–25	$1450000 \times 0.11547 = +167432$
7– 9	21–23	$1850000 \times 0.11547 = +213620$
9–11	19–21	$2150000 \times 0.11547 = +248261$
11–13	17–19	$2350000 \times 0.11547 = +267355$
13–15	15–17	$2450000 \times 0.11547 = +282902$

EXAMPLE III. — Given the same truss as in Example II., loaded at 2, 4, 6, 8, 10, and 12 with 10000 lbs. at each point, the remaining lower joints being loaded with 50000 lbs. at each joint: find the stresses in the members.

EXAMPLE IV. — Given a semi-girder, free at one end (Fig. 112), loaded at 2, 4, and 6 with 10000 lbs., and at 8, 10, and 12 with 5000 lbs.: find the stresses in the members.

TRAVELLING-LOAD.

§ 148. **Half-Lattice Girder: Travelling-Load.** — When a girder is used for a bridge, it is not subjected all the time to the same set of loads.

The load in this case consists of two parts, — one, the dead load, including the bridge weight, together with any permanent load that may rest upon the bridge; and the other, the moving or variable load, also called the travelling-load, such as the weight of the whole or part of a railroad train if it is a railroad bridge, or the weight of the passing teams, etc., if it is a common-road bridge. Hence it is necessary that we should be able to determine the amount and distribution of the loads upon the bridge which will produce the greatest tension or the greatest compression in every member, and the consequent stress produced.

§ 149. **Greatest Stresses in Semi-Girder.** — Wherever the section be assumed in a semi-girder, it is evident that any load placed on the truss at any point between the section and the free end increases both the shearing-force and the bending-moment at that section, and that any load placed between the section and the fixed end has no effect whatever on either the shearing-force or the bending-moment at that section.

Hence every member of a semi-girder will have a greater stress upon it when the entire load is on, than with any partial load.

§ 150. **Greatest Chord Stresses in Girder supported at Both Ends.** — Every load which is placed upon the truss, no matter where it is placed, will produce at any section whatever a *bending-moment* tending to turn the two parts of the truss on the two sides of the section upwards from the supports; i.e., so as to render the truss concave upwards.

Hence every load that is placed upon the truss causes compression in every horizontal upper chord, and tension in every horizontal lower chord. Hence, in order to obtain the greatest chord stresses, we assume the whole of the moving load to be upon the bridge.

§ 151. **Greatest Diagonal Stresses in Girder supported at Both Ends.** — To determine the distribution of the load that will produce the greatest stress of a certain kind (tension or compression) in any given diagonal, let us suppose the diagonal in question to be 7–8 (Fig. 109), through which we take our section *ab*. Now it is evident that any load placed on the truss between *ab* and the left-hand (nearer) support will cause a shearing-force at that section which will tend to slide the part of the girder to the left of the section downwards with reference to the other part, and hence will cause a compressive stress in 7–8; while any load between the section and the right-hand (farther) support will cause a shearing-force of the opposite kind, and hence a tension in the bar 7–8.

Now, the bridge weight itself brings an equal load upon each joint; hence, when the bridge weight is the only load upon the truss, the bar 7–8 is in tension.

Hence, any load placed upon the truss between the section and the farther support tends to increase the shearing-force at that section due to the dead load (provided this is equally distributed among the joints); whereas any load placed between the section and the nearer support tends to decrease the shearing-force at the section due to the dead load, or to produce a shearing-force of the opposite kind to that produced by the dead load at that section.

Hence, if we assume the dead load to be equally distributed among the joints, we shall have the two following propositions true: —

(*a*) In order to determine the greatest stress in any diagonal which is of the same kind as that produced by the dead load, we must assume the moving load to cover all the panel points between the section and the farther abutment, and no other panel points.

(*b*) In order to determine the greatest stress in any diagonal of the opposite kind to that produced by the dead load, we must assume the moving load to cover all the panel points between the section and the nearer abutment, and no others.

This will be made clear by an example.

EXAMPLE I. — Given the truss shown in Fig. 113. Length of chord = length of diagonal = 10 feet. Dead load = 8000 lbs. applied at each upper panel point. Moving load = 30000 lbs. applied at each upper panel point. Find the greatest stresses in the members.

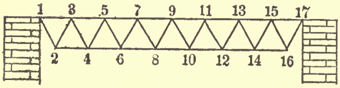


FIG. 113.

Solution. (*a*) *Chord Stresses.* — Assume the whole load to be upon the bridge: this will give 38000 lbs. at each upper panel point; i.e., omitting 1 and 17, where the load acts directly on the support, and not on the truss.

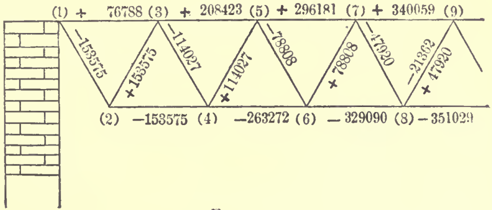


Fig. 114.

Hence, considering the bridge so loaded, we shall have the following results for the chord stresses: —

$$\text{Each supporting force} = 38000\left(\frac{7}{2}\right) = 133000.$$

Section at		Bending-Moment, in foot-lbs.
2	16	$133000 \times 5 = 665000$
3	15	$133000 \times 10 = 1330000$
4	14	$133000 \times 15 - 38000 \times 5 = 1805000$
5	13	$133000 \times 20 - 38000 \times 10 = 2280000$
6	12	$133000 \times 25 - 38000(5 + 15) = 2565000$
7	11	$133000 \times 30 - 38000(10 + 20) = 2850000$
8	10	$133000 \times 35 - 38000(5 + 15 + 25) = 2945000$
9		$133000 \times 40 - 38000(10 + 20 + 30) = 3040000$

Numbers of Chords.		Stresses in Upper Chords.
1–3	15–17	$665000 \times 0.11547 = +76788$
3–5	13–15	$1805000 \times 0.11547 = +208423$
5–7	11–13	$2565000 \times 0.11547 = +296181$
7–9	9–11	$2945000 \times 0.11547 = +340059$

Numbers of Chords.		Stresses in Lower Chords.
2– 4	14–16	$-1330000 \times 0.11547 = -153575$
4– 6	12–14	$-2280000 \times 0.11547 = -263272$
6– 8	10–12	$-2850000 \times 0.11547 = -329090$
8–10		$-3040000 \times 0.11547 = -351029$

Next, as to the diagonals, take, for instance, the diagonal 7–8. When the dead load alone is on the bridge, the diagonal 7–8 is in tension. From the preceding, we see that the greatest tension is produced in this bar when the moving load is on the points 9, 11, 13, and 15, and the dead load only on the points 3, 5, 7. Now, a load of 38000 lbs. at 13, for instance, causes a shearing-force of $\frac{4}{16}(38000) = 9500$ lbs. at any section to the left of 13; and this shearing-force tends to cause the part to the left of the section to slide upwards, and that to the right downwards.

On the other hand, with the same load at the same place, there is produced a shearing-force of $\frac{12}{16}(38000) = 28500$ lbs. at any section to the right of 13; and this shearing-force tends to cause the part to the left to slide downwards, and that to the right upwards. Paying attention to this fact, we shall have, when the loads are distributed as above described, a shearing-force at the bar 7–8 causing tension in this bar; the magnitude of this shearing-force being

$$\frac{38000}{16}(2 + 4 + 6 + 8) - \frac{8000}{16}(2 + 4 + 6) = 41500.$$

Hence, we may arrange the work as follows:—

Numbers of Diagonals.		Greatest Shearing-Forces producing Stresses of Same Kind as Dead Load.	Greatest Stresses in Diagonals of Same Kind as those due to Dead Load.
1–2	17–16	$\frac{38000}{16}(2+4+6+8+10+12+14) = 133000$	-153575
2–3	16–15	$\frac{38000}{16}(2+4+6+8+10+12+14) = 133000$	$+153575$
3–4	15–14	$\frac{38000}{16}(2+4+6+8+10+12) - \frac{8000}{16}(2) = 98750$	-114027
4–5	14–13	$\frac{38000}{16}(2+4+6+8+10+12) - \frac{8000}{16}(2) = 98750$	$+114027$
5–6	13–12	$\frac{38000}{16}(2+4+6+8+10) - \frac{8000}{16}(2+4) = 68250$	-78808
6–7	12–11	$\frac{38000}{16}(2+4+6+8+10) - \frac{8000}{16}(2+4) = 68250$	$+78808$
7–8	11–10	$\frac{38000}{16}(2+4+6+8) - \frac{8000}{16}(2+4+6) = 41500$	-47920
8–9	10–9	$\frac{38000}{16}(2+4+6+8) - \frac{8000}{16}(2+4+6) = 41500$	$+47920$

Numbers of Diagonals.		Greatest Shearing-Forces producing Stresses of Kind Opposite from Dead Load.	Greatest Stresses in Diagonals of Kind Opposite from Dead Load.
8–9	10–9	$\frac{38000}{16}(2+4+6) - \frac{8000}{16}(2+4+6+8) = 18500$	-21362
7–8	11–10	$\frac{38000}{16}(2+4+6) - \frac{8000}{16}(2+4+6+8) = 18500$	$+21362$

The diagonals 7–8, 8–9, 9–10, and 10–11 are the only ones that, under any circumstances, can have a stress of the kind opposite to that to which they are subjected under the dead load alone.

Fig. 114 exhibits the manner of writing the stresses on the diagram.

§ 152. **General Application of this Method.** — It is plain that the method used above will apply to any single system of bridge-truss with horizontal chords and diagonal bracing, whatever be the inclination of the braces.

When seeking the stress in a diagonal, the section must be so taken as to cut that diagonal; and, as far as this stress alone is concerned, it may be equally well taken at any point, as well as near a joint, provided only it cuts that diagonal which is in action under the load that produces the greatest stress in this one, and no other.

On the other hand, when we seek the stress in a horizontal chord, the section might very properly be taken through the joint opposite that chord.

Taking it very near the joint, only serves to make one section answer both purposes simultaneously.

§ 153. **Bridge-Trusses with Vertical and Diagonal Bracing.** — When, as in Figs. 111 and 112, there are both vertical and diagonal braces, and also horizontal chords, we may determine the stresses in the diagonals and in the chords just as before; only we must take the section just to one side of a joint, and never through the joint.

As to the verticals, in order to determine the stress in any vertical, we must impose the conditions of equilibrium between the vertical components of the forces acting at one end of that vertical: thus, if the loads are at the upper joints in Fig. 111, then the stress in vertical 3–2 must be equal and opposite to the vertical component of the stress in diagonal 1–2, as these stresses are the only vertical forces acting at joint 2.

Vertical 5–4 has for its stress the vertical component of the stress in 3–4, etc. Thus

Stress in 3–2 = shearing-force in panel 1–3,
Stress in 5–4 = shearing-force in panel 3–5, etc.

On the other hand, if the loads be applied at the lower joints, then

Stress in 3–2 = shearing-force in panel 3–5,
Stress in 5–4 = shearing-force in panel 5–7, etc.

EXAMPLE. — Given the truss shown in Fig. 111. Given panel length = height of truss = 10 feet, dead load per panel point = 12000 lbs., moving load per panel point = 23000 lbs.; load applied at upper joints.

Solution. (*a*) *Chord Stresses.* — Assume the entire load on the bridge, i.e., 35000 lbs. per panel point. Hence

Total load on truss = 13 (35000) = 455000 lbs.,
Each supporting force = 227500 lbs.

Joint near which Section is taken.		Bending-Moment at the Section very near the Joint, on Either Side of the Joint.	
1	28	0	
3	27	227500 × 10	= 2275000
5	25	227500 × 20 − 35000 × 10	= 4200000
7	23	227500 × 30 − 35000(10 + 20)	= 5775000
9	21	227500 × 40 − 35000(10 + 20 + 30)	= 7000000
11	19	227500 × 50 − 35000(10 + 20 + 30 + 40)	= 7875000
13	17	227500 × 60 − 35000(10 + 20 + 30 + 40 + 50)	= 8400000
15	–	227500 × 70 − 35000(10 + 20 + 30 + 40 + 50 + 60)	= 8575000

To find any chord stress, divide the bending-moment at a section cutting the chord, and passing close to the opposite joint, by the height of the girder, which in this case is 10. Hence we have for the chord stresses (denoting, as before, compression by +, and tension by −): —

Stresses in Upper Chords.			Stresses in Lower Chords.		
1– 3	27–28	+227500	2– 4	24–26	−227500
3– 5	25–27	+420000	4– 6	22–24	−420000
5– 7	23–25	+577500	6– 8	20–22	−577500
7– 9	21–23	+700000	8–10	18–20	−700000
9–11	19–21	+787500	10–12	16–18	−787500
11–13	17–19	+840000	12–14	14–16	−840000
13–15	15–17	+857500			

Diagonals. — It is evident, that, for the diagonals, the same rule holds as in the case of the Warren girder: i.e., the greatest stress of the same kind as that produced by the dead load occurs when the moving load is on all the joints between the diagonal in question and the farther abutment; whereas the greatest stress of the opposite kind occurs when the moving load covers all the joints between the diagonal in question and the nearer abutment.

The work of determining the greatest shearing-forces may be arranged as in tables on p. 191.

Counterbraces. — If the truss were constructed with those diagonals only that slope downwards towards the centre, and which may be called the main braces, the diagonals 11–12, 13–14, 14–17, and 16–19 would sometimes be called upon to bear a thrust, and the verticals 12–13 and 17–16 a pull: this would necessitate making these diagonals sufficiently strong to resist the greatest thrust to which they are liable, and fixing the verticals in such a way as to enable them to bear a pull.

In order to avoid this, the diagonals 10–13, 12–15, 15–16, and 17–18 are inserted, which are called counterbraces, and which come into action only when the corresponding main braces would otherwise be subjected to thrust. They also prevent any tension in the verticals.

Diagonals.		Greatest Shearing-Forces of the Same Kind as those produced by Dead Load.
1- 2	28-26	$\frac{35000}{14}(1+2+3+\ldots+13) = 227500$
3- 4	27-24	$\frac{35000}{14}(1+2+3+\ldots+12)-\frac{12000}{14}(1) = 194143$
5- 6	25-22	$\frac{35000}{14}(1+2+3+\ldots+11)-\frac{12000}{14}(1+2) = 162429$
7- 8	23-20	$\frac{35000}{14}(1+2+3+\ldots+10)-\frac{12000}{14}(1+2+3) = 132357$
9-10	21-18	$\frac{35000}{14}(1+2+3+\ldots+9)-\frac{12000}{14}(1+2+\ldots+4) = 103929$
11-12	19-16	$\frac{35000}{14}(1+2+3+\ldots+8)-\frac{12000}{14}(1+2+\ldots+5) = 77143$
13-14	17-14	$\frac{35000}{14}(1+2+3+\ldots+7)-\frac{12000}{14}(1+2+\ldots+6) = 52000$

Diagonals.		Greatest Shearing-Forces of the Opposite Kind to those produced by Dead Load.
13-14	17-14	$\frac{35000}{14}(1+2+3+\ldots+6)-\frac{12000}{14}(1+2+\ldots+7) = 28500$
11-12	19-16	$\frac{35000}{14}(1+2+\ldots+5)-\frac{12000}{14}(1+2+\ldots+8) = 6643$
9-10	21-18	$\frac{35000}{14}(1+2+\ldots+4)-\frac{12000}{14}(1+2+\ldots+9) = -13571$

The main braces and counterbraces of a panel are never in action simultaneously. Hence we have, for the greatest stresses in the diagonals, the following results, obtained by multiplying the corresponding shearing-forces by $\frac{1}{\cos 45^\circ} = 1.414$.

In the following I have used this number to three decimal places, as being sufficiently accurate for practical purposes.

Stresses in Main Braces.			Stresses in Counterbraces.		
1– 2	28–26	–321685	15–12	15–16	–40299
3– 4	27–24	–274518	13–10	17–18	– 9393
5– 6	25–22	–229675			
7– 8	23–20	–187153			
9–10	21–18	–146956			
11–12	19–16	–109080			
13–14	17–14	– 73528			

Vertical Posts. — Since the loads are applied at the upper joints, the conditions of equilibrium at the lower joints require that the thrust in any vertical post shall be equal to the vertical component of the tension in that diagonal which, being in action at the time, meets it at its lower end.

Hence it is equal to the shearing-force in that panel where the acting diagonal meets it at its lower end.

We therefore have, for the posts, the following as the greatest thrusts: —

STRESSES IN VERTICALS.

3– 2	27–26	+227500
5– 4	25–24	+194143
7– 6	23–22	+162429
9– 8	21–20	+132357
11–10	19–18	+103929
13–12	17–16	+ 77143
15–14		+ 52000

Fig. 115 shows the stresses marked on the diagram.

§ 154. **Manner of Concentrating the Load at the Joints.** — In using the methods given above, we are assuming that all the loads are concentrated at the joints, and that none are distributed over any of the pieces. As far as the moving load is concerned, and also all of the dead load except the weight of the truss itself, this always is, or ought to be, effected; and it is accomplished in a manner similar to that adopted in the case of roof-trusses. This method is shown in the figure (Fig. 116); floor-beams being laid across from girder to girder at the joints, on top of which are laid longitudinal beams, and on these the sleepers if it is a railroad bridge, or the floor if it is a road bridge. The weight of the *truss* itself is so small a part of what the bridge is called upon to bear, that it can, without appreciable error, be considered as concentrated at the joints either of the upper chord, of the lower chord, or of both, according to the manner in which the rest of the load is distributed.

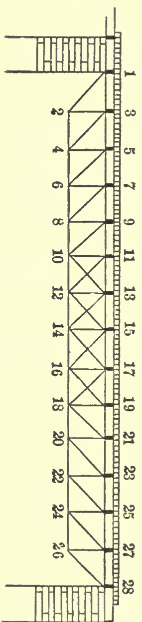


Fig. 116.

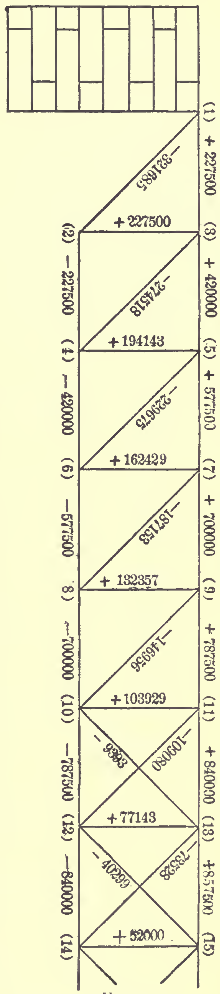


Fig. 115.

§ 155. **Closer Approximation to Actual Shearing-Force.** — In our computations of greatest shearing-force, we make an approximation which is generally considered to be sufficiently close, and which is always on the safe side. To illustrate it, take the case of panel 3–5 of the last example. In determining its greatest shearing-force, we considered a load of 35000 lbs. per panel point to rest on all the joints from the right-hand support to joint 5, inclusive, and the dead load to rest on all the other joints of the truss. Now, it is impossible, if the load is distributed uniformly on the floor of the bridge, to have a load of 35000 lbs. on 5 and 12000 on 3 simultaneously; for, if the moving load extended on the bridge floor only up to 5, the load on 5 would be only $12000 + \frac{1}{2}(23000) = 23500$ lbs., and that on 3 would then be 12000 lbs. If, on the other hand, the moving load extends beyond 5 at all, as it must if the load on 5 is to be greater than 23500 lbs., then part of it will rest on 3, and the load on 3 will then be greater than 12000 lbs.; for whatever load there is between 3 and 5 is supported at 3 and 5.

Moreover, we know that the effect of increasing the load on 5 is to increase the shearing-force, provided we do not at the same time increase that on 3 so much as to destroy the effect of increasing that on 5.

Hence, there must be some point between 3 and 5 to which the moving load must extend in order to render the shearing-force in panel 3–5 a maximum.

Let the distance of this point from 5 be x; then, if we let

$w = \frac{23000}{10} =$ moving load per foot of length,

$$\text{Moving load on panel} = wx,$$

$$\text{Part supported at 3} = \frac{wx^2}{20},$$

$$\text{Part supported at 5} = wx - \frac{wx^2}{20}.$$

Hence, portion of shearing-force due to the moving load on panel 3–5 equals

$$\frac{12}{14}\left(wx - \frac{wx^2}{20}\right) - \frac{1}{14}\frac{wx^2}{20} = \frac{w}{14}\left(12x - \frac{13x^2}{20}\right).$$

This becomes a maximum when its first differential co-efficient becomes zero, i.e., when

$$12 - \tfrac{13}{10}x = 0;$$

therefore

$$x = 9'.23.$$

Hence, when the moving load extends to a distance of 9.23 feet from 5, then the shearing-force in panel 3–5, and hence the stress in diagonal 3–4, is a maximum.

Panels.		Portion of Shearing-Force due to Moving Load on Panel.	Value of x, in feet.	Portion of Load at Joints named below.		Portion of Load at Joints named below.	
1– 3	27–28	$\frac{w}{14}\left(13x - \frac{13x^2}{20}\right)$	10.00	1	11500	3	11500
3– 5	25–27	$\frac{w}{14}\left(12x - \frac{13x^2}{20}\right)$	9.23	3	9797	5	11432
5– 7	23–25	$\frac{w}{14}\left(11x - \frac{13x^2}{20}\right)$	8.46	5	8230	7	11227
7– 9	21–23	$\frac{w}{14}\left(10x - \frac{13x^2}{20}\right)$	7.69	7	6801	9	10886
9–11	19–21	$\frac{w}{14}\left(9x - \frac{13x^2}{20}\right)$	6.92	9	5507	11	10409
11–13	17–19	$\frac{w}{14}\left(8x - \frac{13x^2}{20}\right)$	6.15	11	4350	13	9795
13–15	15–17	$\frac{w}{14}\left(7x - \frac{13x^2}{20}\right)$	5.38	13	3329	15	9045

To show how the adoption of this method would affect the resulting stresses in the diagonals and verticals, I have given the work above, and shown the difference between these and the former results. In this table x = distance covered by load from end of panel nearest the centre.

Panels.		Greatest Shearing-Force of Same Kind as that due to Dead Load.
1- 3	27-28	$\frac{35000}{14}(1+\ldots+13) = 227500$
3- 5	25-27	$\frac{35000}{14}(1+\ldots+11)+\frac{12}{14}(34932)-\frac{1}{14}(21797) = 193385$
5- 7	23-25	$\frac{35000}{14}(1+\ldots+10)+\frac{11}{14}(34727)-\frac{2}{14}(20230)-\frac{1}{14}(12000) = 161038$
7- 9	21-23	$\frac{35000}{14}(1+\ldots+9)+\frac{10}{14}(34386)-\frac{3}{14}(18801)-\frac{12000}{14}(1+2) = 130461$
9-11	19-21	$\frac{35000}{14}(1+\ldots+8)+\frac{9}{14}(33909)-\frac{4}{14}(17507)-\frac{12000}{14}(1+2+3) = 101654$
11-13	17-19	$\frac{35000}{14}(1+\ldots+7)+\frac{8}{14}(33295)-\frac{5}{14}(16350)-\frac{12000}{14}(1+\ldots+4) = 74616$
13-15	15-17	$\frac{35000}{14}(1+\ldots+6)+\frac{7}{14}(32545)-\frac{6}{14}(15329)+\frac{12000}{14}(1+\ldots+5) = 49345$

Hence, for stresses in main braces, we have

Diagonals.		Stresses.
1- 2	28-26	−321685
3- 4	27-24	−273446
5- 6	25-22	−227708
7- 8	23-20	−184472
9-10	21-18	−143739
11-12	19-16	−105507
13-14	17-14	− 69774

Moreover, for the shearing-forces of opposite kind from those due to dead load, we have, if $x =$ distance from end of panel nearest support which is covered by moving load, —

Panels.		Portion of Shear due to Moving Load on Panel.	Value of x.	Portion of Load at Joints named below.		Portion of Load at Joints named below.	
13–15	17–15	$\frac{w}{14}\left(6x - \frac{13x^2}{20}\right)$	4.62	15	2455	13	8171
11–13	19–17	$\frac{w}{14}\left(5x - \frac{13x^2}{20}\right)$	3.84	13	1695	11	7137

Panels.		Greatest Shearing-Forces of Opposite Kind from those due to Dead Load.
13–15	17–15	$\frac{35000}{14}(1+\ldots+5)+\frac{6}{14}(31671)-\frac{7}{14}(14455)-\frac{12000}{14}(1+\ldots+6) = 25846$
11–13	19–17	$\frac{35000}{14}(1+\ldots+4)+\frac{5}{14}(30637)-\frac{8}{14}(13695)-\frac{12000}{14}(1+\ldots+7) = 4116$

Hence we have the following as the stresses in the counter-braces: —

Counter-Braces.		Stresses.
15–12	15–16	− 36546
13–10	17–18	− 5820

And, for the verticals, we have the new, instead of the old, shearing-forces.

The following table compares the results: —

Diagonals.		Stress, Ordinary Method.	Stress, New Method.	Difference.
1– 2	28–26	−321685	−321685	
3– 4	27–24	−274518	−273446	1072
5– 6	25–22	−229675	−227708	1967
7– 8	23–20	−187153	−184472	2681
9–10	21–18	−146956	−143739	3217
11–12	19–16	−109080	−105507	3573
13–14	17–14	− 73528	− 69774	3754
15–12	15–16	− 40299	−36546	3753
13–10	17–18	− 9393	− 5820	3573

Verticals.		Stress, Ordinary Method.	Stress, New Method.	Difference.
3– 2	27–26	+227500	+227500	0
5– 4	25–24	+194143	+193385	758
7– 6	23–22	+162429	+161038	1391
9– 8	21–20	+132357	+130461	1896
11–10	19–18	+103929	+101654	2275
13–12	17–16	+ 77143	+ 74616	2527
15–14		+ 28500	+ 49345	2655

§ 156. **Compound Bridge-Trusses.** — The trusses already discussed have contained but a single system of latticing, or at least only one system that comes in play at one time; so that a vertical section never cuts more than three bars that are in action simultaneously, the main brace having no stress upon it when the counterbrace is in action, and *vice versa.*

We may, however, have bridge-trusses with more than one system of lattices; and, in determining the stresses in their members, we must resolve them into their component systems, and determine the greatest stress in each system separately, and then, for bars which are common to the two systems, add together the stresses brought about by each.

In some cases, the design is such that it is possible to resolve the truss into systems in more than one way, and then there arises an uncertainty as to which course the stresses will actually pursue.

In such cases, the only safe way is to determine the greatest stress in each piece with every possible mode of resolution of the systems, and then to design each piece in such a way as to be able to resist that stress.

Generally, however, such ambiguity is an indication of a waste of material; as it is most economical to put in the bridge only those pieces that are absolutely necessary to bear the stresses, as other pieces only add so much weight to the structure, and are useless to bear the load.

The mode of proceeding can be best explained by some examples.

EXAMPLE I. — Given the lattice-girder shown in Fig. 117, loaded at the lower panel points only. Dead load = 7200 lbs. per panel point, moving load = 18000 lbs per panel point; let the entire length of bridge be 60 feet; let the angle made by braces with horizontal = 60°.

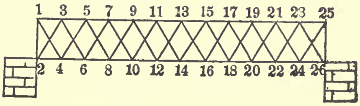


FIG. 117.

This truss evidently consists of the two single trusses shown in Figs. 117*a* *and* 117*b*; and we can compute the greatest stress of each kind in each member of these trusses, and thus obtain at once all the diagonal stresses, and then, by addition, the greatest chord stresses.

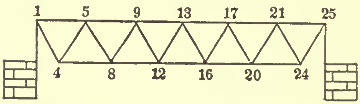


FIG. 117*a*.

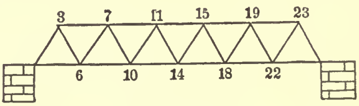


FIG. 117*b*.

Thus the stress in 1–3 (Fig. 117) is the same as the stress in 1–5 (Fig. 117*a*).

The stress in 3–5 = stress in 1–5 (Fig. 117*a*) + stress in 3–7 (Fig. 117*b*).

The stress in 5–7 = stress in 5–9 (Fig. 117*a*) + stress in 3–7 (Fig. 117*b*).

The results are given on the diagram (Fig. 117*c*); the work being left for the student, as it is similar to that done heretofore.

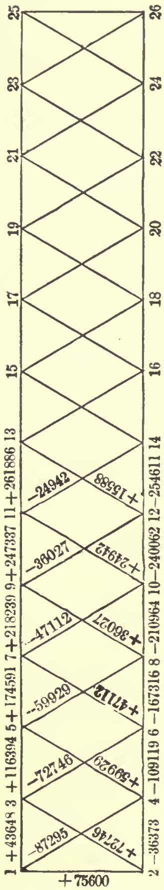


FIG. 117*c*.

EXAMPLE II. — Given the lattice-girder shown in Fig. 118. Given, as before, Dead load = 7200 lbs. per panel point, moving load = 18000 lbs. per panel point, entire length of bridge = 25 feet; load applied at lower panel points.

Solution. — In this case, there are two possible modes of resolving it into systems. The first is shown in Figs. 118*a* and 118*b*: and this is necessarily the mode of division that must hold whenever the load is unevenly distributed, or when the travelling-load covers only a part of the bridge; for a single load at 6 is necessarily put in communication with the support at 2 by means of the diagonals 6–3 and 3–2, and with the support at 12 by means of the diagonals 6–7, 7–10, 10–11, and the vertical 11–12, and can cause no stress in the other diagonals

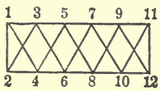


Fig. 118.

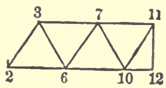


Fig. 118*a*.

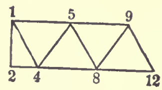


Fig. 118*b*.

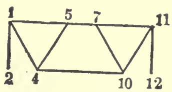


Fig. 118*c*.

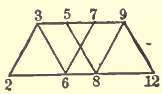


Fig. 118*d*.

When, however, the whole travelling-load is on the bridge, it is perfectly possible to divide it into the two trusses shown in Figs. 118*c* and 118*d*, the diagonals 4–5, 7–10, 6–7, and 5–8 having no stress upon them.

When the load is unevenly distributed, we have certainly the first method of division; and when evenly, we are not sure which will hold.

Hence we must compute the greatest stresses with each mode of division, and use for each member the greatest; for thus only shall we be sure that the truss is made strong enough.

We shall thus have the following results:—

FIRST MODE OF DIVISION (FIGS. 118*a* AND 118*b*).

Diagonals.		Greatest Shearing-Force of One Kind.	Greatest Shearing-Force of Opposite Kind.	Corresponding Stresses.	
Fig. 118*a*.	Fig. 118*b*.				
2- 3	12-9	$\frac{25200}{5}(3+1) = 20160$	0	+23279	− 0
3- 6	9-8	$\frac{25200}{5}(3+1) = 20160$	0	−23279	+ 0
6- 7	8-5	$\frac{25200}{5} - \frac{7200}{5}(2) = 2160$	$\frac{25200}{5}(2) - \frac{7200}{5} = 8640$	+ 2494	− 9976
7-10	5-4	$\frac{25200}{5} - \frac{7200}{5}(2) = 2160$	$\frac{25200}{5}(2) - \frac{7200}{5} = 8640$	− 2494	+ 9976
10-11	4-1	$0 = 0$	$\frac{25200}{5}(2+4) = 30240$	0	−34918

Chords.

Supporting force at 2 (Fig. 118*a*) or 12 (Fig. 118*b*)

$$= \frac{25200}{5}(3+1) = 20160,$$

Supporting force at 12 (Fig. 118*a*) or 2 (Fig. 118*b*)

$$= \frac{25200}{5}(2+4) = 30240.$$

Section.		Bending-Moment.	Chords.		Maximum Stresses in Separate Trusses.	Chords.		Components of Stresses.	Greatest Resultant Stresses.
Fig. 118*a*.	Fig. 118*b*.		Fig. 118*a*.	Fig. 118*b*.					
3	9	20160× 5=100800	2- 6	8-12	−11639	1-3	9-11	0+1-5	+17459
6	8	20160×10=201600	3- 7	5- 9	+23279	3-5	7- 9	3- 7+1-5	+40738
7	5	20160×15−25200 ×5=176400	6-10	4- 8	−20369	5-7		3- 7+5-9	+46558
10	4	30240× 5=151200	7-11	1- 5	+17459	2-4	10-12	2- 6+2-4	−11639
			10-12	2- 4	0	4-6	8-10	2- 6+4-8	−32008
						6-8		6-10+4-8	−40738

SECOND METHOD OF DIVISION (FIGS. 118*c* AND 118*d*).

Diagonals (Fig. 118*c*).

Diagonals.		Maximum Shear.	Corresponding Stresses.
1–4	10–11	25200	−29098
4–5	7–10	0	0

Fig. 118*d*.

Diagonals.		Maximum Shear.	Corresponding Stresses.
2–3	9–12	25200	+29098
3–6	8– 9	25200	−29098
6–7	5– 8	0	0

Chords.

Each supporting force in either figure = 25200.

Fig. 118*c*.

Bending-moment anywhere between 4 and 10 = (25200) (5) = 126000;

$\therefore$ Stress in 1–11 = +14549,
$\therefore$ Stress in 4–10 = −14549.

Fig. 118*d*.

Bending-moment at 3 or 9 = 126000,
Bending-moment anywhere between 6 and 8 = 252000;

$\therefore$ Stress in 3–9 = +29098,
Stress in 2–6 or 8–12 = −14549,
Stress in 6–8 = −29098.

Hence we have for chord stresses, with this second division, —

Chords.			Stresses.
1–3	9–11	1–11 + 0	+14549
3–5	7– 9	1–11 + 3–9	+43647
5–7	–	1–11 + 3–9	+43647
2–4	10–12	0 + 2–6	−14549
4–6	8–10	4–10 + 2–6	−29098
6–8	–	4–10 + 6–8	−43647

Hence, selecting for each bar the greatest, we shall have, as the stresses which the truss must be able to resist, —

1–4	10–11	+ 0	−34918	1–3	9–11	+17459
2–3	12– 9	+29098	− 0	3–5	7– 9	+43647
3–6	9– 8	+ 0	−29098	5–7	–	+46558
4–5	10– 7	+ 9976	− 2494	2–4	10–12	−14549
5–8	7– 6	+ 2494	− 9976	4–6	8–10	−32008
				6–8		−43647

These results are recorded in Fig. 118*e*.

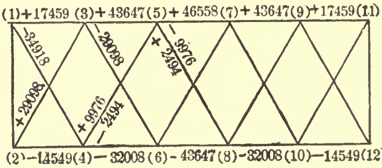


FIG. 118*e*.

§ 157. **Other Trusses.** — In Figs. 119, 120, and 121, we have examples of the double-panel system with the load placed at the lower panel points only. When, as in 119 and 120, the number of panels is odd, the same ambiguity arises as took place in Fig. 118. When, on the other hand, the number of panels is even, as shown in Fig. 121, there is only one mode of division into systems possible. The diagrams speak for themselves, and need no explanation.

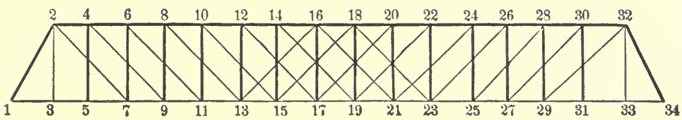


FIG. 119.

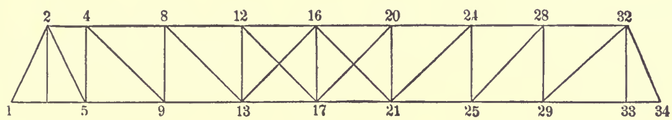


FIG. 119*a*.

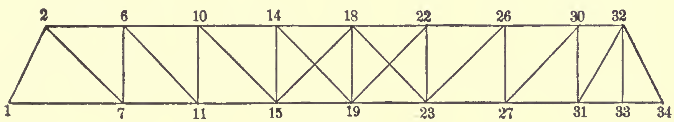


FIG. 119*b*.

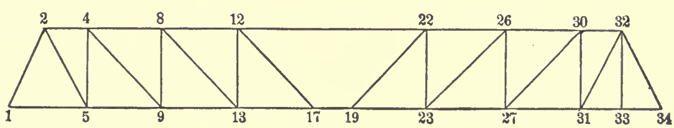


FIG. 119*c*.

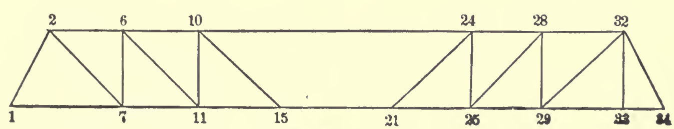


FIG. 119*d*.

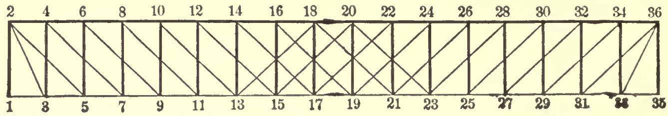


FIG. 120.

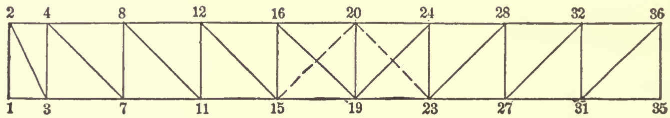


FIG. 120*a*.

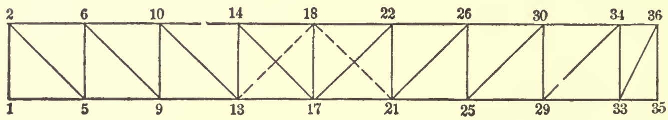


FIG. 120*b*.

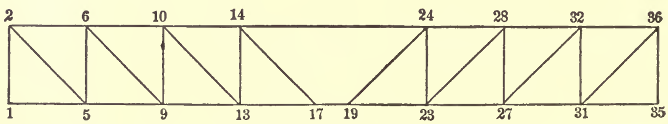


FIG. 120*c*.

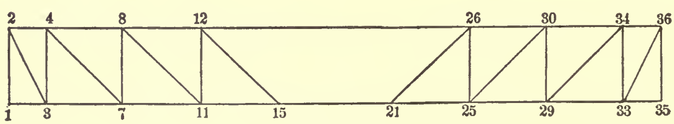


FIG. 120*d*.

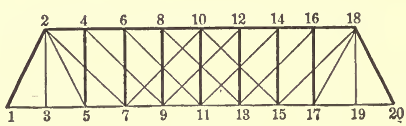


FIG. 121.

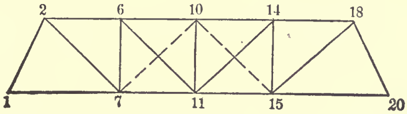


FIG. 121*a*.

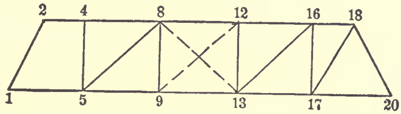


FIG. 121*b*.

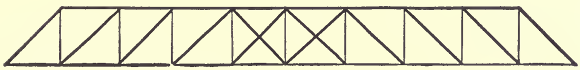


FIG. 122.

The trusses given above may be considered as examples, to be solved by the student by assuming the dead and the moving load per panel point respectively.

§ 158. **Fink's Truss.** — The description of this truss will be evident from the figure. There is, first, the primary truss 1–8–16; then on each side of 9–8 (the middle post of this truss) is a secondary truss (1–4–9 on the left, and 9–12–16 on the right).

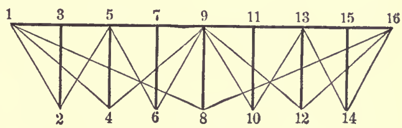


FIG. 123.

Each of these secondary trusses contains a pair of smaller secondary trusses, and the division might be continued if the segments into which the upper chord is thus divided were too long.

Of the inclined ties, there is none in which any load tends to produce compression; in other words, every load either increases the tension in the tie, or else does not affect it. Hence the greatest stresses in all the members will be attained when the entire travelling-load is on the truss, and we need only consider that case.

The determination of the stress in any one member can readily be obtained by determining, by means of the triangle of forces, the stress in that member due to the presence of the total load per panel point, at each point, and then adding the results. This will be illustrated by a few diagonals.

Let angle 8–1–9 $= i$,
Let angle 4–1–5 $= i_1$,
Let angle 2–1–3 $= i_2$;

we shall have, if $w + w_1 =$ entire load per panel point, —

Designation of Ties.	Effect of Loads at							Resultant Tensions.
	3	5	7	9	11	13	15	
1–2	$\frac{w + w_1}{2 \sin i_2}$	0	0	0	0	0	0	$\frac{w + w_1}{2 \sin i_2}$
2–5	$\frac{w + w_1}{2 \sin i_2}$	0	0	0	0	0	0	$\frac{w + w_1}{2 \sin i_2}$
5–6	0	0	$\frac{w + w_1}{2 \sin i_2}$	0	0	0	0	$\frac{w + w_1}{2 \sin i_2}$
6–9	0	0	$\frac{w + w_1}{2 \sin i_2}$	0	0	0	0	$\frac{w + w_1}{2 \sin i_2}$
1–4	$\frac{w + w_1}{4 \sin i_1}$	$\frac{w + w_1}{2 \sin i_1}$	$\frac{w + w_1}{4 \sin i_1}$	0	0	0	0	$\frac{w + w_1}{\sin i_1}$
4–9	$\frac{w + w_1}{4 \sin i_1}$	$\frac{w + w_1}{2 \sin i_1}$	$\frac{w + w_1}{4 \sin i_1}$	0	0	0	0	$\frac{w + w_1}{\sin i_1}$
1–8	$\frac{w + w_1}{8 \sin i}$	$\frac{w + w_1}{4 \sin i}$	$\frac{3}{8}\frac{w+w_1}{\sin i}$	$\frac{w + w_1}{2 \sin i}$	$\frac{3}{8}\frac{w+w_1}{\sin i}$	$\frac{w + w_1}{4 \sin i}$	$\frac{w + w_1}{8 \sin i}$	$\frac{2(w+w_1)}{\sin i}$

The stresses in all the other members may be found in a similar manner.

§ 159. **Bollman's Truss.** — The description of this truss is made sufficiently clear by the figure. The upper chord is made in separate pieces; and the short diagonals 2–5, 3–4, 4–7, 5–6, 7–8, 6–9, 8–11, and 9–10 are only needed to prevent a bending of the upper chord at the joints.

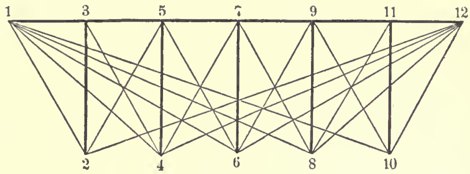


FIG. 124.

When this is their only object, the stress upon them cannot be calculated: indeed, it is zero until bending takes place; and then it is the less, the less the bending. Hence, in this case, the stress is wholly taken up by the principal ties; and these have their greatest stress when the whole load is on the bridge.

The computation of the stresses is made in a similar manner to that used in the Fink.

§ 160. **General Remarks.** — The methods already explained are intended to enable the student to solve any case of a bridge-truss where there is no ambiguity as to the course pursued by the stresses.

In cases where a large number of trusses of one given type are to be computed, it would, as a rule, be a saving of labor to determine formulæ for the stresses in the members, and then substitute in these formulæ.

Such formulæ may be deduced by using letters to denote the load and dimensions, instead of inserting directly their numerical values; and then, having deduced the formulæ for the type of truss, we can apply it to any case by merely substituting for the letters their numerical values corresponding to that case.

Such sets of formulæ would apply merely to specific styles of trusses, and any variation in these styles would require the formulæ to be changed.

In order to show how such formulæ are deduced, a few will be deduced for such a bridge as is shown in Fig. 111.

Let the load be applied at the upper panel points only; let dead load per panel point $= w$, moving load per panel point $= w_1$. Let the whole number of panels be N, N being an even number. Let the length of one panel $=$ height of truss $= l$. Then length of entire span $= Nl$.

Consider the $(n+1)^{\text{th}}$ panel from the middle.

The stress in the main tie is greatest when the moving load is on all the panel points from the farther abutment up to the panel in question, $(n+1)^{\text{th}}$.

Hence, for the n^{th} panel from the middle, the greatest shearing-force that causes tension in the main tie is equal to

$$\frac{w+w_1}{N}\left\{1+2+3+\ldots+\left(\frac{N}{2}+n\right)\right\}-\frac{w}{N}\left\{1+2+3+\ldots+\left(\frac{N}{2}-n-1\right)\right\}$$
$$=\frac{1}{2N}\left\{w_1\left[\left(\frac{N}{2}+n\right)^2+\frac{N}{2}+n\right]+w(2n+1)N\right\}.$$

Hence stress in main tie

$$=\frac{\sqrt{2}}{2N}\left\{w_1\left[\left(\frac{N}{2}+n\right)^2+\frac{N}{2}+n\right]+w(2n+1)N\right\}. \quad (1)$$

For the counterbrace, we should obtain, in a similar way, the formula

$$\frac{\sqrt{2}}{2N}\left\{w_1\left[\left(\frac{N}{2}-n\right)^2-\frac{N}{2}+n\right]-wN(2n+1)\right\},$$

which represents tension when it is positive. Proceed in a similar way for the other members.

When there is more than one system, we must divide the truss into its component systems; and when there is ambiguity, we must use, in determining the dimensions of each member, the greatest stress that can possibly come upon it.

CHAPTER V.

CENTRE OF GRAVITY.

§ 161. The centre of gravity of a body or system of bodies, is that point through which the resultant of the system of parallel forces that constitutes the weight of the body or system of bodies always passes, whatever be the position in which the body is placed with reference to the direction of the forces.

§ 162. **Centre of Gravity of a System of Bodies.** — If we have a system of bodies whose weights are W_1, W_2, W_3, etc., the co-ordinates of their individual centres of gravity being (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), etc., respectively, and if we denote by x_0, y_0, z_0, the co-ordinates of the centre of gravity of the system, we should obtain, just as in the determination of the centre of any system of parallel forces, —

1°. By turning all the forces parallel to OZ, and taking moments about OY,

$$(W_1 + W_2 + W_3 + \text{etc.})x_0 = W_1x_1 + W_2x_2 + W_3x_3 + \text{etc.},$$

or

$$x_0\Sigma W = \Sigma Wx;$$

and, taking moments about OX,

$$(W_1 + W_2 + W_3 + \text{etc.})y_0 = W_1y_1 + W_2y_2 + W_3y_3 + \text{etc.},$$

or

$$y_0\Sigma W = \Sigma Wy.$$

2°. By turning all the forces parallel to OX, and taking moments about OY,

$$(W_1 + W_2 + W_3 + \text{etc.})z_0 = W_1z_1 + W_2z_2 + W_3z_3 + \text{etc.},$$

or

$$z_0\Sigma W = \Sigma Wz.$$

Hence we have, for the co-ordinates of the centre of gravity of the system,

$$x_0 = \frac{\Sigma Wx}{\Sigma W}, \quad y_0 = \frac{\Sigma Wy}{\Sigma W}, \quad z_0 = \frac{\Sigma Wz}{\Sigma W}.$$

EXAMPLES.

1. Suppose a rectangular, homogeneous plate of brass (Fig. 125), where $AD = 12$ inches, $AB = 5$ inches, and whose weight is 2 lbs., to have weights attached at the points A, B, C, and D respectively, equal to 8, 6, 5, and 3 lbs.; find the centre of gravity of the system.

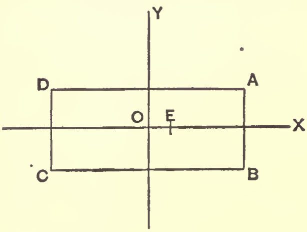


FIG. 125.

Solution.

Assume the origin of co-ordinates at the centre of the rectangle, and we have

$$\begin{array}{lllll}
W_1 = 2, & W_2 = 8, & W_3 = 6, & W_4 = 5, & W_5 = 3, \\
x_1 = 0, & x_2 = 6, & x_3 = 6, & x_4 = -6, & x_5 = -6, \\
y_1 = 0, & y_2 = \frac{5}{2}, & y_3 = -\frac{5}{2}, & y_4 = -\frac{5}{2}, & y_5 = \frac{5}{2};
\end{array}$$

$$\begin{aligned}
\therefore\ \Sigma Wx &= 0 + 48 + 36 - 30.0 - 18.0 = 36, \\
\Sigma Wy &= 0 + 20 - 15 - 12.5 + 7.5 = 0, \\
\Sigma W &= 2 + 8 + 6 + 5.0 + 3.0 = 24;
\end{aligned}$$

$$\therefore\ x_0 = \frac{36}{24} = 1.5, \quad y_0 = \frac{0}{24} = 0.$$

Hence the centre of gravity is situated at a point E on the line OX, where $OE = 1.5$.

2. Given a uniform circular plate of radius 8, and weight 3 lbs. (Fig. 126). At the points A, B, C, and D, weights are attached equal to 10, 15, 25, and 23 lbs. respectively, also given $AB = 45°$, $BC = 105°$, $CD = 120°$; find the centre of gravity of the system.

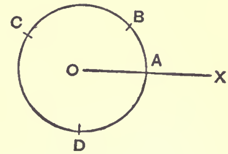


Fig. 126.

§ 163. **Centre of Gravity of Homogeneous Bodies.** — For the case of a single homogeneous body, the formulæ have been already deduced in § 44. They are

$$x_0 = \frac{\int x dV}{\int dV}, \quad y_0 = \frac{\int y dV}{\int dV}, \quad z_0 = \frac{\int z dV}{\int dV};$$

and for the weight of the body,

$$W = w\int dV,$$

where x_0, y_0, z_0, are the co-ordinates of the centre of gravity of the body, W its weight, and w its weight per unit of volume.

From these formulæ we can readily deduce those for any special cases; thus, —

(*a*) *For a volume* referred to rectangular co-ordinate axes, $dV = dxdydz$.

$$x_0 = \frac{\int\int\int x dxdydz}{\int\int\int dxdydz}, \quad y_0 = \frac{\int\int\int y dxdydz}{\int\int\int dxdydz}, \quad z_0 = \frac{\int\int\int z dxdydz}{\int\int\int dxdydz}.$$

(*b*) *For a flat plate of uniform thickness*, t, the centre of gravity is in the middle layer; hence only two co-ordinates are required to determine it. If it be referred to a system of rectangular axes in the middle plane, $dV = tdxdy$,

$$x_0 = \frac{\int\int x dxdy}{\int\int dxdy}, \quad y_0 = \frac{\int\int y dxdy}{\int\int dxdy}.$$

The centre of gravity of such a thin plate is also called the centre of gravity of the plane area that constitutes the middle plane section; hence —

(*c*) *For a plane area* referred to rectangular co-ordinate axes in its own plane,

$$x_0 = \frac{\int\int x dx dy}{\int\int dx dy}, \quad y_0 = \frac{\int\int y dx dy}{\int\int dx dy}.$$

(*d*) *For a slender rod of uniform sectional area, a,* if x, y, z, be the co-ordinates of points on the axis (straight or curved) of the rod, we shall have $dV = ads = a\sqrt{(dx)^2 + (dy)^2 + (dz)^2}$,

$$x_0 = \frac{\int x ds}{\int ds} = \frac{\int x\sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2}dx}{\int\sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2}dx},$$

$$y_0 = \frac{\int y ds}{\int ds} = \frac{\int y\sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2}dx}{\int\sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2}dx},$$

$$z_0 = \frac{\int z ds}{\int ds} = \frac{\int z\sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2}dx}{\int\sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2}dx}.$$

(*e*) *For a slender rod whose axis lies wholly in one plane,* the centre of gravity lies, of course, in the same plane; and if our co-ordinate axes be taken in this plane, we shall have $z = 0$ $\therefore \frac{dz}{dx} = 0$, and also $z_0 = 0$. Hence we need only two co-ordinates to determine the centre of gravity, hence $dV = ads = a\sqrt{(dx)^2 + (dy)^2}$.

$$x_0 = \frac{\int xds}{\int ds} = \frac{\int x\sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx}{\int\sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx},$$

$$y_0 = \frac{\int yds}{\int ds} = \frac{\int y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx}{\int\sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx}.$$

(f) *For a line,* straight or curved, which lies entirely in one plane, we shall have, again,

$$x_0 = \frac{\int xds}{\int ds} = \frac{\int x\sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx}{\int\sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx},$$

$$y_0 = \frac{\int yds}{\int ds} = \frac{\int y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx}{\int\sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx}.$$

Whenever the body of which we wish to determine the centre of gravity is made up of simple figures, of which we already know the positions of the centres of gravity, the method explained in § 162 should be used, and not the formulæ that involve integration; i.e., taking moments about any given line will give us the perpendicular distance of the centre of gravity from that line.

In the case of the determination of the strength and stiffness of beams, it is necessary to know the distance of a horizontal line passing through the centre of gravity of the section, from the top or the bottom of the section; but it is of no practical importance to know the position of the centre of gravity on this line. In most of the examples that follow, therefore, the results given are these distances. These examples should be worked out by the student.

In the case of wrought-iron beams of various sections, on account of the thinness of the iron, a sufficiently close approximation is often obtained by considering the cross-section as composed of its central lines; the area of any given portion being found by multiplying the thickness of the iron by the corresponding length of line, the several areas being assumed to be concentrated in single lines.

EXAMPLES.

1. *Straight Line* AB (Fig. 127). — The centre of gravity is evidently at the middle of the line, as this is a point of symmetry.

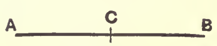


FIG. 127.

2. *Combination of Two Straight Lines.* — The centre of gravity in each case lies on the line OO_1, Figs. 128, 129, 130, and 131.

(*a*) *Angle-Iron of Unequal Arms* (Fig. 128). — Length $AB = b$, length $BC = h$, area $AB = A$, area $BC = B$;

$$\therefore \quad BE = DE = \frac{1}{2}\frac{bh}{\sqrt{b^2 + h^2}}.$$

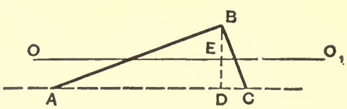


FIG. 128.

(*b*) *Angle-Iron of Equal Arms* (Fig. 129). — Length $AB = BC = b$;

$$\therefore \quad BE = DE = \frac{b}{2\sqrt{2}} = \frac{b}{4}\sqrt{2}.$$

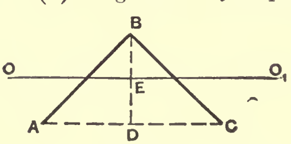


FIG. 129.

(*c*) *Cross of Equal Arms* (Fig. 130).—$AB = OO_1 = h$;

$$\therefore \quad AC = BC = \frac{h}{2}.$$

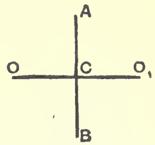


FIG. 130.

(*d*) T-*Iron* (Fig. 131).—Area $AB = A$, area $CE = B$, length $CE = h$;

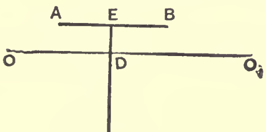


FIG. 131.

$$\therefore \quad DE = \frac{Bh}{2(A+B)},$$

$$CD = \frac{2A+B}{2(A+B)}h.$$

3. *Combination of Three Lines.*—OO_1 = line passing through the centre of gravity.

(*a*) *Thin Isosceles Triangular Cell* (Fig. 132).—Length $AB = BC = a$, length $AC = b$, area $AB = BC = A$, area $AC = B$;

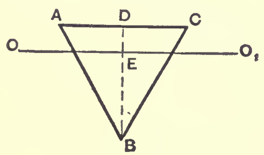


FIG. 132.

$$\therefore \quad DB = \sqrt{a^2 - \frac{b^2}{4}}$$

$$= \tfrac{1}{2}\sqrt{(2a-b)(2a+b)}$$

$$\therefore \quad DE = \frac{A}{2(2A+B)}\sqrt{(2a-b)(2a+b)},$$

$$BE = \frac{A+B}{2(2A+B)}\sqrt{(2a-b)(2a+b)}.$$

(*b*) *Same in Different Position* (Fig. 133).

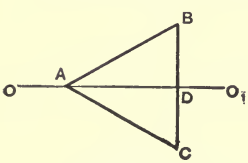


FIG. 133.

$$BD = DC = \frac{a}{2}$$

(*c*) *Channel-Iron* (Fig. 134). — Area of flanges $= A$, area of web $= B$, depth of flanges $+ \frac{1}{2}$ thickness of web $= h$;

Fig. 134.

$$\therefore \quad CE = \frac{1}{2}\frac{Ah}{A+B},$$

$$DE = \frac{h}{2}\frac{A+2B}{A+B}.$$

(*d*) I-*Beam* (Fig. 135). — Area of upper flange $= A_1$, area of lower flange $= A_2$, area of web $= B$, height $= h$.

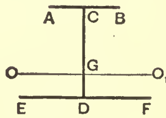


Fig. 135.

$$CG = \frac{h}{2}\frac{2A_2 + B}{A_1 + A_2 + B},$$

$$GD = \frac{h}{2}\frac{2A_1 + B}{A_1 + A_2 + B}.$$

4. *Combination of Four Lines.* — $OO_1 =$ line passing through the centre of gravity.

(*a*) *Thin Rectangular Cell* (Fig. 136). — Length $AB = h$;

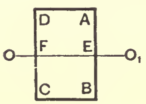


Fig. 136.

$$\therefore \quad AE = BE = \frac{h}{2}.$$

(*b*) *Thin Square Cell* (Fig. 137). — $AB = BC = h$;

$$\therefore \quad BE = CE = \frac{h}{2}.$$

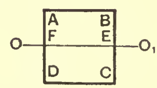


Fig. 137.

5. *Circular Arcs.*

(*a*) *Circular Arc* AB (Fig. 138). — Angle $AOB = \theta_1$, radius $= r$. Use formula

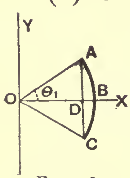


Fig. 138.

$$x_0 = \frac{\int x ds}{\int ds}, \quad y_0 = \frac{\int y ds}{\int ds};$$

but use polar co-ordinates, where

$$ds = r d\theta, \quad x = r\cos\theta, \quad y = r\sin\theta,$$

$$\therefore \quad x_0 = \frac{r^2\int_0^{\theta_1}\cos\theta d\theta}{r\int_0^{\theta_1}d\theta} = r\frac{(\sin\theta_1)}{\theta_1},$$

$$y_0 = \frac{r^2\int_0^{\theta_1}\sin\theta d\theta}{r\int_0^{\theta_1}d\theta} = r\frac{(1-\cos\theta_1)}{\theta_1} = 2r\frac{\sin^2\frac{1}{2}\theta_1}{\theta_1}.$$

(*b*) *Circular Arc* AC (same figure).

$$x_0 = \frac{r\sin\theta_1}{\theta_1}, \quad y_0 = 0.$$

(*c*) *Quarter-Arc of Circle* AB, *Radius* r (Fig. 139).

$$x_0 = \frac{r^2\int_0^{\frac{\pi}{2}}\cos\theta d\theta}{r\int_0^{\frac{\pi}{2}}d\theta} = \frac{2r}{\pi}.$$

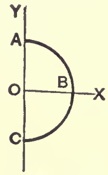


FIG. 139.

$$y_0 = \frac{2r}{\pi}.$$

(*d*) *Semi-circumference* ABC (same figure).

$$x_0 = \frac{2r}{\pi}, \quad y_0 = 0.$$

6. *Combination of Circles and Straight Lines.*

Barlow Rail (Fig. 140). — Two quadrants, radius r, and web, whose area $= \frac{3}{11}$ the united area of the quadrants. Let united area of quadrants $= A$, area of web $= \frac{3}{11}A$; let $EF = x_0$:

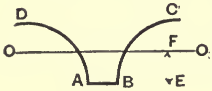


FIG. 140.

$$\pi = \frac{22}{7},$$

$$\therefore \quad \frac{14}{11}Ax_0 = A\left(\frac{2r}{\pi}\right) = \frac{7}{11}Ar \qquad \therefore \quad x_0 = \frac{r}{2} = EF_1.$$

7. *Areas.*

(*a*) *T-Section* (Fig. 141).—Let length $AB = B$, $EF = b$, entire height $= H$, $GE = h$. Let distance of centre of gravity below $AB = x_1$; therefore, taking moments about AB as an axis,

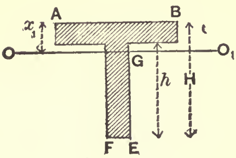


FIG. 141.

$$x_1\{BH - h(B - b)\}$$
$$= \tfrac{1}{2}BH^2 - h(B - b)\left(H - \frac{h}{2}\right),$$

whence we can readily derive x_1.

(*b*) *I-Section* (Fig. 142).—Let $AB = B$, $GH = b$, $MN = b_1$, entire height $= H$, $BC = H - h$, $EH = h_1$; and let $x_1 =$ distance of centre of gravity below AB.

Hence, taking moments about AB, we have

$$x_1\{B(H - h) + b_1(h - h_1) + bh_1\}$$
$$= \frac{B}{2}(H - h)^2 + bh_1\left(H - \frac{h_1}{2}\right)$$
$$+ b_1(h - h_1)\left(H - h + \frac{h - h_1}{2}\right),$$

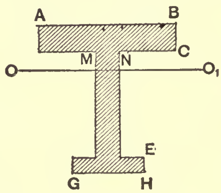


FIG. 142.

whence we can deduce x_1.

(*c*) *Triangle* (Fig. 143).—If we consider the triangle OBC as composed of an indefinite number of narrow strips parallel to the side CB, of which $FLHK$ is one, the centre of gravity of each one of these strips will be on the line OD drawn from O to the middle point of the side CB; hence the centre of gravity of the entire triangle must be on the line OD. For a similar reason, it must be on the median line CE; hence the centre of gravity must be at the intersection of the median lines, and hence

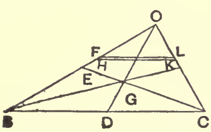


FIG. 143.

$$x_0 = OG = \tfrac{2}{3}OD. \quad \text{Moreover, area} = \frac{BC \,.\, OD \sin ODC}{2}.$$

(*d*) *Trapezoid* (Fig. 144).

First Solution.—Bisect AB in O, and CE in D; let g_1 be the centre of gravity of CEB, and g_2 that of ABC. Then will G, the centre of gravity of the trapezoid, be on the line g_1g_2, and

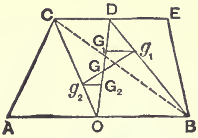


FIG. 144.

$$\frac{Gg_1}{Gg_2} = \frac{ABC}{CEB}.$$

But it must be on the line OD; hence it is at their intersection.

From the similarity of GG_1g_1 GG_2g_2, we have

$$\frac{GG_1}{GG_2} = \frac{Gg_1}{Gg_2} = \frac{ABC}{BEC} = \frac{AB}{CE} = \frac{B}{b};$$

$$\therefore \quad \frac{GG_2}{G_1G_2} = \frac{b}{B+b}, \quad \text{and since} \quad G_1G_2 = \frac{OD}{3},$$

$$\therefore \quad OG = OG_2 + GG_2 = \frac{OD}{3} + GG_2 = \frac{OD}{3}\left(1 + \frac{b}{B+b}\right).$$

Second Solution.—Fig. 144 (*a*). Let O be the point of intersection of the non-parallel sides AC and BE. Let $OF = x_1$, $OD = x_2$, $OG = x_0$. Take moments about an axis through O, and perpendicular to OF, and we readily obtain

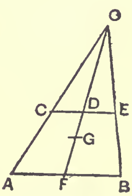


FIG. 144*a*.

$$x_0 = \frac{2}{3}\frac{x_1^3 - x_2^3}{x_1^2 - x_2^2}.$$

(*e*) *Parabolic Half-Segment* OAB (Fig. 145).—Let $OA = x_1$, $AB = y_1$; let x_0, y_0, be the co-ordinates of the centre of gravity; let the equation of the parabola be $y^2 = 4ax$:

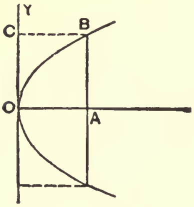


FIG. 145.

$$\therefore \quad x_0 = \frac{\int_0^{x_1}\int_0^{2a^{\frac{1}{2}}x^{\frac{1}{2}}} xdxdy}{\int_0^{x_1}\int_0^{2a^{\frac{1}{2}}x^{\frac{1}{2}}} dxdy} = \frac{2a^{\frac{1}{2}}\int_0^{x_1} x^{\frac{3}{2}}dx}{2a^{\frac{1}{2}}\int_0^{x_1} x^{\frac{1}{2}}dx}$$

$$= \frac{\frac{2}{5}x_1^{\frac{5}{2}}}{\frac{2}{3}x_1^{\frac{3}{2}}} = \frac{3}{5}x_1,$$

$$y_0 = \frac{\int_0^{x_1}\int_0^{2a^{\frac{1}{2}}x^{\frac{1}{2}}} ydxdy}{\int_0^{x_1}\int_0^{2a^{\frac{1}{2}}x^{\frac{1}{2}}} dxdy} = \frac{3}{4}a^{\frac{1}{2}}x_1^{\frac{1}{2}} = \frac{3}{8}y_1,$$

$$\text{Area} = \int_0^{2a^{\frac{1}{2}}x^{\frac{1}{2}}}\int_0^{x_1} dxdy = \frac{4}{3}a^{\frac{1}{2}}x_1^{\frac{3}{2}} = \frac{2}{3}(2a^{\frac{1}{2}}x_1^{\frac{1}{2}})x_1 = \frac{2}{3}x_1y_1.$$

(*f*) *Parabolic Spandril* OBC (Fig. 145).—Let x_0, y_0, be co-ordinates of centre of gravity of the spandril.

$$x_0 = \frac{\int_0^{x_1}\int_{2a^{\frac{1}{2}}x^{\frac{1}{2}}}^{y_1} xdxdy}{\int_0^{x_1}\int_{2a^{\frac{1}{2}}x^{\frac{1}{2}}}^{y_1} dxdy} = \frac{3}{10}x_1,$$

$$y_0 = \frac{\int_0^{x_1}\int_{2a^{\frac{1}{2}}x^{\frac{1}{2}}}^{y_1} ydxdy}{\int_0^{x_1}\int_{2a^{\frac{1}{2}}x^{\frac{1}{2}}}^{y_1} dxdy} = \frac{3}{4}y_1,$$

$$\text{Area} = x_1y_1 - \frac{2}{3}x_1y_1 \qquad = \frac{1}{3}x_1y_1.$$

(*g*) *Circular Sector* OAC (Fig. 146).—Let $OA = r$, $AOX = \theta_1$, x_0, y_0, be the co-ordinates of the centre of gravity :

$$\therefore \quad y_0 = 0,$$

$$x_0 = \frac{\int_{r\cos\theta_1}^{r}\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} xdxdy + \int_0^{r\cos\theta_1}\int_{-x\tan\theta_1}^{x\tan\theta_1} xdxdy}{\frac{1}{2}r(2r\theta_1)}$$

$$= \frac{2}{3}r\frac{\sin\theta_1}{\theta_1},$$

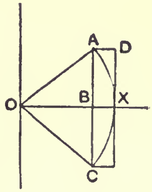


FIG. 146.

$$\text{Area} = \tfrac{1}{2}r(2r\theta_1) = r^2\theta_1.$$

Second Solution.

Consider the sector to be made up of an indefinite number of narrow rings ; let ρ be the variable radius, and $d\rho$ the thickness :

$$\therefore \quad \text{Elementary area} = 2\rho\theta_1 d\rho,$$

and centre of gravity of this elementary area is on OX, at a distance from O equal to $\rho\frac{\sin\theta_1}{\theta_1}$ [see Example 5 (*b*)] ;

$$\therefore \quad x = \frac{\int_0^r \left\{\rho\frac{\sin\theta_1}{\theta_1}\right\}\left\{2\rho\theta_1 d\rho\right\}}{\int_0^r 2\rho\theta_1 d\rho} = \frac{2\sin\theta_1\int_0^r \rho^2 d\rho}{2\theta_1\int_0^r \rho d\rho} = \frac{2}{3}r\frac{\sin\theta_1}{\theta_1}.$$

(*h*) *Circular Half-Segment* ABX (Fig. 146),

$$x_0 = \frac{\int_{r\cos\theta_1}^{r}\int_0^{\sqrt{r^2-x^2}} xdxdy}{\text{Sector minus triangle}} = \frac{\int_{r\cos\theta_1}^{r} x\sqrt{r^2-x^2}dx}{\frac{1}{2}r^2\theta_1 - \frac{1}{2}r^2\sin\theta_1\cos\theta_1}$$

$$= \frac{2}{3}r\frac{\sin^3\theta_1}{\theta_1 - \sin\theta_1\cos\theta_1},$$

$$y_0 = \frac{\int_{r\cos\theta_1}^{r}\int_0^{\sqrt{r^2-x^2}} ydxdy}{\frac{1}{2}r^2(\theta_1 - \sin\theta_1\cos\theta_1)} = \frac{1}{3}r\frac{4\sin^2\frac{1}{2}\theta_1 - \sin^2\theta_1\cos\theta_1}{\theta_1 - \sin\theta_1\cos\theta_1}.$$

§ 164. **Pappus's Theorems.**—The following two theorems serve often to simplify the determination of the **centres of** gravity of lines and areas. They are as follows:—

THEOREM I.—If a plane curve lies wholly on one side of a straight line in its own plane, and, revolving about that line, generates thereby a surface of revolution, the area of the surface is equal to the product of the length of the revolving line, and of the path described by its centre of gravity.

Proof.—Let the curve lie in the xy plane, and let the axis of y be the line about which it revolves. We have, from what precedes, § 163 (e), $x_0 = \frac{\int xds}{\int ds}$;

$$\therefore \quad x_0 \int ds = \int xds,$$

where x_0 equals the perpendicular distance of the centre of gravity of the curve from OY, ds = elementary arc,

$$\therefore \quad 2\pi x_0 \int ds = \int (2\pi x) ds;$$

or, reversing the equation,

$$\int (2\pi x) ds = (2\pi x_0) s.$$

But $\int (2\pi x) ds$ = surface described in one revolution, while s = length of arc, and $2\pi x_0$ = path described by the centre of gravity in one revolution. Hence follows the proposition.

THEOREM II.—If a plane area lying wholly on the same side of a straight line in its own plane revolves about that line, and thereby generates a solid of revolution, the volume of the solid thus generated is equal to the product of the revolving area, and of the path described by the centre of gravity of the plane area during the revolution.

Proof. — Let the area lie in the xy plane, and let the axis OY be the axis of revolution. We then have, from what has preceded, if x_0 = perpendicular distance of the centre of gravity of the plane area from OY, the equation, § 163 (*b*),

$$x_0 = \frac{\int\int x dx dy}{\int\int dx dy}.$$

Hence

$$x_0 \int\int dx dy \quad = \int\int x dx dy;$$

$$\therefore \quad (2\pi x_0)\int\int dx dy = \int\int (2\pi x) dx dy$$

or

$$\int\int (2\pi x) dx dy \;=\; 2\pi x_0 \int\int dx dy.$$

But $\int\int(2\pi x)dxdy$ = volume described in one revolution, and $2\pi x_0$ = path described by the centre of gravity in one revolution. Hence follows the proposition.

The same propositions hold true for any part of a revolution, as well as for an entire revolution, since we might have multiplied through by the circular measure θ, instead of by 2π.

It is evident that the first of these two theorems may be used to determine the centre of gravity of a line, when the length of the line, and the surface described by revolving it about the axis, are known; and so also that the second theorem may be used to determine the centre of gravity of a plane area whenever the area is known, and also the volume described by revolving it around the axis.

EXAMPLES.

1. *Circular Arc* AC (Fig. 138). — Length of arc = $s = 2r\theta$, surface of zone described by revolving it about OY = circumference of a great circle multiplied by the altitude = $(2\pi r)(2r\sin\theta_1)$;

$$\therefore \quad (2\pi x_0)(2r\theta_1) = (2\pi r)(2r\sin\theta_1) \qquad \therefore \quad x_0\theta_1 = r\sin\theta_1$$

$$\therefore \quad x = r\frac{\sin\theta_1}{\theta_1}.$$

2. *Semicircular Arc* (Fig. 139).—Length of arc $= \pi r$, surface of sphere described $= 4\pi r^2$;

$$\therefore \quad 2\pi x_0(\pi r) = 4\pi r^2 \qquad \therefore \quad x_0 = \frac{2r}{\pi}.$$

3. *Trapezoid* (Fig. 147).—Let $AD = b$, $BC = b$; let it revolve around AD: it generates two cones and a cylinder.

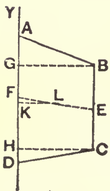


Fig. 147.

$$\text{Area of trapezoid} = \frac{AD + BC}{2} BG,$$

$$\text{Volume} = \frac{\pi(GB)^2}{3}(AG + HD) + \pi(GB)^2 . BC$$

$$= \frac{\pi(GB)^2}{3}(AG + HD + 3BC)$$

$$= \frac{\pi(GB)^2}{3}(AD + BC + BC)$$

$$\therefore \quad (2\pi x_0)\left(\frac{AD + BC}{2}\right) . GB = \frac{\pi(GB)^2}{3}\{(AD + BC) + BC\}$$

$$\therefore \quad x_0 = \frac{GB}{3}\left(1 + \frac{BC}{AD + BC}\right) = \frac{GB}{3}\left(1 + \frac{b}{B + b}\right) = KL$$

$$\therefore \quad FL = \frac{FE}{3}\left(1 + \frac{b}{B + b}\right).$$

4. *Circular Sector* ACO (Fig. 146).—Area of sector $= r^2\theta_1$, volume described $= \frac{1}{3}r$(surface of zone) $= \frac{1}{3}r(2\pi r)(2r \sin \theta_1) = \frac{4}{3}\pi r^3 \sin \theta_1$;

$$\therefore \quad (2\pi x_0)(r^2\theta_1) = \frac{4}{3}\pi r^3 \sin \theta_1 \qquad \therefore \quad x_0 = \frac{2}{3}r\frac{\sin \theta_1}{\theta_1}.$$

§ 165. **Centre of Gravity of Solid Bodies.**—The general formulæ furnish, in most cases, a very complicated solution, and hence we generally have recourse to some simpler method. A few examples will be given in this and the next section.

Tetrahedron ABCD (Fig. 148). — The plane ABE, containing the edge AB and the middle point E of the edge CD, bisects all lines drawn parallel to CD, and terminating in the faces ABD and ABC: hence a similar reasoning to that used in the case of the triangle will show that the centre of gravity of the pyramid must be in the plane ABE; in the same way it may be shown that it must lie in the plane ACF. Hence it must lie in their intersection, or in the line AG joining the vertex A with the centre of gravity (intersection of the medians) of the opposite face. In the same way it can be shown that the centre of gravity of the triangular pyramid must lie in the line drawn from the vertex B to the centre of gravity of the face ACD. Hence the centre of gravity of the tetrahedron will be found on the line AG at a distance from G equal to $\frac{1}{4}AG$.

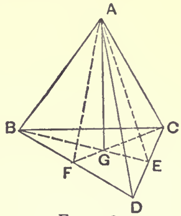


Fig. 148.

§ 166. **Centre of Gravity of Bodies which are Symmetrical with Respect to an Axis.** — Such solids may be generated by the motion of a plane figure, as $ABCD$ (Fig. 149), of variable dimensions, and of any form whose centre G remains upon the axis OX; its plane being always perpendicular to OX, and its variable area X being a function of x, its distance from the origin.

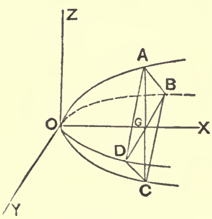


Fig. 149.

Here the centre of gravity will evidently lie on the axis OX, and the elementary volume will be the volume of a thin plate whose area is X and thickness Δx; hence the elementary volume will be $X\Delta x$.

Take moments about OY, and we shall have

$$x_0 \int X dx = \int X x dx \quad \text{and} \quad \text{Volume} = \int X dx,$$

or

$$x_0 = \frac{\int X x dx}{\int X dx}, \qquad V = \int X dx.$$

EXAMPLES.

1. *Ellipsoid* $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ (Fig. 150).—Find centre of gravity of the half to the right of the x plane. Let $OK = x$. Now if, in the equation of the ellipsoid, we make $y = 0$, we have $\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1$;

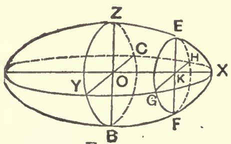


FIG. 150.

$$\therefore \quad z = \frac{c}{a}\sqrt{a^2 - x^2},$$

where $z = EK$.

Make $z = 0$ in the equation of the ellipsoid, and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$;

$$\therefore \quad y = \frac{b}{a}\sqrt{a^2 - x^2},$$

where $y = KG$;

$$\therefore \quad EK = \frac{c}{a}\sqrt{a^2 - x^2}, \qquad KG = \frac{b}{a}\sqrt{a^2 - x^2},$$

are the semi-axes of the variable ellipse $EGFH$, which, by moving along OX, generates the ellipsoid. Hence

$$\text{Area } EGFH = \pi(EK \,.\, GK) = \frac{\pi bc}{a^2}(a^2 - x^2) = X;$$

hence

$$\text{Elementary volume} = \frac{\pi bc}{a^2}(a^2 - x^2)\Delta x$$

$$\therefore \quad x_0 = \frac{\frac{\pi bc}{a^2}\int_0^a (a^2x - x^3)\,dx}{\frac{\pi bc}{a^2}\int_0^a (a^2 - x^2)\,dx} = \frac{\left\{\frac{a^2x^2}{2} - \frac{x^4}{4}\right\}_0^a}{\left\{a^2x - \frac{x^3}{3}\right\}_0^a} = \tfrac{3}{8}a.$$

$$V = \frac{\pi bc}{a^2}\int_0^a (a^2 - x^2)\,dx = \tfrac{2}{3}\pi abc.$$

2. *Hemisphere.*—Make $a = b = c$, and $x_0 = \frac{3}{8}a$, $V = \frac{2}{3}\pi a^3$.

If the section X were oblique to OX, making an angle θ with it, the elementary volume would not be Xdx, but $Xdx \sin\theta$, and we should have

$$x_0 = \frac{\sin\theta \int Xxdx}{\sin\theta \int Xdx} = \frac{\int Xxdx}{\int Xdx} \quad \text{and} \quad V = \sin\theta \int Xdx.$$

3. *Oblique Cone* (Fig. 151).—Let $OA = h$; let area of base be A, and let the angle made by OX with the base be θ;

$$\therefore \quad \frac{X}{A} = \frac{x^2}{h^2} \qquad \therefore \quad X = \frac{A}{h^2}x^2$$

$$\therefore \quad x_0 = \frac{\frac{A}{h^2}\int_0^h x^3dx}{\frac{A}{h^2}\int_0^h x^2dx} = \frac{\frac{h^4}{4}}{\frac{h^3}{3}} = \tfrac{3}{4}h.$$

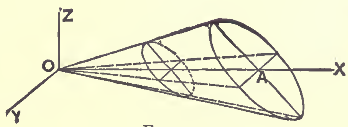


FIG. 151.

$$V = \sin\theta \int Xdx = \frac{A}{h^2}\sin\theta \int_0^h x^2dx = \tfrac{1}{3}Ah\sin\theta.$$

4. *Truncated Cone* (Fig. 151).—Let height of entire cone be $h = OA$; let height of portion cut off be h_1;

$$\therefore \quad x_0 = \frac{\frac{A}{h^2}\int_{h_1}^h x^3dx}{\frac{A}{h^2}\int_{h_1}^h x^2dx} = \frac{\frac{h^4 - h_1^4}{4}}{\frac{h^3 - h_1^3}{3}} = \frac{3}{4}\frac{h^4 - h_1^4}{h^3 - h_1^3}.$$

$$V \frac{A}{h^2}\sin\theta \int_{h_1}^h x^2dx = \frac{Ah\sin\theta}{3h^3}(h^3 - h_1^3) = \tfrac{1}{3}Ah\sin\theta\left(1 - \frac{h_1^3}{h^3}\right).$$

CHAPTER VI.

STRENGTH OF MATERIALS.

§ 167. **Stress, Strain, and Modulus of Elasticity.** — When a body is subjected to the action of external forces, if we imagine a plane section dividing the body into two parts, the force with which one part of the body acts upon the other at this plane is called the *stress* on the plane; it may be a tensile, a compressive, or a shearing stress, or it may be a combination of either of the two first with the last. In order to know the stress completely, we must know its distribution and its direction at each point of the plane. If we consider a small area lying in this plane, including the point O, and represent the stress on this area by p, whereas the area itself is represented by a, then will the limit of $\frac{p}{a}$ as a approaches zero be the intensity of the stress on the plane under consideration at the point O.

When a body is subjected to the action of external forces, and, in consequence of this, undergoes a change of form, it will be found that lines drawn within the body are changed, by the action of these external forces, in length, in direction, or in both; and the entire change of form of the body may be correctly described by describing a sufficient number of these changes.

If we join two points, A and B, of a body before the external forces are applied, and find, that, after the application of the external forces, the line joining the same two points of the body has undergone a change of length $\Delta(AB)$, then is the limit of the ratio $\frac{\Delta(AB)}{AB}$ as AB approaches zero called the strain of the body at the point A in the direction AB.

If $AB + \Delta(AB) > AB$, the strain is one of tension.
If $AB + \Delta(AB) < AB$, the strain is one of compression.

Suppose a straight rod of uniform section A to be subjected to a pull P in the direction of its length, and that this pull is uniformly distributed over the cross-section: then will the intensity of the stress on the cross-section be

$$p = \frac{P}{A}.$$

If P be measured in pounds, and A in square inches, then will p be measured in pounds per square inch.

If the length of the rod before the load is applied be l, and its length after the load is applied be $l + e$, then is e the elongation of the rod; and if this elongation is uniform throughout the length of the rod, then is $\frac{e}{l}$ the elongation of the rod per unit of length, or the strain.

Hence, if α represent the strain due to the stress p per unit of area, we shall have

$$\alpha = \frac{e}{l}.$$

The *Modulus of Elasticity* is commonly defined as the ratio of the stress per unit of area to the strain, or

$$E = \frac{p}{\alpha};$$

and this is expressed in units of weight per unit of area, as in pounds per square inch.

This definition is true, however, only for stresses for which Hooke's law "The stress is proportional to the strain" holds.

For greater stresses the permanent set must first be deducted from the strain, and the remainder be used as divisor.

The *limit of elasticity* of any material is the *stress* above which the stresses are no longer proportional to the strains.

The modulus of elasticity was formerly defined as the weight that would stretch a rod one square inch in section to double its length, if Hooke's law held up to that point, and the rod did not break.

EXAMPLES.

1. A wrought-iron rod 10 feet long and 1 inch in diameter is loaded in the direction of its length with 8000 lbs.; find (1) the intensity of the stress, (2) the elongation of the rod; assuming the modulus of the iron to be 28000000 lbs. per square inch.

2. What would be the elongation of a similar rod of cast-iron under the same load, assuming the modulus of elasticity of cast-iron to be 17000000 lbs. per square inch?

3. Given a steel bar, area of section being 4 square inches, the length of a certain portion under a load of 25000 lbs. being 10 feet, and its length under a load of 100000 lbs. being 10′ 0″.075; find the modulus of elasticity of the material.

4. What load will be required to stretch the rod in the first example $\frac{1}{10}$ inch?

§ 168. **Resistance to Stretching and Tearing.** — The most-used criterion of safety against injury for a loaded piece is, that the greatest intensity of the stress to which any part of it is subjected shall nowhere exceed a certain fixed amount, called the working-strength of the material; this working-strength being a certain fraction of the breaking-strength determined by practical considerations.

The more correct but less used criterion is, that the greatest strain in any part of the structure shall nowhere exceed the working-strain; the greatest allowable amount of strain being a fixed quantity determined by practical considerations.

This is equivalent to limiting the allowable elongation or compression to a certain fraction of its length, or the deflection of a beam to a certain fraction of the span.

If the stress on a plane surface be uniformly distributed, its resultant will evidently act at the centre of gravity of the surface, as has been already shown in § 42 to be the case with any uniformly distributed force.

If a straight rod of uniform section and material be subjected to a pull in the direction of its length, and if the resultant of the pull acts along a line passing through the centres of gravity of the sections of the rod, it is assumed in practice that the stress is uniformly distributed throughout the rod, and hence that for any section we shall obtain the stress per square inch by dividing the total pull by the number of square inches in the section.

If, on the other hand, the resultant of the pull does not act through the centres of gravity of the sections, the pull is not uniformly distributed; and while

$$p = \frac{P}{A}$$

will express the mean stress per square inch, the actual intensity of the stress will vary at different points of the section, being greater than $\frac{P}{A}$ at some points and less at others. How to determine its greatest intensity in such cases will be shown later.

With good workmanship and well-fitting joints, the first case, or that of a uniformly distributed stress, can be practically realized; but with ill-fitting joints or poor workmanship, or with a material that is not homogeneous, the resultant of the pull is liable to be thrown to one side of the line passing through the centres of gravity of the sections, and thus there is set up a bending-action in addition to the direct tension, and therefore an unevenly distributed stress.

It is of the greatest importance in practice to take cognizance of any such irregularities, and determine the greatest intensity of the stress to which the piece is subjected: though it is too often taken account of merely by means of a factor of safety; in other words, by guess.

Leaving, then, this latter case until we have studied the stresses due to bending, we will confine ourselves to the case of the uniformly distributed stress.

If the total pull on the rod in the direction of its length be P, and the area of its cross-section A, we shall have, for the intensity of the pull,

$$p = \frac{P}{A}.$$

On the other hand, if the working-strength of the material per unit of area be f, we shall have, for the greatest admissible load to be applied,

$$P = fA.$$

If f be the working-strength of the material per square inch, and E the modulus of elasticity, then is the greatest admissible strain equal to

$$\alpha = \frac{f}{E}.$$

Thus, assuming 12000 lbs. per square inch as the working tensile strength of wrought-iron, and 28000000 lbs. per square inch as its modulus of elasticity, its working-strain would be

$$\alpha = \frac{12000}{28000000} = \frac{3}{7000}.$$

Hence the greatest safe elongation of the bar would be $\frac{3}{7000}$ of its length. Hence a rod 10 feet long could safely be stretched $\frac{3}{700}$ of a foot $= 0.0514''$.

§ 169. **Approximate Values of Breaking Strength, and of Modulus of Elasticity.**—In a later part of this book the attempt will be made to give an account of the experiments that have been made to determine the strength and elasticity of the materials ordinarily used in construction, in such a way as to enable the student to decide for himself, in any special case, upon the proper values of the constants that he ought to use.

For the present, however, the following will be given as a rough approximation to some of these quantities, which we may make use of in our work until we reach the above-mentioned account.

(*a*) *Cast-Iron.*

Breaking tensile strength per square inch, of common qualities, 14000 to 20000 lbs.; of gun iron, 30000 to 33000 lbs.

Modulus of elasticity for tension and for compression, about 17000000 lbs. per square inch.

(*b*) *Wrought-Iron.*

Breaking tensile strength per square inch, from 40000 to 60000 lbs.

Modulus of elasticity for tension and for compression, about 28000000.

(*c*) *Mild Steel.*

Breaking tensile strength per square inch, 55000 to 70000 lbs.

Modulus of elasticity for tension and for compression, from 28000000 to 30000000 lbs. per square inch.

(*d*) *Wood.*

Breaking compressive strength per square inch:—

Oak, green	3000 lbs.
Oak, dry	3000 to 6000 lbs.
Yellow pine, green	3000 to 4000 lbs.
Yellow pine, dry	4000 to 7000 lbs.

Modulus of elasticity for compression (average values): —

Oak	1300000 lbs. per square inch.
Yellow pine	1600000 lbs. per square inch.

§ 170. **Sudden Application of the Load.** — If a wrought-iron rod 10 feet long and 1 square inch in section be loaded with 12000 pounds in the direction of its length, and if the modulus of elasticity of the iron be 28000000, it will stretch $0.0514''$ provided the load be gradually applied: thus, the rod begins to stretch as soon as a small load is applied; and, as the load gradually increases, the stretch increases, until it reaches $0.0514''$.

If, on the other hand, the load of 12000 lbs. be suddenly applied (i.e., put on all at once) without being allowed to fall through any height beforehand, it would cause a greater stretch at first, the rod undergoing a series of oscillations, finally settling down to an elongation of $0.0514''$.

To ascertain what suddenly applied load will produce at most the elongation $0.0514''$, observe, that, in the case of the gradually applied load, we have a load gradually increasing from

$$0 \text{ to } 12000 \text{ lbs.}$$

Its mean value is, therefore, $\frac{1}{2}(12000) = 6000$ lbs.; and this force descends through a distance of

$$0.0514''.$$

Hence the amount of mechanical work done on the rod by the gradually applied load in producing this elongation is

$$(6000)(0.0514) = 308.4 \text{ inch-lbs.}$$

Hence, if we are to perform upon the rod 308.4 inch-lbs. of work with a constant force, and if the stretch is to be $0.0514''$, the magnitude of the force must be

$$\frac{308.4}{0.0514''} = 6000 \text{ lbs.}$$

Hence a suddenly applied load will produce double the strain that would be produced by the same load gradually applied; and, moreover, a suddenly applied load should be only half as great as one gradually applied if it is to produce the same strain.

§ 171. **Resilience of a Tension-Bar.** — The resilience of a tension-rod is the mechanical work done in stretching it to the same amount that it would stretch under the greatest allowable gradually applied load, and is found by multiplying the greatest allowable load by half the corresponding elongation.

Thus, suppose a load of 100 lbs. to be dropped upon the rod described above in such a way as to cause an elongation not greater than $0.0514''$, it would be necessary to drop it from a height not greater than $3.08''$.

EXAMPLES.

1. A wrought-iron rod is 12 feet long and 1 inch in diameter, and is loaded in the direction of its length; the working-strength of the iron being 12000 lbs. per square inch, and the modulus of elasticity 28000000 lbs. per square inch.

Find the working-strain.
Find the working-load.
Find the working-elongation.
Find the working-resilience.

From what height can a 50-pound weight be dropped so as to produce tension, without stretching it more than the working-elongation?

2. Do the same for a cast-iron rod, where the working-strength is 5000 pounds per square inch, and the modulus of elasticity 17000000; the dimensions of the rod being the same.

§ 172. **Results of Wöhler's Experiments on Tensile Strength.** — According to the experiments of Wöhler, of which an account will be given later, the breaking-strength of a piece depends, not only on whether the load is gradually or suddenly applied, but also on the extreme variations of load that the piece is called upon to undergo, and the number of changes to which it is to be submitted during its life.

For a piece which is always in tension, he determines the following two constants ; viz., t, the carrying-strength per square inch, or the greatest quiescent stress that the piece will bear, and u, the primitive safe strength, or the greatest stress per square inch of which the piece will bear an indefinite number of repetitions, the stress being entirely removed in the intervals.

This primitive safe strength, u, is used as the breaking-strength when the stress varies from 0 to u every time. Then, by means of Launhardt's formula, we are able to determine the ultimate strength per square inch for any different limits of stress, as for a piece that is to be alternately subjected to 80000 and 6000 pounds.

Thus, for Phœnix Company's axle iron, Wöhler finds

$$t = 3290 \text{ kil. per sq. cent.} = 46800 \text{ lbs. per sq. in.},$$
$$u = 2100 \text{ kil. per sq. cent.} = 30000 \text{ lbs. per sq. in.}$$

Launhardt's formula for the ultimate strength per unit of area is

$$a = u\left\{1 + \frac{t-u}{u}\,\frac{\text{least stress}}{\text{greatest stress}}\right\}.$$

Hence, with these values of t and u, we should have, for the ultimate strength per square inch,

$$a = 2100\left\{1 + \frac{1}{2}\,\frac{\text{least stress}}{\text{greatest stress}}\right\} \text{kil. per sq. cent.},$$

or

$$a = 30000\left\{1 + \frac{1}{2}\,\frac{\text{least stress}}{\text{greatest stress}}\right\} \text{lbs. per sq. in.}$$

Thus, if least stress = 6000, and greatest = 80000, we should have

$$a = 30000\{1 + \tfrac{1}{2} \cdot \tfrac{6}{80}\} = 30000\{1 + \tfrac{3}{80}\} = 31125;$$

if least stress = 60000, and greatest = 80000,

$$a = 30000\{1 + \tfrac{1}{2} \cdot \tfrac{6}{8}\} = 30000\{1 + \tfrac{3}{8}\} = 41250;$$

if least stress = greatest stress = 80000,

$$a = 30000\{1 + \tfrac{1}{2}\} = 45000 = \text{carrying-strength}.$$

Hence, instead of using, as breaking-strength per square inch in all cases, 45000, we should use a set of values varying from 45000 down to 30000, according to the variation of stress which the piece is to undergo.

For working-strength, Weyrauch divides this by 3: thus obtaining, for working-strength per square inch,

$$b = 10000\left\{1 + \frac{1}{2}\,\frac{\text{least stress}}{\text{greatest stress}}\right\}\text{lbs. per sq. in.};$$

For Krupp's cast-steel, notwithstanding the fact that Wöhler finds

$$t = 7340 \text{ kil. per sq. cent.} = 104400 \text{ lbs. per sq. in.},$$
$$u = 3300 \text{ kil. per sq. cent.} = 46900 \text{ lbs. per sq. in.},$$

Weyrauch recommends

$$a = 3300\left\{1 + \frac{9}{11}\,\frac{\text{least stress}}{\text{greatest stress}}\right\}\text{kil. per sq. cent.},$$

or

$$a = 46900\left\{1 + \frac{9}{11}\,\frac{\text{least stress}}{\text{greatest stress}}\right\}\text{lbs. per sq. in.},$$

$$\therefore\quad b = 15633\left\{1 + \frac{9}{11}\,\frac{\text{least stress}}{\text{greatest stress}}\right\}\text{lbs. per sq. in.}$$

EXAMPLES.

Find the breaking-strength per square inch for a wrought-iron tension rod.

1. Extreme loads are 75000 and 6000 lbs.
2. Extreme loads are 120000 and 100000 lbs.
3. Extreme loads are 300000 and 10000 lbs.

Find the safe section for the rod in each case.

§ 173. **Suspension-Rod of Uniform Strength.** — In the case of a long suspension-rod, the weight of the rod itself sometimes becomes an important item. The upper section must, of course, be large enough to bear the weight that is hung from the rod plus the weight of the rod itself; but it is sometimes desirable to diminish the sections as they descend. This is often accomplished in mines by making the rod in sections, each section being calculated to bear the weight below it plus its own weight.

Were the sections gradually diminished, so that each section would be just large enough to support the weight below it, we should, of course, have a curvilinear form; and the equation of this curve could be found as follows, or, rather, the area of any section at a distance from the bottom of the rod.

Let $W =$ weight hung at O (Fig. 152),

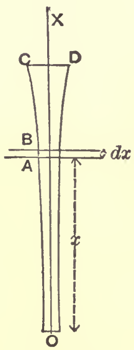


Fig. 152.

Let w = weight per unit of volume of the rod,

Let x = distance AO,

Let S = area of section A,

Let $x + dx$ = distance BO,

Let $S + dS$ = area of section at B,

Let f = working-strength of the material per square inch.

1°. The section at O must be just large enough to sustain the load W;

$$\therefore \quad S_0 = \frac{W}{f}.$$

2°. The area in dS must be just enough to sustain the weight of the portion of the rod between A and B.

The weight of this portion is $wSdx$;

$$\therefore \quad dS = \frac{wSdx}{f}$$

$$\therefore \quad \frac{dS}{S} = \frac{w}{f}dx \qquad \therefore \quad \log_e S = \frac{w}{f}x + \text{a constant.}$$

When $x = 0$, $S = \dfrac{W}{f}$;

$$\therefore \quad \log_e \frac{W}{f} = \text{the constant} \qquad \therefore \quad \log_e S - \log_e\left(\frac{W}{f}\right) = \frac{w}{f}x$$

$$\therefore \quad \frac{S}{\left(\frac{W}{f}\right)} = e^{\frac{wx}{f}} \qquad \therefore \quad S = \frac{W}{f}e^{\frac{wx}{f}}.$$

This gives us the means of determining the area at any distance x from O.

EXAMPLES.

1. A wrought-iron tension-rod 200 feet long is to sustain a load of 2000 lbs. with a factor of safety of 4, and is to be made in 4 sections, each 50 feet long; find the diameter of each section, the weight of the wrought iron being 480 lbs. per cubic foot.

2. Find the diameter needed if the rod were made of uniform section, also the weight of the extra iron necessary to use in this case.

3. Find the equation of the longitudinal section of the rod, assuming a square cross-section, if it were one of uniform strength, instead of being made in 4 sections.

§ 174. **Thin Hollow Cylinders subjected to an Internal Normal Pressure.** — Let p denote the uniform intensity of the pressure exerted by a fluid which is confined within a hollow cylinder of radius r and of thickness t (Fig. 153), the thickness being small compared with the radius. Let us consider a unit of length of the cylinder, and let us also consider the forces acting on the upper half-ring CED.

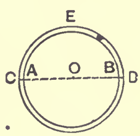


FIG. 153.

The total upward force acting on this half-ring, in consequence of the internal normal pressure, will be the same as that acting on a section of the cylinder made by a plane passing through its axis, and the diameter CD. The area of this section will be $2r \times 1 = 2r$: hence the total upward force will be $2r \times p = 2pr$; and the tendency of this upward force is to cause the cylinder to give way at A and B, the upper part separating from the lower.

This tendency is resisted by the tension in the metal at the sections AC and BD; hence at each of these sections, there has to be resisted a tensile stress equal to $\frac{1}{2}(2pr) = pr$. This stress is really not distributed uniformly throughout the cross-section of the metal; but, inasmuch as the metal is thin, no serious error will be made if it be accounted as distributed uniformly. The area of each section, however, is $t \times 1 = t$; therefore, if T denote the intensity of the tension in the metal in a tangential direction (i.e., the intensity of the hoop tension), we shall have

$$T = \frac{pr}{t}.$$

Hence, to insure safety, T must not be greater than f, the working-strength of the material for tension ; hence, putting

$$f = \frac{pr}{t},$$

we shall have

$$t = \frac{pr}{f}$$

as the proper thickness, when p = normal pressure per square inch, and radius $= r$.

The above are the formulæ in common use for the determination of the thickness of the shell of a steam-boiler; for in that case the steam-pressure is so great that the tension induced by any shocks that are likely to occur, or by the weight of the boiler, is very small in comparison with that induced by the steam-pressure. On the other hand, in the case of an ordinary water-pipe, the reverse is the case.

To provide for this case, Weisbach directs us to add to the thickness we should obtain by the above formulæ, a constant minimum thickness.

The following are his formulæ, d being the diameter in inches, p the internal normal pressure in atmospheres, and t the thickness in inches. For tubes made of

Sheet-iron	$t = 0.00086\,pd + 0.12$
Cast-iron	$t = 0.00238\,pd + 0.34$
Copper	$t = 0.00148\,pd + 0.16$
Lead	$t = 0.00507\,pd + 0.21$
Zinc	$t = 0.00242\,pd + 0.16$
Wood	$t = 0.03230\,pd + 1.07$
Natural stone	$t = 0.03690\,pd + 1.18$
Artificial stone	$t = 0.05380\,pd + 1.58$

§ 175. **Resistance to Direct Compression.** — When a piece is subjected to compression, the distribution of the compressive stress on any cross-section depends, first, upon whether the resultant of the pressure acts along the line containing the centres of gravity of the sections, and, secondly, upon the dimensions of the piece; thus determining whether it will bend or not.

In the case of an eccentric load, or of a piece of such length that it yields by bending, the stress is not uniformly distributed; and, in order to proportion the piece, we must determine the greatest intensity of the stress upon it, and so proportion it that this shall be kept within the working-strength of the material for compression.

Either of these cases is not a case of direct compression.

In the case of direct compression (i.e., where the stress over each section is uniformly distributed), the intensity of the stress is found by dividing the total compression by the area of the section; so that, if P be the total compression, and A the area of the section, and p the intensity of the compressive stress,

$$p = \frac{P}{A}.$$

On the other hand, if f is the compressive working-strength of the material per square inch, and A the area of the section in square inches, then the greatest allowable load on the piece subjected to compression is

$$P = fA.$$

The same remarks as were made in regard to a suddenly applied load and resilience, in the case of direct tension, apply in the case of direct compression.

§ 176. **Results of Wöhler's Experiments on Compressive Strength.** — Wöhler also made experiments in regard to pieces subjected to alternate tension and compression, taking, in the experiments themselves, the case where the metal is subjected to alternate tensions and compressions of equal amount.

The greatest stress of which the piece would bear an indefinite number of changes under these conditions, is called the vibration safe strength, and is denoted by s.

Weyrauch deduces a formula similar to that of Launhardt for the greatest allowable stress per unit of area on the piece when it is subjected to alternate tensions and compressions of different amounts.

Thus, for Phœnix Company's axle iron, Wöhler deduces

t = 3290 kil. per sq. cent. = 46800 lbs. per sq. in.,
u = 2100 kil. per sq. cent. = 30000 lbs. per sq. in.,
s = 1170 kil. per sq. cent. = 16600 lbs. per sq. in.

Weyrauch's formula for the ultimate strength per unit of area is

$$a = u\left\{1 - \frac{u - s}{u}\,\frac{\text{least maximum stress}}{\text{greatest maximum stress}}\right\};$$

and, with these values of u and s, it gives

$$a = 2100\left\{1 - \frac{1}{2}\,\frac{\text{least maximum stress}}{\text{greatest maximum stress}}\right\}\text{kil. per sq. cent.,}$$

or

$$a = 30000\left\{1 - \frac{1}{2}\,\frac{\text{least maximum stress}}{\text{greatest maximum stress}}\right\}\text{lbs. per sq. in.}$$

With a factor of safety of 3, we should have, for the greatest admissible stress per square inch,

$$b = 10000\left\{1 - \frac{1}{2}\,\frac{\text{least maximum stress}}{\text{greatest maximum stress}}\right\}\text{lbs.}$$

For Krupp's cast-steel,

t = 7340 kil. per sq. cent. = 104400 lbs. per sq. in.,
u = 3300 kil. per sq. cent. = 46900 lbs. per sq. in. approximately,
s = 2050 kil. per sq. cent. = 29150 lbs. per sq. in. approximately.

We have, therefore, for the breaking-strength per unit of area, according to Weyrauch's formula,

$$a = 3300\left\{1 - \frac{5}{11}\,\frac{\text{least maximum stress}}{\text{greatest maximum stress}}\right\}\text{kil. per sq. cent.,}$$

or

$$a = 46900\left\{1 - \frac{5}{11}\,\frac{\text{least maximum stress}}{\text{greatest maximum stress}}\right\}\text{lbs. per sq. in.;}$$

and, using a factor of safety of 3, we have, for the greatest admissible stress per square inch,

$$b = 15630\left\{1 - \frac{5}{11}\,\frac{\text{least maximum stress}}{\text{greatest maximum stress}}\right\}\text{lbs. per sq. in.}$$

The principles respecting an eccentric compressive load, and those respecting the giving-way of long columns so far as they are known, can only be treated after we have studied the resistance of beams to bending; hence this subject will be deferred until that time.

EXAMPLES.

Find the proper working and breaking strength per square inch to be used for a wrought-iron rod, the extreme stresses being —

1. 80000 lbs. tension and 6000 lbs. compression.
2. 100000 lbs. tension and 100000 lbs. compression.
3. 70000 lbs. tension and 60000 lbs. compression.

Do the same for a steel rod.

§ 177. **Resistance to Shearing.** — One of the principal cases where the resistance to shearing comes into practical use is that where the members of a structure, which are themselves subjected to direct tension or compression or bending, are united by such pieces as bolts, rivets, pins, or keys, which are subjected to shearing. Sometimes the shearing is combined with tension or with bending; and whenever this is the case, it is necessary to take account of this fact in designing the pieces. It is important that the pins, keys, etc., should be equally strong with the pieces they connect.

Probably one of the most important modes of connection is by means of rivets. In order that there may be only a shearing action, with but little bending of the rivets, the latter must fit very tightly. The manner in which the riveting is done will necessarily affect very essentially the strength of the joints;

hence the only way to discuss fully the strength of riveted joints is to take into account the manner of effecting the riveting, and hence the results of experiments. These will be spoken of later; but the ordinary theories by which the strength and proportions of some of the simplest forms of riveted joints are determined will be given, which theories are necessary also in discussing the results of experiments thereon.

The principle on which the theory is based, in these simple cases, is that of making the resistance of the joint to yielding equal in the first three, and also in either the fourth or the fifth of the ways in which it is possible for it to yield, as enumerated on pages 258 and 259.

A single-riveted lap-joint is one with a single row of rivets, as shown in Fig. 154.

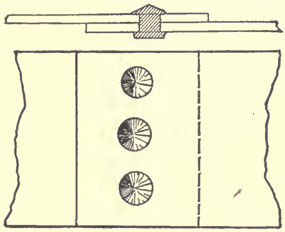


FIG. 154.

A single-riveted butt-joint with one covering plate is shown in Fig. 155.

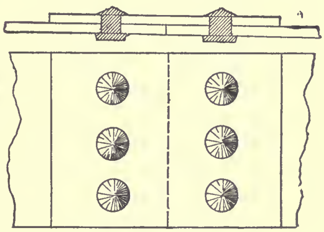


FIG. 155.

A single-riveted butt-joint with two covering plates is shown in Fig. 156.

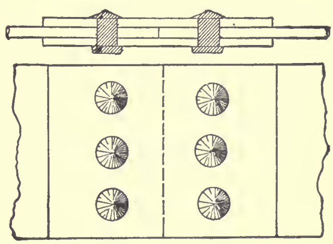


FIG. 156.

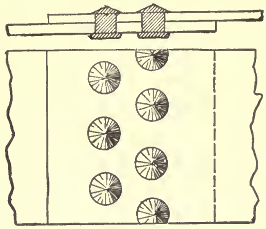


FIG. 157.

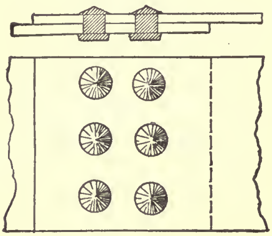


FIG. 158.

A double-riveted lap-joint with the rivets staggered is shown in Fig. 157; one with chain riveting, in Fig. 158.

Taking the case of the single-riveted lap-joint shown in Fig. 154, it may yield in one of five ways:—

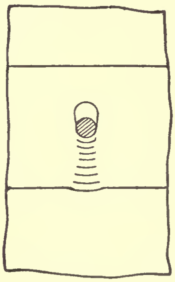


FIG. 159.

1°. By the crushing of the plate in front of the rivet (Fig. 159).

FIG. 160.

2°. By the shearing of the rivet (Fig. 160).

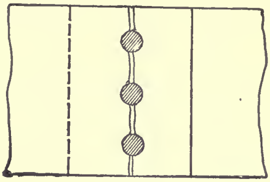


FIG. 161.

3°. By the tearing of the plate between the rivet-holes (Fig. 161).

4°. By the rivet breaking through the plate (Fig. 162).

5°. By the rivet shearing out the plate in front of it.

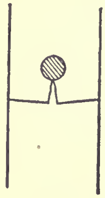


FIG. 162.

Let us call

d the diameter of a rivet.

p the pitch of the rivets; i.e., their distance apart from centre to centre.

t the thickness of the plate.

l the lap of the plate; i e., the distance from the centre of a rivet-hole to the outer edge of the plate.

$h = l - \frac{d}{2}$.

f_t the ultimate tensile strength of the iron.

f_s the ultimate shearing-strength of the rivet-iron.

$f_{s'}$ the ultimate shearing-strength of the plate.

f_c the ultimate crushing-strength of the iron.

We shall then have —

1°. Resistance of plate in front of rivet to crushing $= f_c td$.

2°. Resistance of one rivet to shearing $= f_s\left(\frac{\pi d^2}{4}\right)$.

3°. Resistance of plate between two rivet-holes to tearing $= f_t t(p - d)$.

4°. Resistance of plate to being broken through $= a\frac{th^2}{d}$, where a is a constant depending on the material. This may be taken as empirical for the present.

A reasonable value of this constant is $\frac{4}{3}f_t$.

5°. Resistance of plate in front of the rivet to shearing $= 2f_{s'}lt$.

Assuming that we know the thickness of the plate to start with, we obtain, by equating the first two resistances,

$$f_c td = f_s\frac{\pi d^2}{4} \qquad \therefore \quad d = \frac{4t}{\pi}\frac{f_c}{f_s},$$

which determines the diameter of the rivet.

Equating 3° and 2°, we obtain

$$f_t t(p - d) = f_s\frac{\pi d^2}{4} \qquad \therefore \quad p = d + \frac{f_s}{f_t}\frac{\pi d^2}{4t},$$

which gives the pitch of the rivets in terms of the diameter of the rivet, and the thickness of the plate.

Equating, next, 4° and 1°, we have

$$a\frac{th^2}{d} = f_c td \qquad \therefore \quad h = d\sqrt{\frac{f_c}{a}},$$

which gives the lap of the plate needed in order that it may not break through.

By equating 5° and 1°, we find the lap needed that it may not shear out in front of the rivet.

A similar method of reasoning would enable us to determine the corresponding quantities in the cases of double-riveted joints, etc.

There are a number of practical considerations which modify more or less the results of such calculations, and which can only be determined experimentally. A fuller account of this subject from an experimental point of view will be given later.

§ 178. **Intensity of Stress.**—Whenever the stress over a plane area is uniformly distributed, we obtain its intensity at each point by dividing the total stress by the area over which it acts, thus obtaining the amount per unit of area. When, however, the stress is not uniformly distributed, or when its intensity varies at different points, we must adopt a somewhat different definition of its *intensity at a given point.* In that case, if we assume a small area containing that point, and divide the stress which acts on that area by the area, we shall have, in the quotient, an approximation to the intensity required, which will approach nearer and nearer to the true value of the intensity at that point, the smaller the area is taken.

Hence the intensity of a variable stress at a given point is, —

The limit of the ratio of the stress acting on a small area containing that point, to the area, as the latter grows smaller and smaller.

By dividing the total stress acting on a certain area by the entire area, we obtain the mean intensity of the stress for the entire area.

§ 179. **Graphical Representation of Stress.** — A convenient mode of representing stress graphically is the following: —

Let AB (Fig. 163) be the plane surface upon which the stress acts; let the axes OX and OY be taken in this plane, the axis OZ being at right angles to the plane.

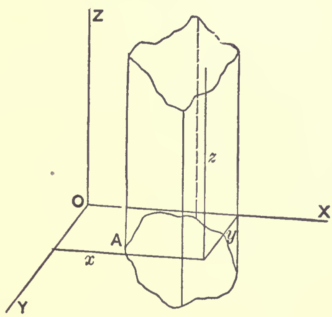


Fig. 163.

Conceive a portion of a cylinder whose elements are all parallel to OZ, bounded at one end by the given plane surface, and at the other by a surface whose ordinate at any point contains as many units of length as there are units of force in the intensity of the stress at that point of the given plane surface where the ordinate cuts it.

The volume of such a figure will evidently be

$$V = \int\int z dx dy = \int\int p dx dy,$$

where $z = p =$ intensity of the stress at the given point.

Hence the volume of the cylindrical figure will contain as many units of volume as the total stress contains units of force; or, in other words, the total stress will be correctly represented by the volume of the body.

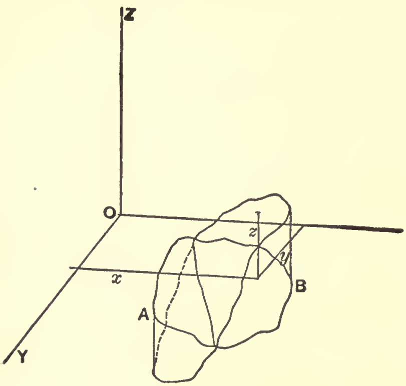


Fig. 164.

If the stress on the plane figure is partly tension and partly compression, the surface whose ordinates represent the intensity of the stress will lie partly on one side of the given plane surface and partly on the other; this surface and the plane surface on which the stress acts, cutting each other in some line, straight or curved, as shown in Fig. 164. In that case, the magnitude of the resultant stress $P = V = \int\int zdxdy$ will be equal to the difference of the wedge-shaped volumes shown in the figure.

It will be observed that the above method of representing stress graphically represents, 1°, the intensity at each point of the surface to which it is applied; and, 2°, the total amount of the stress on the surface. It does not, however, represent its direction, except in the case when the stress is normal to the surface on which it acts.

In this latter case, however, this is a complete representation of the stress.

The two most common cases of stress are, 1°, uniform stress, and, 2°, uniformly varying stress. These two cases are represented respectively in Figs. 165 and 166; the direction also being correctly represented when, as is most frequently the case, the stress is normal to the surface of action. In Fig. 165, AB is supposed to be the surface on which the stress acts; the stress is supposed to be uniform, and normal to the surface on which it acts; the bounding surface in this case becomes a plane parallel to AB; the intensity of the stress at any point, as P, will be represented by PQ; while the whole cylinder will contain as many units of volume as there are units of force in the whole stress.

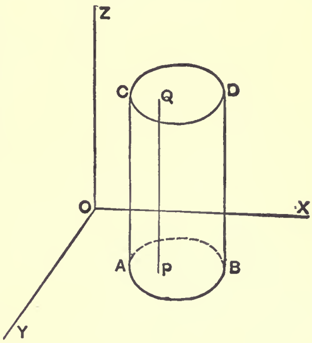


FIG. 165.

Fig. 166 represents a uniformly varying stress. Here, again, AB is the surface of action, and the stress is supposed to vary at a uniform rate from the axis OY. The upper bounding surface of the cylindrical figure which represents the stress becomes a plane inclined to the XOY plane, and containing the axis OY.

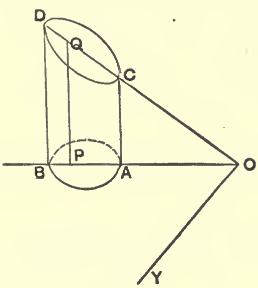


FIG. 166.

In this case, if a represent the intensity of the stress at a unit's distance from OY, the stress at a distance x from OY will be $p = ax$, and the total amount of the stress will be

$$P = \int\int p\,dx\,dy = a\int\int x\,dx\,dy.$$

When a stress is oblique to the surface of action, it may be represented correctly in all particulars, except in direction, in the above-stated way.

§ 180. **Centre of Stress.** — The centre of stress, or the point of the surface at which the resultant of the stress acts, often becomes a matter of practical importance. If, for convenience, we employ a system of rectangular co-ordinate axes, of which the axes OX and OY are taken in the plane of the surface on which the stress acts, and if we let $p = \phi(x, y)$ be the intensity of the stress at the point (x, y), we shall have, for the co-ordinates of the centre of stress,

$$x_1 = \frac{\int\int xpdxdy}{\int\int pdxdy}, \quad y_1 = \frac{\int\int ypdxdy}{\int\int pdxdy},$$

(see § 42), where the denominator, or $\int\int pdxdy$, represents the total amount of the stress.

When the stress is positive and negative at different parts of the surface, as in Fig. 164, the case may arise when the positive and negative parts balance each other, and hence the stress on the surface constitutes a statical couple. In that case

$$\int\int pdxdy = 0.$$

§ 181. **Uniform Stress.** — In the case of uniform stress, we have —

1°. The intensity of the stress is constant, or $p =$ a constant.

2°. The volume which represents it graphically becomes a cylinder with parallel and equal bases, as in Fig. 165.

3°. The centre of stress is at the centre of gravity of the surface of action; for the formulæ become, when p is constant,

$$x_1 = \frac{p\int\int xdxdy}{p\int\int dxdy} = \frac{\int\int xdxdy}{\int\int dxdy} = x_0,$$

$$y_1 = \frac{p\int\int ydxdy}{p\int\int dxdy} = \frac{\int\int ydxdy}{\int\int dxdy} = y_0,$$

where x_0, y_0, are the co-ordinates of the centre of gravity of the surface.

Examples of uniform stress have already been given in the cases of direct tension, direct compression, and, in the case of riveted joints, for the shearing-force on the rivet.

§ 182. **Uniformly Varying Stress.** — Uniformly varying stress has already been defined as a stress whose intensity varies uniformly from a given line in its own plane; and this line will be called the *Neutral Axis*. Thus, if the plane be taken as the XOY plane (Fig. 166), and the given line be taken as OY, we shall have, if a denotes the intensity of the stress at a unit's distance from OY, and x the distance of any special point from OY, that the intensity of the stress at the point will be

$$p = ax.$$

The total amount of the stress will be

$$P = a\int\int xdxdy.$$

The total moment of the stress about OY will be found by multiplying each elementary stress by its leverage. This leverage is, in the case of normal stress, x; hence in that case the moment of any single elementary force will be

$$(ax\Delta x\Delta y)x = ax^2\Delta x\Delta y,$$

and the total moment of the stress will be

$$M = a\int\int x^2dxdy = aI.$$

In the case of oblique stress, this result has to be modified, as the leverage is no longer x. Confining ourselves to stress normal to the plane of action, we have, for the co-ordinates of the centre of stress,

$$x_1 = \frac{\int\int pxdxdy}{\int\int pdxdy} = \frac{a\int\int x^2dxdy}{P} = \frac{\int\int x^2dxdy}{\int\int xdxdy} = \frac{\int\int x^2dxdy}{x_0A} = \frac{I}{x_0A}$$

$$y_1 = \frac{\int\int pydxdy}{\int\int pdxdy} = \frac{a\int\int xydxdy}{P} = \frac{\int\int xydxdy}{\int\int xdxdy} = \frac{\int\int xydxdy}{x_0A},$$

since

$$P = a\int\int xdxdy = ax_0A,$$

where x_0, y_0, are the co-ordinates of the centre of gravity, and A is the area of the surface of action.

§ 183. **Case of a Uniformly Varying Stress which amounts to a Statical Couple.** — Whenever $P = 0$, we have

$$a\int\int xdxdy = 0 \quad \therefore \int\int xdxdy = 0 \quad \therefore x_0A = 0 \quad \therefore x_0 = 0.$$

In this case, therefore, we have —

1°. There is no resultant stress, and hence the whole stress amounts to a statical couple.

2°. Since $x_0 = 0$, the centre of gravity of the surface of action is on the axis OY, which is the neutral axis.

Hence follows the proposition: —

When a uniformly varying stress amounts to a statical couple, the neutral axis contains (passes through) the centre of gravity of the surface of action.

In this case there is no single resultant of the stress; but the moment of the couple will be, as has been already shown,

$$M = a\int\int x^2dxdy.$$

§ 184. **Example of Uniformly Varying Stress.** — One of the most common examples of uniformly varying stress is that of the pressure of water upon the sides of the vessel containing it.

Thus, let Fig. 167 represent the vertical cross-section of a reservoir wall, the water pressing against the vertical face AB. It is a fact established by experiment, that the intensity of the pressure of any body of water at any point is proportional to the depth of the point below the free upper level of the water, and normal to the surface pressed upon. Hence, if we suppose the free upper level of the water to be even with the top of the wall, the intensity of the pressure there will be zero; and if we represent by CB the intensity of the pressure at the bottom, then, joining A and

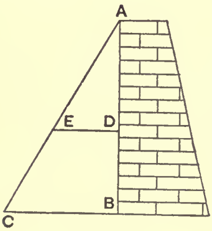


Fig. 167.

C we shall have the intensity of the pressure at any point, as D, represented by ED, where

$$ED : CB = AD : AB.$$

Here, then, we have a case of uniformly varying stress normal to the surface on which it acts.

§ 185. **Fundamental Principles of the Common Theory of the Stresses in Beams under a Transverse Load.** — Fig. 168 shows a beam fixed at one end and loaded at the other, while Fig. 169 shows a beam supported at the ends and loaded at the middle. Let, in each case, the plane of the paper contain a vertical longitudinal section of the beam. In Fig. 168, it is evident that the upper fibres are lengthened, while the lower ones are shortened, and *vice versa* in Fig. 169. In either case, there is, somewhere between the upper and lower fibres, a fibre which is neither elongated nor compressed.

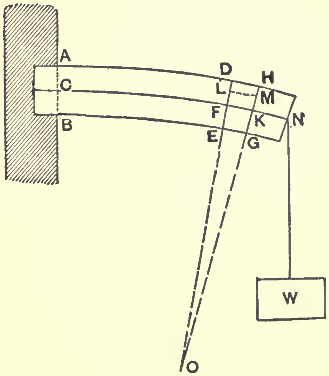


Fig. 168.

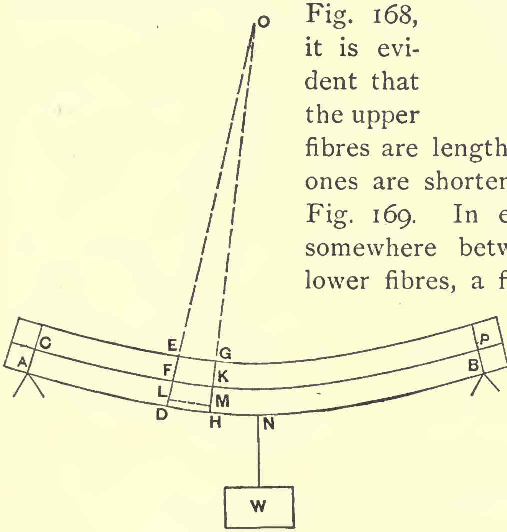


Fig. 169.

Let CN represent that fibre, Fig. 168, and CP, Fig. 169. This line may be called the neutral line of the longitudinal section; and, if a section be made at any point at right angles to this line, the horizontal line which lies in the cross-section, and cuts the neutral lines of all the longitudinal sections, or, in other words, the locus of the points where the neutral lines of the longitudinal sections cut the cross-section, is called the *Neutral Axis* of the cross-section. In the ordinary theory of the stresses in beams, a number of assumptions are made, which will now be enumerated.

ASSUMPTIONS MADE IN THE COMMON THEORY OF BEAMS.

ASSUMPTION NO. 1. — If, when a beam is not loaded, a plane cross-section be made, this cross-section will still be a plane after the load is put on, and bending takes place. From this assumption, we deduce, as a consequence, that, if a certain cross-section be assumed, the elongation or shortening per unit of length of any fibre at the point where it cuts this cross-section, is proportional to the distance of the fibre from the neutral axis of the cross-section.

Proof. — Imagine two originally parallel cross-sections so near to each other that the curve in which that part of the neutral line between them bends may, without appreciable error, be accounted circular. Let ED and GH (Fig. 168 or Fig. 169) be the lines in which these cross-sections cut the plane of the paper, and let O be the point of intersection of the lines ED and GH. Let $OF = r$, $FL = y$, $FK = l$, $LM = l + \alpha l$, in which α is the strain or elongation per unit of length of a fibre at a distance y from the neutral line, y being a variable; then, because FK and LM are concentric arcs subtending the same angle at the centre, we shall have the proportion

$$\frac{r + y}{r} = \frac{l + \alpha l}{l} \quad \text{or} \quad 1 + \alpha = 1 + \frac{y}{r},$$

$$\therefore \alpha = \frac{y}{r} \quad \text{or} \quad \alpha = \left(\frac{1}{r}\right)y;$$

but as y varies for different points in any given cross-section, while r remains the same for the same section, it follows, that, if a certain cross-section be assumed, *the strain of any fibre at the point where it cuts this cross-section is proportional directly to the distance of this fibre from the neutral axis of the cross-section.*

Assumption No. 2. — This assumption is that commonly known as *Hooke's Law.* It is as follows: "*Ut tensio sic vis;*" i.e., The stress is proportional to the strain, or to the elongation or compression per unit of length. As to the evidence in favor of this law, experiment shows, that, as long as the material is not strained beyond safe limits, this law holds. Hence, making these two assumptions, we shall have: *At a given cross-section of a loaded beam, the direct stress on any fibre varies directly as the distance of the fibre from the neutral axis.* Hence it is a uniformly varying stress, and we may represent it graphically as follows: Let $ABCD$, Fig. 170, be the cross-section of a beam, and KL the neutral axis. Assume this for axis OY, and draw the other two axes, as in the figure. If, now, EA be drawn to represent the intensity of the direct (normal) stress at A, then will the pair of wedges $AEFBKL$ and $DCHGKL$ represent the stress graphically, since it is uniformly varying.

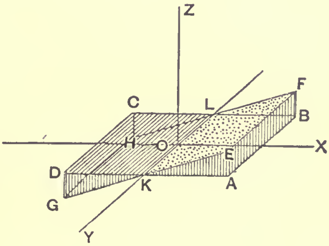


Fig. 170.

POSITION OF NEUTRAL AXIS.

Assumption No. 3.—It will next be shown that, on the two assumptions made above, and from the further assumption that the deformation of each fibre of the beam parallel to its longitudinal axis is due to the forces acting on its ends alone and that it suffers no traction from neighboring fibres, it follows that the neutral axis must pass through the centre of gravity of the cross-section.

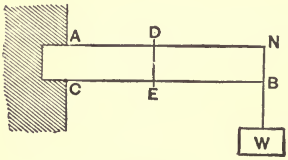


FIG. 171.

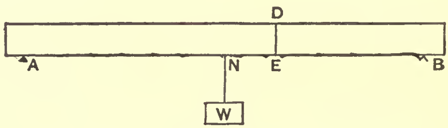


FIG. 172.

Since the curvatures in Figs. 168 and 169 are exaggerated in order to render them visible, Figs. 171 and 172 have been drawn. If, now, we assume a section DE, such that $AD = x$ (Fig. 171) and $NE = x$ (Fig. 172), and consider all the forces acting on that part of the beam which lies to the right of DE (i.e., both the external forces and the stresses which the other parts of the beam exert on this part), we must find them in equilibrium. The external forces are, in Fig. 172, —

1°. The loads acting between B and E; in this case there are none.

2°. The supporting force at B; in this case it is equal to $\frac{W}{2}$, and acts vertically upwards.

In Fig. 171 they are, —

The loads between D and N; in this case there is only the one, W at N.

The internal forces are merely the stresses exerted by the other parts of the beam on this part: they are, —

1°. The resistance to shearing at the section, which is a vertical stress.

2°. The direct stresses, which are horizontal.

Now, since the part of the beam to the right of DE is at rest, the forces acting on it must be in equilibrium; and, since they are all parallel to the plane of the paper, we must have the three following conditions; viz., —

1°. The algebraic sum of the vertical forces must be zero.

2°. The algebraic sum of the horizontal forces must be zero.

3°. The algebraic sum of the moments of the forces about any axis perpendicular to the plane of the paper must be zero.

But, on the above assumptions, the only horizontal forces are the direct stresses: hence the algebraic sum of these direct stresses must be zero; or, in other words, the direct stresses must be equivalent to a statical couple.

Now, it has already been shown, that, whenever a uniformly varying stress amounts to a statical couple, the neutral axis must pass through the centre of gravity of the surface acted upon. Hence in a loaded beam, if the three preceding assumptions be made, it follows that the neutral axis of any cross-section must contain the centre of gravity of that section.

By way of experimental proof of this conclusion, Barlow has shown by experiment, that, in a cast-iron beam of rectangular section, the neutral axis does pass through the centre of gravity of the section.

RÉSUMÉ.

The conclusions arrived at from the foregoing are as follows: —

1°. That at any section of a loaded beam, if a horizontal line be drawn through the centre of gravity of the section, then the fibres lying along this line will be subjected neither to tension nor to compression; in other words, this line will be the neutral axis of the section.

2°. The fibres on one side of this line will be subjected to tension, those on the other side being subjected to compression; the tension or compression of any one fibre being proportional to its distance from the neutral axis.

The first of the three assumptions of the common theory was not accepted by St. Venant, who developed by means of the methods of the Theory of Elasticity a theory of beams based upon the second and third assumptions only. A study of St. Venant's theory involves, however, far more complication, and requires a good previous knowledge of the Theory of Elasticity. Moreover the results of the two theories as far as the determination of the outside fibre-stresses and of the deflections are practically in agreement, while, on the other hand, the intensities of the shearing-forces as computed by the two theories are not in agreement.

The St. Venant theory may be found in several treatises upon the Theory of Elasticity.

§ 186. **Shearing-Force and Bending-Moment.** — In determining the strength of a beam, or the proper dimensions of a beam to bear a certain load, when we assume the neutral axis to pass through the centre of gravity of the cross-section, we have imposed the second of the three last-mentioned conditions of equilibrium. The remaining two conditions may otherwise be stated as follows: —

1°. The total force tending to cause that part of the beam that lies to one side of the section to slide by the other part, must be balanced by the resistance of the beam to shearing at the section.

2°. The resultant moment of the external forces acting on that part of the beam that lies to one side of the section, about a horizontal axis in the plane of the section, must be balanced by the moment of the couple formed by the resisting stresses.

The shearing-force at any section is the force with which the part of the beam on one side of the section tends to slide by the part on the other side. In a beam free at one end, it is equal to the sum of the loads between the section and the free end. In a beam supported at both ends, it is equal in magnitude to the difference between the supporting force at either end, and the sum of the loads between the section and that support.

The bending-moment at any section is the resultant moment of the external forces acting on the part of the beam to one side of the section, these moments being taken about a horizontal axis in the section.

In a beam free at one end, it is equal to the sum of the moments of the loads between the section and the free end, about a horizontal axis in the section.

In a beam supported at both ends, it is the difference between the moment of either supporting force, and the sum of the moments of the loads between the section and that support; all the moments being taken about a horizontal axis in the section.

Hence the two conditions of equilibrium may be more briefly stated as follows: —

1°. The shearing-force at the section must be balanced by the resistance opposed by the beam to shearing at the section.

2°. The bending-moment at the section must be balanced by the moment of the couple formed by the resisting stresses.

It is necessary, therefore, in determining the strength of a beam, to be able to determine the shearing-force and bending-moment at any point, and also the greatest shearing-force and the greatest bending-moment, whatever be the loads.

A table of these values for a number of ordinary cases will now be given; but I should recommend that the table be merely considered as a set of examples, and that the rules already given for finding them be followed in each individual case.

Let, in each case, the length of the beam be l, and the total load W. When the beam is fixed at one end and free at the other, let the origin be taken at the fixed end; when it is supported at both ends, let it be taken directly over one support. Let x be the distance of any section from the origin. Then we shall have the results given in the following table:—

Description of Beam.	Distribution of the Load.		Shearing-Force. At Distance x from Origin.	Shearing-Force. Greatest.	Bending-Moment. At Distance x from Origin.	Bending-Moment. Greatest.
Beam fixed at one end, free at the other,	Single load at free end,		W	W	$W(l-x)$	Wl
	Load uniformly distributed,		$\frac{W}{l}(l-x)$	W	$\frac{W}{2l}(l-x)^2$	$\frac{Wl}{2}$
Beam supported at both ends,	Single load at middle,	Between origin and middle,	$\frac{W}{2}$	$\frac{W}{2}$	$\frac{W}{2}x$	$\frac{Wl}{4}$
		Beyond middle,	$-\frac{W}{2}$		$\frac{W}{2}(l-x)$	
	Load uniformly distributed,		$\frac{W}{l}\left(\frac{l}{2}-x\right)$	$\frac{W}{2}$	$\frac{W}{2l}(lx-x^2)$	$\frac{Wl}{8}$
	Single load at distance a from origin,	Between origin and load,	$\frac{W(l-a)}{l}$	$\frac{W(l-a)}{l}$	$\frac{W(l-a)}{l}x$	$\frac{Wa(l-a)}{l}$
		Beyond load,	$-\frac{Wa}{l}$	$\frac{Wa}{l}$	$\frac{Wa}{l}(l-x)$	

In a beam fixed at one end and free at the other, the greatest shearing-force, and also the greatest bending-moment, are at the fixed end. In a beam supported at both ends, and loaded at the middle, or with a uniformly distributed load, the greatest shearing-force is at either support, the greatest bending-moment being at the middle. In the last case (i.e., that of a beam supported at the ends, and having a single load not at the middle), the greatest bending-moment is at the load; the greatest shearing-force being at that support where the supporting force is greatest.

§ 187. **Moments of Inertia of Sections.** — In the usual methods of determining the strength of a beam or column, it is necessary to know, 1°, the distance from the neutral axis of the section to the most strained fibres; 2°, the moment of inertia of the section about the neutral axis. The manner of finding the moments of inertia has been explained in Chap. II.

In the following table are given the areas of a large number of sections, and also their moments of inertia about the neutral axis, which is the axis YY in each case. These results should be deduced by the student.

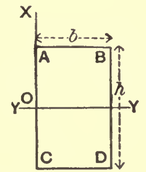
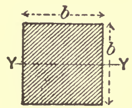
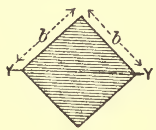
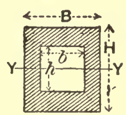
Figure.		Description.	*A.*	*I.*	Distance of *YY* from Extreme Fibres.
	173	*Rectangle*, — Height $= h$ Breadth $= b$	bh	$\frac{bh^3}{12}$	$\frac{h}{2}$
	174	*Square*, — Side $= b$	b^2	$\frac{b^4}{12}$	$\frac{b}{2}$
	175	*Square*, — Side $= b$	b^2	$\frac{b^4}{12}$	$\frac{1}{2}b\sqrt{2}$
	176	*Rectangular cell*, — Outside dimensions $= B$ and H Inside dimensions $= b$ and h	$BH - bh$	$\frac{1}{12}(BH^3 - bh^3)$	$\frac{H}{2}$

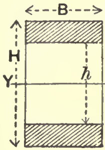
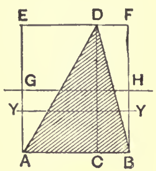
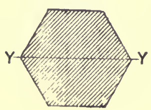
Figure.		Description.	A.	I.	Distance of YY from Extreme Fibres.
	177	*Hollow rectangle*, — Outside dimensions, B and H Inside dimensions, B and h	$B(H-h)$	$\frac{1}{12}B(H^3-h^3)$	$\frac{H}{2}$
	178	*Triangle*, — $AB=b$, $DC=h$	$\frac{bh}{2}$	$\frac{bh^3}{36}$	$YE=\frac{2}{3}h$ $YA=\frac{1}{3}h$
		When axis is AB		$\frac{bh^3}{12}$	
		When axis is EF		$\frac{bh^3}{4}$	
	179	*Regular hexagon*, — Side $=a$	$\frac{3a^2\sqrt{3}}{2}$	$\frac{5a^4\sqrt{3}}{16}=0.541a^4$	$\frac{a\sqrt{3}}{2}=0.866a$

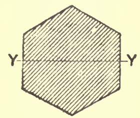
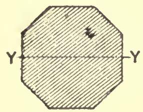
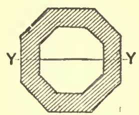
Figure.		Description.	A.	I.	Distance of YY from Extreme Fibres.
	180	*Regular hexagon*	$\frac{3a^2\sqrt{3}}{2}$	$\frac{5a^4\sqrt{3}}{16} = 0.541a^4$	a
	181	*Regular octagon*, — Radius of circumscribed circle $= a$ Length of one side $= 2a \sin 22\frac{1}{2}°$ $= a\sqrt{2 - \sqrt{2}}$ $= 0.765a$	$2a^2\sqrt{2} = 2.828a^2$	$\frac{a^4}{6}(1 + 2\sqrt{2}) = 0.638a^4$	$\frac{a}{2}\sqrt{2 + \sqrt{2}}$ $= 0.924a$ $= a \cos 22\frac{1}{2}°$
	182	*Octagonal cell*, — Radius of circle circumscribed around outer octagon $= a_1$ Radius of circle circumscribed around inner octagon $= a_2$	$(a_1^2 - a_2^2)2\sqrt{2}$ $= (a_1^2 - a_2^2)(2.828)$	$\frac{1 + 2\sqrt{2}}{6}(a_1^4 - a_2^4)$ $= 0.638(a_1^4 - a_2^4)$	$\frac{a}{2}\sqrt{2 + \sqrt{2}}$

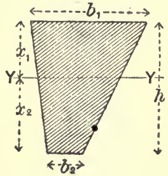
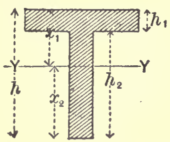
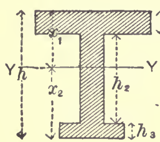
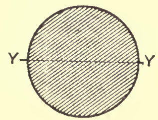
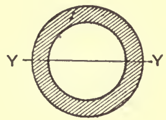
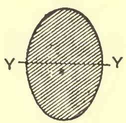
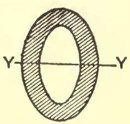
Figure.		Description.	A.	I.	Distance of YY from Extreme Fibres.
	183	*Trapezoid*, — Parallel sides, b_1 and b_2 respectively Height h	$(b_1 + b_2)\frac{h}{2}$	$\frac{h^3}{36} \cdot \frac{b_1^2 + 4b_1b_2 + b_2^2}{b_1 + b_2}$	$x_1 = \frac{b_1 + 2b_2}{b_1 + b_2} \cdot \frac{h}{3}$ $x_2 = \frac{2b_1 + b_2}{b_1 + b_2} \cdot \frac{h}{3}$
	184	**T**-*section*, — Area of flange $= A_1$ Area of web $= A_2$ Total depth $= h$ $= h_1 + h_2$ $A = A_1 + A_2$	$A_1 + A_2$	$\frac{A_1h_1^2 + A_2h_2^2}{12} + \frac{A_1A_2(h_1 + h_2)^2}{4(A_1 + A_2)}$	$x_1 = \frac{h}{2} - \frac{A_1h_2 - A_2h_1}{2(A_1+A_2)}$ $x_2 = \frac{h}{2} + \frac{A_1h_2 - A_2h_1}{2(A_1 + A_2)}$
	185	**I**-*section*, — Area of upper flange $= A_1$ Area of web $= A_2$ Area of lower flange $= A_3$	$A_1+A_2+A_3$	$\frac{A_1h_1^2+A_2h_2^2+A_3h_3^2}{12} + A_1\left(x_1-\frac{h_1}{2}\right)^2 + A_2\left(x_1-h_1-\frac{h_2}{2}\right)^2 + A_3\left(x_2-\frac{h_3}{2}\right)^2$	$x_1 = \frac{A_1\frac{h_1}{2}+A_2\left(h_1+\frac{h_2}{2}\right)+A_3\left(h_1+h_2+\frac{h_3}{2}\right)}{A_1+A_2+A_3}$

Figure.	Description.	*A.*	*I.*	Distance of *YY* from Extreme Fibres.
 186	*Circle,* — Radius r	πr^2	$\frac{\pi r^4}{4}$	r
 187	*Hollow circle,* — Outer radius $= r$ Inner radius $= r_1$	$\pi(r^2 - r_1^2)$	$\frac{\pi(r^4 - r_1^4)}{4}$	r
 188	*Ellipse,* — Vertical axis $= h = 2a$ Horizontal axis $= b_1 = 2b$	$\pi ab = \frac{\pi b_1 h}{4}$	$\frac{\pi a^3 b}{4} = \frac{\pi b_1 h^3}{64}$	$a = \frac{h}{2}$
 189	*Hollow ellipse,* — Outer semi-axes a and b Inner semi-axes a_1 and b_1	$\pi(ab - a_1 b_1)$	$\frac{\pi a^3 b}{4} - \frac{\pi a_1^3 b_1}{4}$	a

In the following table, the cross-sections are considered as composed of their central lines; the area of any given portion being found by multiplying the thickness of the iron by the corresponding length of line, just as was done in the corresponding cases under "Centre of Gravity."

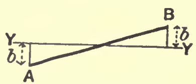
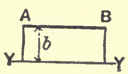
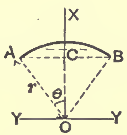
Figure.		Description.	A.	I.	Distance of Axis YY from Extreme Fibres.
	190	Straight line AB about an axis YY through one end	A	$\frac{1}{3}Ab^2$	
	191	Straight line AB about an axis YY through the middle	A	$\frac{1}{3}Ab^2$	b
	192	Straight line AB about about an axis parallel to it	A	Ab^2	
	193	Circular arc AB about an axis YY through the centre of the circle, and parallel to the chord AB; $t =$ thickness of iron	$A = 2r\theta t$	$tr^3(\theta + \sin\theta\cos\theta)$ $= \frac{Ar^2}{2}\left[1 + \frac{\sin\theta\cos\theta}{\theta}\right]$	

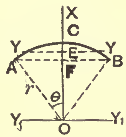
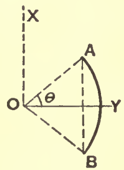
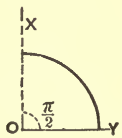
Figure.		Description.	A.	I.	Distance of Axis YY from Extreme Fibres.
	194	Circular arc AB about an axis YY through the centre of gravity, and parallel to the chord	$A = 2r\theta t$	$tr^3\left[\theta + \sin\theta\cos\vartheta - \frac{2\sin^2\theta}{\theta}\right]$ $= \frac{Ar^2}{2}\left[1 + \frac{\sin\theta\cos\theta}{\theta} - \frac{2\,s.r^2\,\theta}{\theta^2}\right]$	$CE = r\left(1 - \frac{\sin\theta}{\theta}\right)$ $FE = r\left(\frac{\sin\theta}{\theta} - \cos\theta\right)$
	195	Circular arc AB about an axis OY at right angles to the chord, and passing through the centre of the circle	$A = 2r\theta t$	$tr^3(\theta - \sin\theta\cos\theta)$ $= \frac{Ar^2}{2}\left[1 - \frac{\sin\theta\cos\theta}{\theta}\right]$	$r\sin\theta$
	196	Quarter-arc of circle about an axis through one end and the centre of the circle	$A = \frac{\pi r}{2}$	$\frac{Ar^2}{2}$	

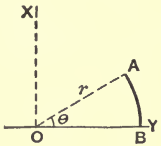
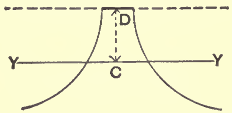
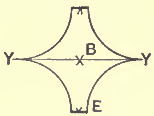
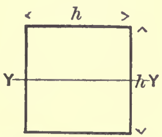
Figure.		Description.	A.	I.	Distance of Axis YY from Extreme Fibres.
	197	Circular arc AB	$A = tr\theta$	$\frac{tr^3}{2}(\theta - \sin\theta\cos\theta)$ $= \frac{Ar^2}{2}\left[1 - \frac{\sin\theta\cos\theta}{\theta}\right]$	
	198	*Barlow rail,*— Two quadrants of circular arcs and a web. If area of arcs $= A$, then web $= \frac{3}{11}A$. YY goes through the centre of gravity of the rail	$\frac{14}{11}A$	$\frac{2}{11}Ar^2$, calling $\pi = \frac{22}{7}$	$\frac{r}{2}$
	199	*Double Barlow rail,*— Joint area of arcs $= 2A$, and of webs $\frac{6}{11}A$	$\frac{28}{11}A$	Ar^2	r
	200	Square cell	$A = 4ht$	$\frac{2}{3}h^3t = \frac{1}{6}Ah^2$	$\frac{h}{2}$

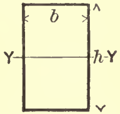
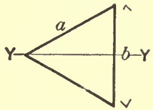
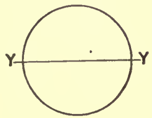
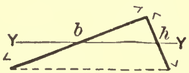
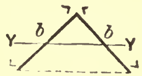
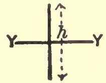
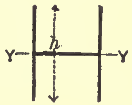
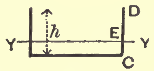
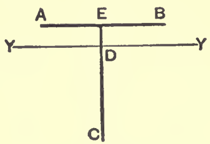
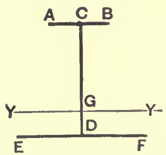
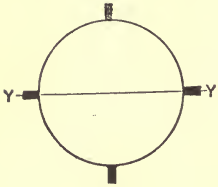
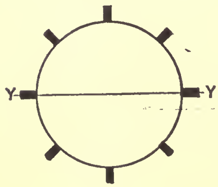
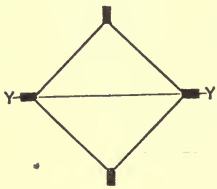
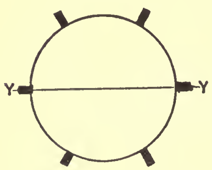
Figure.		Description.	A.	I.	Distance of Axis YY from Extreme Fibres.
	201	Rectangular cell	$A = 2t(b+h)$	$\frac{th^2}{6}(3b + h)$	$\frac{h}{2}$
	202	Triangular cell	$A = t(b+2a)$	$\frac{b^2t}{12}(2a + b) = \frac{Ab^2}{12}$	$\frac{b}{2}$
	203	*Circular cell,*— Radius r	$2\pi rt$	$\pi r^3t = \frac{Ar^2}{2}$	r
	204	*Angle iron,*— Unequal arms, Thickness t, Length of one arm $= b$, and of other $= h$	$t(b + h)$	$\frac{b^2h^2t\,(b + h)}{12\,(b^2 + h^2)} = \frac{Ab^2h^2}{12\,(b^2 + h^2)}$	$\frac{1}{2}\frac{bh}{\sqrt{b^2 + h^2}}$
	205	Angle of equal arms	$2bt$	$\frac{b^3t}{12} = \frac{Ab^2}{24}$	$\frac{1}{2}b\sqrt{2}$

Figure.		Description.	A.	I.	Distance of Axis YY from Extreme Fibres.
	206	Cross of equal arms	$2ht$	$\frac{h^3t}{12} = \frac{Ah^2}{24}$	$\frac{h}{2}$
	207	H-*section*, — Area of web $= B$ Combined area of flanges $= A$	$A + B$	$\frac{Ah^2}{12}$	$\frac{h}{2}$
	208	*Channel-section*, — Area of web $= B$ Combined area of flanges $= A$	$A + B$	$h^2\left[\frac{A}{12} + \frac{AB}{4(A+B)}\right]$	$DE = \frac{h}{2}\frac{A + 2B}{A + B}$ $CE = \frac{h}{2}\frac{A}{A + B}$
	209	T-*section*, — Area of flange $= A$ Area of web $= B$	$A + B$	$\frac{Bh^2}{12}\left(\frac{4A + B}{A + B}\right)$	$DE = \frac{h}{2}\frac{B}{A + B}$ $DC = \frac{h}{2}\frac{2A + B}{A + B}$
	210	I-*iron*, — Area upper flange $= A$ Area lower flange $= A_1$ Area of web $= B$	$A + A_1 + B$	$\frac{h^2}{12}\left[\frac{12AA_1 + 4B(A + A_1) + B^2}{A + A_1 + B}\right]$	$CG = \frac{h}{2}\frac{2A_1 + B}{A + A_1 + B}$ $GD = \frac{h}{2}\frac{2A + B}{A + A_1 + B}$

§ 188. **Cross-Sections of Phœnix Columns considered as made of Lines.** — It is to be observed that the moments of inertia are the same for all axes passing through the centre. Thickness $= t$, radius of round ones $= r$, area of each flange $= a$, length of each flange $= l$.

Figure.	Description.	A.	I.
 211	Four flanges	$2\pi rt + 4a$	$\pi r^3 t + 2a\left(r + \frac{l}{2}\right)^2$
 212	Eight flanges	$2\pi rt + 8a$	$\pi r^3 t + 4a\left(r + \frac{l}{2}\right)^2$
 213	Square, four flanges, $r =$ radius of circumscribed circle	$4rt\sqrt{2} + 4a$	$\frac{4r^3t\sqrt{2}}{3} + 2a\left(r + \frac{l}{2}\right)^2$
 214	Six flanges	$2\pi rt + 6a$	$\pi r^3 t + 3a\left(r + \frac{l}{2}\right)^2$

§ 189. **Graphical Representation of Bending-Moments.** — The bending-moment at each point of a loaded beam may be represented graphically by lines laid off to scale, as will be shown by examples.

I. Suppose we have the cantilever shown in Fig. 215, loaded at D with a load W: then will the bending-moment at any section, as at F, be obtained by multiplying W by FD; that at AC being $W \times (AB)$. If, now, we lay off CE to scale to represent this, i.e., having as many units of length as there are units of moment in the product $W \times (AB)$, and join E with D, then will the ordinate FG of any point, as G, represent (to the same scale) the bending-moment at a section through F.

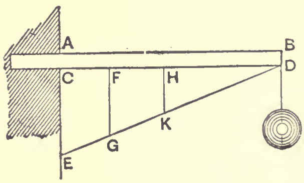


FIG. 215.

II. If we have a uniformly distributed load, we should have, for the line corresponding to CE in Fig. 215, a curve. This is shown in Fig. 216, where we have the uniformly distributed load $EIGF$. If we take the origin at D, as before, we have, for the bending-moment, at a distance x from the origin, as has been shown, $\frac{W}{2l}(l - x)^2$; and by giving x different values, and laying off the corresponding value of the bending-moment, we obtain the curve CA, any ordinate of which will represent the bending-moment at the corresponding point of the beam.

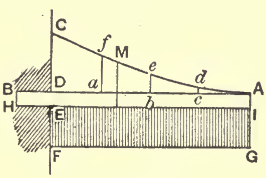


FIG. 216.

When we have more than one load on a beam, we must draw the curve of bending-moments for each load separately, and then find the actual bending-moment at any point of the beam by taking the sum of the ordinates (drawn from that point) of each of these separate curves or straight lines. If we then draw a new curve, whose ordinates are these sums, we shall have the actual curve of bending-moments for the beam as loaded. Some examples will now be given, which will explain themselves.

III. Fig. 217 shows a cantilever with three concentrated loads. The line of bending-moments for the load at C is CE, that for the load at O is OF, and for the load at P is PG. They are combined above the beam by laying off $AH = DE$, $HK = DF$, and $KL = DG$, and thus obtaining the broken line $LMNB$, which is the line of bending-moments of the beam loaded with all three loads.

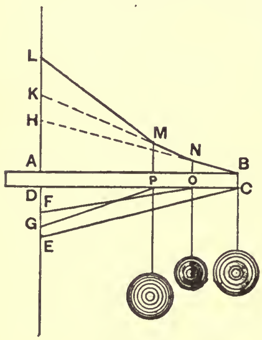


FIG. 217.

IV. Fig. 218 shows the case of a beam supported at both ends, and loaded at a single point D; ALB is the line of bending-moments when the weight of the beam is disregarded, so that $xy =$ bending-moment at x.

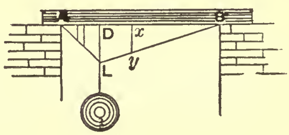


FIG. 218

V. Fig. 219 shows the case of a beam supported at the ends, and loaded with three concentrated loads at the points B, C, and D respectively; the lines of bending-moments for each individual load being respectively AFE, AGE, and AHE, and the actual line of bending-moments being $AKLME$.

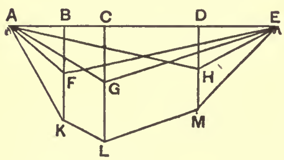


FIG. 219.

VI. Fig. 220 shows the case of a beam supported at the ends, and loaded with a uniformly distributed load; the line of bending-moments being a curve, *ACDB*, as shown in the figure.

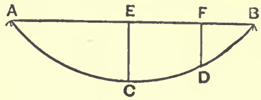


FIG. 220.

VII. In Fig. 221 we have the case of a beam, over a part of which, viz., *EF*, there is a distributed load; the rest of the beam being unloaded. The line of bending-moments is curvilinear between *E* and *F*, and straight outside of these limits. It is *AGSHB;* and, when the curve is plotted, we can find the greatest bending-moment graphically by finding its greatest ordinate. We can also determine it analytically by first determining the bending-moment at a distance x from the origin, and on the side towards the resultant of the load, and then differentiating. This process is shown in the following:—

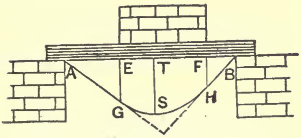


FIG. 221.

Let A (Fig. 222) be the point where the resultant of the load acts, and O the middle of the beam, and let w be the load per unit of length; let $OA = a$, $AB = AC = b$, and $ED = 2c$, so that the whole load $= 2wb$: therefore supporting force at $D = 2wb\dfrac{a+c}{2c} = \dfrac{wb(a+c)}{c}$.

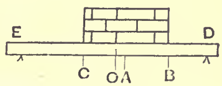


FIG. 222.

If we take a section at a distance x from O to the right, we shall have, for the bending-moment at that section,

$$\frac{wb(a+c)}{c}(c-x) - \frac{w}{2}(a+b-x)^2 = \text{a maximum.}$$

Differentiate, and we have

$$\frac{-wb(a+c)}{c} + w(a+b-x) = 0 \quad \therefore \quad x = \frac{a(c-b)}{c};$$

hence the greatest bending-moment will be

$$\frac{wb(a+c)}{c}\left(c - \frac{a(c-b)}{c}\right) - \frac{w}{2}\left(a+b-a+\frac{ab}{c}\right)^2$$

$$= \frac{wb}{c^2}(a+c)(c^2-ac+ab) - \frac{wb^2}{2c^2}(a^2+2ac+c^2)$$

$$= \frac{wb}{2c^2}(a^2b - 2a^2c + 2c^3 - bc^2).$$

VIII. In Figs. 223 and 224 we have the case of a beam supported at the ends, and loaded with a uniformly distributed load, and also with a concentrated load. In the first figure, the greatest bending-moment is at D, and in the second at C.

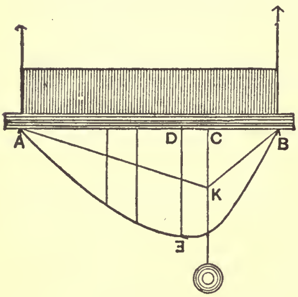


FIG. 223.

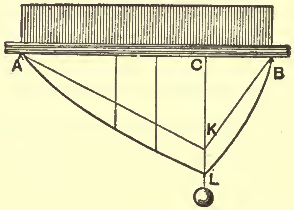


FIG. 224.

IX. In Fig. 225 we have a beam supported at A and B, and loaded at C and D with equal weights; the lengths of AC and BD being equal. We have, consequently, between A and B, a uniform bending-moment; while on the left of A and on the right of B we have a varying bending-moment. The line of bending-moments is, in this case, $CabD$.

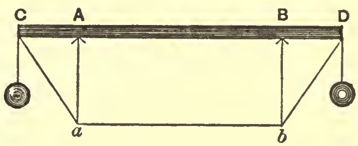


FIG. 225.

We may, in a similar way, derive curves of bending-moment for all cases of loading and supporting beams.

§ 190. Mode of Procedure for Ascertaining the Stresses at Different Parts of a Beam when the Loads and the Dimensions are given, and when no Fibre at the Cross-section under Consideration is Strained beyond the Elastic Limit.—When the dimensions of a beam, the load and its distribution, and the manner of supporting are given, and it is desired to find the actual intensity of the stress on any particular fibre at any given cross-section, we must proceed as follows :—

1°. Find the actual bending-moment (M) at that cross-section.

2°. Find the moment of inertia (I) of the section about its neutral axis.

3°. Observe, that, from what has already been shown, the moment of the couple formed by the tensions and compressions is aI, where $a =$ intensity of stress of a fibre whose distance from the neutral axis is unity, and that this moment must equal the bending-moment at the section in order to secure equilibrium. Hence we must have

$$aI = M.$$

Moreover, if p denote the (unknown) intensity of the stress of the fibre where the stress is desired, and if y denote the distance of this fibre from the neutral axis, we shall have

$$a = \frac{p}{y}, \qquad \therefore \quad \frac{p}{y}I = M, \qquad \therefore \quad p = \frac{My}{I},$$

from which equation we can determine p.

EXAMPLES.

1. Given a beam 18 feet span, supported at both ends, and loaded uniformly (its own weight included) with 1000 lbs. per foot of length. The cross-section is a **T**, where area of flange = 3 square inches, area of web = 4 square inches, height = 10 inches. Find (a) the

bending-moment at 3 feet from one end; (*b*) the greatest bending-moment; (*c*) the greatest intensity of the tension at each of the above sections; (*d*) the greatest intensity of the compression at each of these sections.

2. Given an I-beam with equal flanges, area of each flange = 3 square inches, area of web = 3 square inches, height = 10 inches; the beam is 12 feet long, supported at the ends, and loaded uniformly (its own weight included) with a load of 2000 lbs. per foot of length. Find (*a*) the bending-moment at a section one foot from the end; (*b*) the greatest bending-moment; (*c*) the greatest intensity of the stress at each of the above cross-sections.

§ 191. **Mode of Procedure for Ascertaining the Dimensions of a Beam to bear a Certain Load, or the Load that a Beam of Given Dimensions and Material is Capable of Bearing.** — If we wish to determine the proper dimensions of the beam when the load and its distribution, as well as the manner of supporting, are given, so that it shall nowhere be strained beyond safe limits, or if we wish to determine the greatest load consistent with safety when the other quantities are given, we must impose the condition that the greatest intensity of the tension to which any fibre is subjected shall not exceed the safe working-strength for tension of the material of which the beam is made, and the greatest intensity of the compression to which any fibre is subjected shall not exceed the safe working-strength of the material for compression.

Thus, we must in this case first determine where is the section of greatest bending-moment (this determination sometimes involves the use of the Differential Calculus).

Next we must determine the magnitude of the greatest bending-moment, absolutely if the load and length of the beam are given (if not, in terms of these quantities), and then equate this to the moment of the resisting couple.

Thus, if M_0 is the greatest bending-moment, when the loads are such that no fibre is strained beyond the elastic limit, I_0 the moment of inertia of that section where this greatest bending-moment acts, and if f_t=greatest tensile fibre stress per square inch, f_c= greatest compressive fibre strength per square inch, y_t=distance of most stretched fibre from the neutral axis, and y_c=distance of most compressed fibre from the neutral axis, then will $\frac{f_t}{y_t}$ be the greatest tension per square inch, at a unit's distance from the neutral axis, and $\frac{f_c}{y_c}$ the greatest compression per square inch, at a unit's distance from the neutral axis.

Moreover, in this case, these two ratios are equal, and hence

$$M_0=\frac{f_t}{y_t}I=\frac{f_c}{y_c}I.$$

SAFE OR WORKING-LOAD.

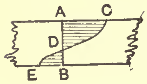
If f_t'=safe working-strength per square inch for tension, f_c'=safe working-strength per square inch for compression, and M_0'=greatest safe working bending-moment, then the ratios, $\frac{f_t'}{y_t}$ and $\frac{f_c'}{y_c}$, are not equal.

Hence, when $\frac{f_t'}{y_t}$ is less than $\frac{f_c'}{y_c}$ we have $M_0'=\frac{f_t'}{y_t}I$, and when $\frac{f_t'}{y_t}$ is greater than $\frac{f_c'}{y_c}$ we have $M_0'=\frac{f_c'}{y_c}I$.

BREAKING-LOAD AND MODULUS OF RUPTURE.

If M is the greatest bending-moment when the beam is subjected to its breaking-load, the formulæ given above do not apply, inasmuch as a portion of the fibres are strained beyond the elastic limit, and Hooke's law no longer holds, since, after the elastic limit is passed, the ratio of stress to strain decreases when the stress increases.

Indeed, the stresses in the different fibres are no longer proportional to the distances of those fibres from the neutral axis. A graphical representation of the stress at different points of any given section AB would be of the character shown in the figure, the form of the curve CDE varying with the shape of the cross-section.



Nevertheless, it is customary to compute the breaking-strength of a beam by means of the formula $f=\frac{My}{I}$, where y is taken as the distance from the neutral axis to that outer fibre which gives way first, i.e., to the most stretched fibre if the beam breaks by tension, or to the most compressed fibre, if it breaks by compression. The quantity f, which may thus be computed from the formula

$$f=\frac{My}{I},$$

is defined as the *Modulus of Rupture.*

Inasmuch as this formula would give the outside fibre stress, if the stress were uniformly varying, it follows that, in the case of materials for which the tensile is less than the compressive strength, the modulus of rupture is greater than the tensile strength, while in that of materials for which the compressive is less than the tensile strength the modulus of rupture is greater than the compressive strength.

For experimental work bearing upon this matter, see an article by Prof. J. Sondericker, in the Technology Quarterly for October, 1888.

WORKING-STRENGTH.

The working-strength per square inch of a material for transverse strength is the greatest stress per square inch to which it is safe to subject the most strained fibre of the beam. It is usually obtained by dividing the modulus of rupture by some factor of safety, as 3 or 4.

§ 192. EXAMPLES.

1. Given a beam (Fig. 226) supported at both ends, and loaded, 1°, with w pounds per unit of length uniformly, and 2°, with a single load W at a distance a from the left-hand support: find the position of the section of greatest bending-moment, and the value of the greatest bending-moment.

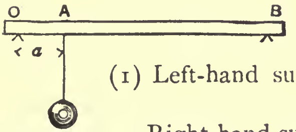


FIG. 226.

Solution.

(1) Left-hand supporting-force $= \dfrac{wl}{2} + \dfrac{W(l-a)}{l}$.

Right-hand supporting-force $= \dfrac{wl}{2} + \dfrac{Wa}{l}$.

(2) Assume a section at a distance x from the left-hand support (this support being the origin), and the bending-moment at that section is,—

when $x < a$, $$\left\{\frac{wl}{2} + \frac{W(l-a)}{l}\right\}x - \frac{wx^2}{2};$$

and when $x > a$,

$$\left\{\frac{wl}{2} + \frac{W(l-a)}{l}\right\}x - \frac{wx^2}{2} - W(x-a).$$

To find the value of x for the section of greatest bending-moment, differentiate each, and put the first differential co-efficient = zero.

We shall thus have, in the first case,

$$\frac{wl}{2} + \frac{W(l-a)}{l} - wx = 0, \text{ or } x = \frac{l}{2} + \frac{W(l-a)}{wl};$$

and in the second case,

$$\frac{wl}{2} + \frac{W(l-a)}{l} - wx - W = 0, \text{ or } x = \frac{l}{2} + \frac{W(l-a)}{wl} - \frac{W}{w}.$$

Now, whenever the first is $< a$, or the second is $> a$, we shall have in that one the value of x corresponding to the section of greatest bending-moment. But if the first is $> a$, *and* the second $< a$, then the greatest bending-moment is at the concentrated load.

These conclusions will be evident on drawing a diagram representing the bending-moments graphically, as in Figs. 223 and 224; and the greatest bending-moment may then be found by substituting, in the corresponding expression for the bending-moment, the deduced value of x.

2. Given an I-beam, 10 feet long, supported at both ends, and loaded, at a distance 2 feet to the left of the middle, with 20000 pounds. Find the bending-moment at the middle, the greatest bending-moment, also the greatest intensity of the tension, and that of the compression at each of these sections.

Given Area of upper flange = 8 sq. in.
Area of lower flange = 5 sq. in.
Area of web = 7 sq. in.
Total depth = 14 in.

§ 193. **Beams of Uniform Strength.** — A beam of uniform strength (technically so called) is one in which the dimensions of the cross-section are varied in such a manner, that, at each cross-section, the greatest intensity of the tension shall be the same, and so also the greatest intensity of the compression.

Such beams are very rarely used; and, as the cross-section varies at different points, it would be decidedly bad engineering to make them of wood, for it would be necessary to cut the wood across the grain, and this would develop a tendency to split.

In making them of iron, also, the saving of iron would generally be more than offset by the extra cost of rolling such a beam. Nevertheless, we will discuss the form of such beams in the case when the section is rectangular.

In all cases we have the general equation

$$M = \frac{p}{y} I$$

applying at each cross-section, where M = bending-moment (section at distance x from origin), I = moment of inertia of same section, y = distance from neutral axis to most strained fibre, and p = intensity of stress on most strained fibre; the condition for this case being that p is a constant for all values of x (i.e., for all positions of the section), while M, I, and y are functions of x.

As we are limiting ourselves to rectangular sections, if we let $b =$ breadth and $h =$ depth of rectangle (one or both varying with x), we shall have

$$M = \frac{p}{6} bh^2$$

as the condition for such a beam, with p a constant for all values of x, when the same load remains on the beam.

We must, therefore, have bh^2 proportional to M. Hence, assuming the origin as before,

1°. Fixed at one end, load at the other, $bh^2 = \left(\frac{6}{p}\right) W(l - x)$.

2°. Fixed at one end, uniformly loaded, $bh^2 = \left(\frac{6}{p}\frac{W}{2l}\right)(l - x)^2$.

3°. Supported at ends, loaded at middle, $\begin{cases} \text{for } x < \frac{l}{2}, & bh^2 = \left(\frac{6}{p}\frac{W}{2}\right)x; \\ \text{for } x > \frac{l}{2}, & bh^2 = \left(\frac{6}{p}\frac{W}{2}\right)(l - x). \end{cases}$

4°. Supported at ends, uniformly loaded, $bh^2 = \left(\frac{6}{p}\frac{W}{2l}\right)(lx - x^2)$.

Now, this variation of section may be accomplished in one of two ways: 1st, by making h constant, and letting b vary; and 2d, by making b constant, and letting h vary. Thus, in the first case above mentioned, if h is constant, we have, for the plan of the beam,

$$b = \left(\frac{6W}{ph^2}\right)(l - x):$$

and if one side be taken parallel to the axis of the beam, this will be the equation of the other side; and, as this is the equation of a straight line, the plan will be a triangle.

If, on the other hand, b be constant, and h vary, we shall have, for the vertical longitudinal section of the beam,

$$h^2 = \left(\frac{6W}{pb}\right)(l - x);$$

and, if one side be taken as a straight line in the direction of the axis, the other will be a parabola.

A similar reasoning will give the plan or elevation respectively in each case; and these can be readily plotted from their equations.

CROSS-SECTION OF EQUAL STRENGTH.

A cross-section of equal strength (technically so called) is one so proportioned that the greatest intensity of the tension shall bear the same ratio to the breaking tensile strength of the material as the greatest intensity of the compression bears to the breaking compressive strength of the material. This is accomplished, as will be shown directly, by so arranging the form and dimensions of the section that the distance of the neutral axis from the most stretched fibre shall bear to its distance from the most compressed fibre the same ratio that the tensile bears to the compressive strength of the material.

Let f_c = breaking-strength per square inch for compression,
f_t = breaking-strength per square inch for tension,
y_c = distance of neutral axis from most compressed fibre,
y_t = distance of neutral axis from most stretched fibre.

If p_c = actual greatest intensity of compression, and p_t = actual greatest intensity of tension, then, for a cross-section of equal strength, we must have, according to the definition, $\frac{p_c}{p_t} = \frac{f_c}{f_t}$; but we have $\frac{p_c}{y_c} = \frac{p_t}{y_t}$ = intensity of stress at a unit's distance from the neutral axis. Hence, combining these two, we obtain

$$\frac{y_c}{y_t} = \frac{f_c}{f_t}.$$

EXAMPLE.

Suppose we have $f_c = 80000$ lbs. per square inch, and $f_t = 20000$ lbs. per square inch.: find the proper proportion between the flange A_1 and the web A_2 of a **T**-section whose depth is h.

§ 194. **Deflection of Beams.**—We have already seen (§ 185), that, in the case of a beam which is bent by a transverse load, we have

$$\alpha = \frac{1}{r}(y),$$

where (having assumed a certain cross-section whose distance from the origin is x) $\alpha =$ the *strain* of a fibre whose distance from the neutral axis is y, and $r =$ radius of curvature of the neutral lamina at the section in question. Hence follows the equation

$$\frac{1}{r} = \frac{\alpha}{y};$$

but from the definition of E, the modulus of elasticity, we shall have

$$\alpha = \frac{p}{E},$$

where $p =$ intensity of the stress at a distance y from the neutral axis.

Hence it follows, assuming Hooke's law, that

$$\frac{1}{r} = \frac{p}{Ey} = \frac{1}{E}\frac{p}{y}$$

We have already seen, that, disregarding signs, $M = \frac{p}{y} I$

(making, of course, the two assumptions already spoken of when this formula was deduced), where $M =$ bending-moment at, and $I =$ moment of inertia of, the section in question; i.e., of that section whose distance from the origin is x. This gives $\frac{p}{y} = -\frac{M}{I}$, if, denoting tension by the $+$ sign, and taking y positive upwards, we call M positive when it tends to cause tension on the lower, and compression on the upper, side; these being the conventions in regard to signs which we shall adopt in future. Hence, by substitution, we have

$$\frac{1}{r} = \frac{p}{Ey} = -\frac{M}{EI}. \qquad (1)$$

Now, if we assume the axis of x coincident with the neutral line of the central longitudinal section of the beam, and the axis of v at right angles to this, and v positive upwards, no matter where the origin is taken, we shall always have, as is shown in the Differential Calculus,

$$\frac{1}{r} = \frac{-\frac{d^2v}{dx^2}}{\left(1 + \left(\frac{dv}{dx}\right)^2\right)^{\frac{3}{2}}}.$$

Hence equation (1) becomes

$$\frac{\frac{d^2v}{dx^2}}{\left(1 + \left(\frac{dv}{dx}\right)^2\right)^{\frac{3}{2}}} = \frac{M}{EI}, \qquad (2)$$

M and I being functions of x: and, when we can integrate this equation, we can obtain v in terms of x, thus having the equation of the elastic curve of the neutral line; and, by computing the value of v corresponding to any assumed value of x, we can obtain the deflection at that point of the beam.

The above equation (2) is, as a rule, too complicated to be integrated, except by approximation; and the approximation usually made is the following: —

Since in a beam not too heavily loaded, the slope, and consequently the tangent of the slope (or angle the neutral line makes with the horizontal at any point), is necessarily small, it follows that $\frac{dv}{dx}$ is very small, and hence $\left(\frac{dv}{dx}\right)^2$ is also very small, and $1 + \left(\frac{dv}{dx}\right)^2$ is nearly equal to unity. Making this substitution, we obtain, in place of equation (2),

$$\frac{d^2v}{dx^2} = \frac{M}{EI}; \qquad (3)$$

and this is the equation with which we always start in computing the slope and deflection of a loaded beam, or in finding the equation of the elastic line.

By one integration (suitably determining the arbitrary constant) we obtain the slope whose tangent is $\frac{dv}{dx}$, and by a second integration we obtain the deflection v at a distance x from the origin; and thus, by substituting any desired value for x, we can obtain the deflection at any point.

§ 195. **Ordinary Formulæ for Slope and Deflection.** — We may therefore write, if i is the circular measure of the slope at a distance x from the origin, since $i = \tan i = \frac{dv}{dx}$ nearly,

$$\frac{d^2v}{dx^2} = \frac{M}{EI},$$

$$i = \frac{dv}{dx} = \int \frac{M}{EI} dx,$$

$$v = \int\int \frac{M}{EI} dx^2.$$

In these equations, of course, E is taken as a constant, M must ALWAYS be expressed in terms of x, and so also must I whenever the section varies at different points. When, however, the section is uniform, I is constant, and the formulæ reduce to

$$i = \frac{1}{EI}\int M dx, \qquad v = \frac{1}{EI}\int\int M dx^2.$$

§ 196. **Special Cases.** — 1°. Let us take a cantilever loaded with a single load at the free end. Assume the origin, as before, at the fixed end, and let the beam be one of uniform section. We then have $M = -W(l - x)$,

$$\therefore \quad i = -\frac{W}{EI}\int (l - x) dx = -\frac{W}{EI}\left(lx - \frac{x^2}{2}\right) + c.$$

To determine c, observe that when $x = 0$, $i = 0$;

$$\therefore \quad c = 0 \qquad \therefore \quad i = -\frac{W}{EI}\left(lx - \frac{x^2}{2}\right) \qquad (1)$$

is the slope at a distance x from the origin.

The deflection at the same point will be

$$v = \int i dx = -\frac{W}{EI}\int\left(lx - \frac{x^2}{2}\right)dx = -\frac{W}{EI}\left(\frac{lx^2}{2} - \frac{x^3}{6}\right) + c;$$

but when $x = 0$, $v = 0$ $\quad\therefore\quad c = 0 \quad\therefore\quad$ the deflection at a distance x from the origin will be

$$v = -\frac{W}{EI}\left(\frac{lx^2}{2} - \frac{x^3}{6}\right). \qquad (2)$$

The equations (1) and (2) give us the means of finding the slope and deflection at any point of the beam.

To find the greatest slope and deflection, we have that both expressions are greatest when $x = l$. Hence, if i_0 and v_0 represent the greatest slope and deflection respectively,

$$i_0 = -\frac{Wl^2}{2EI}, \qquad v_0 = -\frac{Wl^3}{3EI}.$$

2°. Next take the case of a beam supported at both ends and loaded uniformly, the load per unit of length being w.

Assume the origin at the left-hand end; then

$$M = \frac{wl}{2}x - \frac{wx^2}{2} = \frac{w}{2}(lx - x^2) \qquad \text{and} \qquad W = wl$$

$$\therefore \quad i = \frac{w}{2EI}\int (lx - x^2)dx = \frac{w}{2EI}\left(\frac{lx^2}{2} - \frac{x^3}{3}\right) + c.$$

To determine c, we have that when $x = \frac{l}{2}$, then $i = 0$;

$$\therefore \quad 0 = \frac{w}{2EI}\left(\frac{l^3}{8} - \frac{l^3}{24}\right) + c \qquad \therefore \quad c = -\frac{wl^3}{24EI}$$

$$\therefore \quad i = \frac{w}{2EI}\left(\frac{lx^2}{2} - \frac{x^3}{3}\right) - \frac{wl^3}{24EI} = \frac{w}{24EI}(6lx^2 - 4x^3 - l^3) \quad (1)$$

$$\therefore \quad v = \int idx = \frac{w}{24EI}\int (6lx^2 - 4x^3 - l^3)dx$$

$$= \frac{w}{24EI}(2lx^3 - x^4 - l^3x) + c.$$

But when $x = 0$, $v = 0$;

$$\therefore \quad c = 0$$

$$\therefore \quad v = \frac{w}{24EI}(2lx^3 - x^4 - l^3x). \qquad (2)$$

For the greatest slope, we have $x = 0$, or $x = l$;

$$\therefore \quad i_0 = \frac{-wl^3}{24EI} = \frac{-Wl^2}{24EI}.$$

For the greatest deflection, $x = \frac{l}{2}$;

$$\therefore \quad v_0 = \frac{-w}{24EI}\frac{5l^4}{16} = \frac{-5wl^4}{384EI} = \frac{-5Wl^3}{384EI}.$$

3°. Take the case of a beam supported at both ends, and loaded at the middle with a load W.

Assume, as before, the origin at the left-hand support. Then we shall have

$$M = \frac{W}{2}x, \quad x < \frac{l}{2}, \quad \text{and} \quad M = \frac{W}{2}(l - x) \text{ when } x > \frac{l}{2}.$$

Therefore, for the slope up to the middle, we have

$$i = \frac{W}{2EI}\int xdx = \frac{W}{2EI}\frac{x^2}{2} + c.$$

When $x = \frac{l}{2}$, then $i = 0$;

$$\therefore \quad c = -\frac{Wl^2}{16EI};$$

$$\therefore \quad i = \frac{W}{4EI}\left(x^2 - \frac{l^2}{4}\right), \tag{1}$$

and

$$v = \frac{W}{4EI}\int\left(x^2 - \frac{l^2}{4}\right)dx = \frac{W}{4EI}\left(\frac{x^3}{3} - \frac{l^2x}{4}\right) + c.$$

But when $x = 0$, $v = 0$;

$$\therefore \quad c = 0.$$

$$\therefore \quad v = \frac{W}{4EI}\left(\frac{x^3}{3} - \frac{l^2x}{4}\right). \tag{2}$$

The slope is greatest when $x = 0$;

$$\therefore \quad i_0 = \frac{-Wl^2}{16EI}.$$

The deflection is greatest when $x = \frac{l}{2}$;

$$\therefore \quad v_0 = \frac{-Wl^3}{48EI}.$$

4°. In the following table I denotes the moment of inertia of the largest section:

Uniform Cross-Section.	Greatest Slope.	Greatest Deflection.
Fixed at one end, loaded at the other .	$\frac{1}{2}\frac{Wl^2}{EI}$	$\frac{1}{3}\frac{Wl^3}{EI}$
Fixed at one end, loaded uniformly . .	$\frac{1}{6}\frac{Wl^2}{EI}$	$\frac{1}{8}\frac{Wl^3}{EI}$
Supported at ends, load at middle . . .	$\frac{1}{16}\frac{Wl^2}{EI}$	$\frac{1}{48}\frac{Wl^3}{EI}$
Supported at ends, uniformly loaded . .	$\frac{1}{24}\frac{Wl^2}{EI}$	$\frac{5}{384}\frac{Wl^3}{EI}$
Uniform Strength and Uniform Depth, Rectangular Section.		
Fixed at one end, load at the other . .	$\frac{Wl^2}{EI}$	$\frac{1}{2}\frac{Wl^3}{EI}$
Fixed at one end, uniformly loaded . .	$\frac{1}{2}\frac{Wl^2}{EI}$	$\frac{1}{4}\frac{Wl^3}{EI}$
Supported at both ends, load at middle .	$\frac{1}{8}\frac{Wl^2}{EI}$	$\frac{1}{32}\frac{Wl^3}{EI}$
Supported at both ends, uniformly loaded,	$\frac{1}{16}\frac{Wl^2}{EI}$	$\frac{1}{64}\frac{Wl^3}{EI}$
Uniform Strength and Uniform Breadth, Rectangular Section.		
Fixed at one end, loaded at the other,	$2\frac{Wl^2}{EI}$	$\frac{2}{3}\frac{Wl^3}{EI}$
Supported at both ends, load at middle,	$\frac{1}{4}\frac{Wl^2}{EI}$	$\frac{1}{24}\frac{Wl^3}{EI}$
Supported at both ends, uniformly loaded	$0.098\frac{Wl^2}{EI}$	$0.018\frac{Wl^3}{EI}$

§ 197. **Deflection with Uniform Bending-Moment.** — If the bending-moment is uniform, then M is constant; and, if I is also constant, we have

$$i = \frac{M}{EI}\int dx = \frac{Mx}{EI} + c;$$

but when $x = \frac{l}{2}$, then $i = 0$;

$$\therefore \quad c = -\frac{Ml}{2EI}$$

$$\therefore \quad i = \frac{M}{EI}\left(x - \frac{l}{2}\right) = \frac{dv}{dx}$$

$$\therefore \quad v = \frac{M}{EI}\left(\frac{x^2}{2} - \frac{lx}{2}\right),$$

the constant disappearing because $v = 0$ when $x = 0$.

Hence, for a beam where the bending-moment is uniform, we have

$$i = \frac{M}{EI}\left(x - \frac{l}{2}\right), \quad v = \frac{M}{EI}\left(\frac{x^2}{2} - \frac{lx}{2}\right);$$

and for greatest slope and deflection, we have

$$i_0 = \frac{-Ml}{2EI}, \quad v_0 = \frac{M}{EI}\left(\frac{l^2}{8} - \frac{l^2}{4}\right) = -\frac{1}{8}\frac{Ml^2}{EI}.$$

§ 198. **Resilience of a Beam.** — *The resilience of a beam is the mechanical work performed in deflecting it to the amount it would deflect under its greatest allowable gradually applied load.* In the case of a concentrated load, if W is the greatest allowable gradually applied load, and v_1 the corresponding deflection at the point of application of the load, then will the mean value of the load that produces this deflection be $\frac{W}{2}$, and the resilience of the beam will be $\frac{W}{2}v_1$.

§ 199. **Slope and Deflection of a Beam with a Concentrated Load not at the Middle.** — Take, as the next case, a beam (Fig. 228). Let the load at A be W, and distance $OA = a$, and let $a > \frac{l}{2}$.

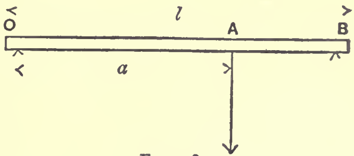


FIG. 228.

$$x < a \quad M = \frac{W(l-a)}{l}x,$$

$$x > a \quad M = \frac{Wa}{l}(l-x),$$

$$\therefore \quad x < a \quad i = \frac{W(l-a)}{lEI}\int x dx = \frac{W(l-a)}{2lEI}x^2 + c.$$

When $x = 0$, $i = i_0 =$ undetermined slope at O;

$$\therefore \quad c = i_0, \qquad \therefore \quad i = \frac{W(l-a)}{2lEI}x^2 + i_0, \qquad (1)$$

and

$$\therefore \quad v = \frac{W(l-a)}{2lEI}\int x^2 dx + i_0\int dx = \frac{W(l-a)}{6lEI}x^3 + i_0 x + c.$$

When $x = 0$, $v = 0$;

$$\therefore \quad v = \frac{W(l-a)}{6lEI}x^3 + i_0 x, \qquad (2)$$

$$x > a \quad i = \frac{Wa}{lEI}\int(l-x)dx = \frac{Wa}{lEI}\left(lx - \frac{x^2}{2}\right) + c.$$

To determine c, observe that when $x = a$, this value of i and that deduced from (1) must be identical.

$$\frac{Wa}{lEI}\left(la - \frac{a^2}{2}\right) + c = \frac{W(l-a)a^2}{2lEI} + i_0 \qquad \therefore \quad c = -\frac{Wa^2}{2EI} + i_0$$

$$\therefore \quad i = \frac{Wa}{lEI}\left(lx - \frac{x^2}{2}\right) - \frac{Wa^2}{2EI} + i_0,$$

or

$$i = \frac{Wa}{2lEI}(2lx - x^2 - la) + i_0, \qquad (3)$$

and

$$v = \frac{Wa}{2lEI}\int(2lx - x^2 - la)dx + i_0\int dx$$

$$= \frac{Wa}{2lEI}\left(lx^2 - \frac{x^3}{3} - lax\right) + i_0x + c.$$

To determine c, observe that when $x = a$, this value of v and (2) must be identical;

$$\therefore \quad \frac{Wa}{2lEI}\left(-\frac{a^3}{3}\right) + i_0a + c = \frac{W(l-a)}{6lEI}a^3 + i_0a$$

$$\therefore \quad c = \frac{W}{6lEI}(la^3 - a^4 + a^4) = \frac{Wla^3}{6lEI} = \frac{Wa^3}{6EI}$$

$$\therefore \quad v = \frac{Wa}{6lEI}(3lx^2 - x^3 - 3lax + la^2) + i_0x. \qquad (4)$$

To determine i_0, we have that when $x = l$, $v = 0$;

$$\therefore \quad 0 = \frac{Wa}{6lEI}(2l^3 - 3al^2 + la^2) + i_0l$$

$$\therefore \quad i_0 = \frac{Wa}{6l^2EI}(3al^2 - 2l^3 - la^2) = \frac{Wa}{6lEI}(3al - 2l^2 - a^2).$$

Substituting this value of i_0 in the equations (1), (2), (3), and (4), we obtain for

$$(1) \quad i = \frac{W(l-a)}{2lEI}x^2 + \frac{Wa}{6lEI}(3al - 2l^2 - a^2),$$

$$(2) \quad v = \frac{W(l-a)}{6lEI}x^3 + \frac{Wa}{6lEI}(3al - 2l^2 - a^2)x,$$

$$(3)\ i = \frac{Wa}{2lEI}(2lx - x^2 - al) + \frac{Wa}{6lEI}(3al - 2l^2 - a^2),$$

$$(4)\ v = \frac{Wa}{6lEI}(3lx^2 - x^3 - 3lax + la^2) + \frac{Wa}{6lEI}(3al - 2l^2 - a^2)x.$$

To find the greatest deflection, differentiate (2), and place the first differential co-efficient equal to zero: or, which is the same thing, place $i = 0$ in (1), and find the value of x; then substitute this value in (2), and we shall have the greatest deflection.

We thus obtain

$$(l - a)x^2 = \frac{-a}{3}(3al - 2l^2 - a^2) \qquad \therefore\ x^2 = \frac{a}{3}\left(\frac{2l^2 - 3al + a^2}{l - a}\right),$$

or

$$x^2 = \frac{a}{3}(2l - a) \qquad \therefore\ x = \frac{\sqrt{2al - a^2}}{\sqrt{3}};$$

and the greatest deflection becomes

$$v_0 = -\frac{Wa(l - a)(2l - a)}{9lEI}\frac{\sqrt{2al - a^2}}{\sqrt{3}}.$$

§ 200. EXAMPLES.

1. In example 1, p. 294, find the greatest deflection of the beam when it is loaded with $\frac{1}{4}$ of its breaking-load, assuming $E = 1200000$.

2. In the same case, find what load will cause it to deflect $\frac{1}{400}$ of its span.

3. What will be the stress at the most strained fibre when this occurs.

4. In example 3, p. 294, find the load the beam will bear without deflecting more than $\frac{1}{400}$ of its span, assuming $E = 24000000$.

5. Find the stress at the most strained fibre when this occurs.

6. In example 6, p. 295, find the greatest deflection under a load $\frac{1}{4}$ the breaking-load.

§ 201. **Deflection and Slope under Working-Load.** — If we take the four cases of deflection given in the first part of the table on p. 305, and calling f the working strength of the material, and y the distance of the most strained fibre from the neutral axis, and if we make the applied load the working-load, we shall have respectively —

$$1°. \quad Wl = \frac{fI}{y} \qquad \therefore \quad W = \frac{fI}{ly}.$$

$$2°. \quad \frac{Wl}{2} = \frac{fI}{y} \qquad \therefore \quad W = \frac{2fI}{ly}.$$

$$3°. \quad \frac{Wl}{4} = \frac{fI}{y} \qquad \therefore \quad W = \frac{4fI}{ly}.$$

$$4°. \quad \frac{Wl}{8} = \frac{fI}{y} \qquad \therefore \quad W = \frac{8fI}{ly}.$$

And the values of slope and deflection will become respectively,

	Slope.	Deflection.		Slope.	Deflection.
1°.	$\frac{1}{2}f\frac{l}{Ey}$	$\frac{1}{3}f\frac{l^2}{Ey}$	3°.	$\frac{1}{4}f\frac{l}{Ey}$	$\frac{1}{12}\frac{fl^2}{Ey}$
2°.	$\frac{1}{3}f\frac{l}{Ey}$	$\frac{1}{4}f\frac{l^2}{Ey}$	4°.	$\frac{1}{3}f\frac{l}{Ey}$	$\frac{5}{48}f\frac{l^2}{Ey}$

From these values, and those given on p. 305, we derive the following two propositions: —

1°. If we have a series of beams differing only in length and we apply the same load in the same manner to each, their greatest slopes will vary as the squares of their lengths, and their greatest deflections as the cubes of their lengths.

2°. If, however, we load the same beams, not with the same load, but each one with its working-load, as determined by allowing a given greatest fibre stress, then will their greatest slopes vary as the lengths, and their greatest deflections as the squares of their lengths.

§ 202. **Slope and Deflection of Rectangular Beams.** — If the beams are rectangular, so that $I = \frac{bh^3}{12}$ and $y = \frac{h}{2}$, the values of slope and deflection above referred to become further simplified, and we have the following tables: —

	Given Load W.		Working-Load. Greatest Fibre Stress $= f$.	
	Slope.	Deflection.	Slope.	Deflection.
1°.	$\frac{6Wl^2}{Ebh^3}$	$\frac{4Wl^3}{Ebh^3}$	$\frac{fl}{Eh}$	$\frac{2}{3}\frac{fl^2}{Eh}$
2°.	$\frac{2Wl^2}{Ebh^3}$	$\frac{3}{2}\frac{Wl^3}{Ebh^3}$	$\frac{2}{3}\frac{fl}{Eh}$	$\frac{1}{2}\frac{fl^2}{Eh}$
3°.	$\frac{3}{4}\frac{Wl^2}{Ebh^3}$	$\frac{1}{4}\frac{Wl^3}{Ebh^3}$	$\frac{1}{2}\frac{fl}{Eh}$	$\frac{1}{6}\frac{fl^2}{Eh}$
4°.	$\frac{1}{2}\frac{Wl^2}{Ebh^3}$	$\frac{5}{32}\frac{Wl^3}{Ebh^3}$	$\frac{2}{3}\frac{fl}{Eh}$	$\frac{5}{24}\frac{fl^2}{Eh}$

So that, in the case of rectangular beams similarly loaded and supported, we may say that —

Under a given load W, the slopes vary as the squares of the lengths, and inversely as the breadths and the cubes of the depths; while the deflections vary as the cubes of the lengths, and inversely as the breadths and the cubes of the depths.

On the other hand, under their working-loads, the slopes vary directly as the lengths, and inversely as the depths; while the deflections vary as the squares of the lengths, and inversely as the depths.

§ 203. **Beams Fixed at the Ends.** — The only cases which we shall discuss here are the two following; viz., —

1°. Uniform section loaded at the middle.

2°. Uniform section, load uniformly distributed.

CASE I. — *Uniform Section loaded at the Middle.* — The fixing at the ends may be effected by building the beam for some distance into the wall, as shown in Fig. 229. The same result, as far as the effect on the beam is concerned, might be effected as follows: Having merely supported it, and placed upon it the loads it has to bear, load the ends outside of the supports just enough to make the tangents at the supports horizontal.

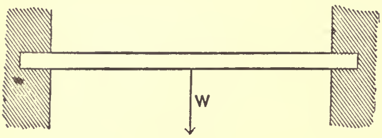


FIG. 229.

These loads on the ends would, if the other load was removed, cause the beam to be convex upwards: and, moreover, the bending-moment due to this load would be of the same amount at all points between the supports; i.e., a uniform bending-moment. Moreover, since the effect of the central load and the loads on the ends is to make the tangents over the supports horizontal, it follows that the upward slope at the support due to the uniform bending-moment above described must be just equal in amount to the downward slope due to the load at the middle, which occurs when the beam is only supported.

Hence the proper method of proceeding is as follows: —

1°. Calculate the slope at the support as though the beam were supported, and not fixed, at the ends; and we shall have, if we represent this slope by i_1, the equation

$$i_1 = -\frac{Wl^2}{16EI}. \qquad (1)$$

2°. Determine the uniform bending-moment which would produce this slope.

To do this, we have, if we represent this uniform bending-moment by M_1, that the slope which it would produce would be

$$-\frac{M_1 l}{2EI}; \qquad (2)$$

and, since this is equal to i_1, we shall have the equation

$$-\frac{M_1 l}{2EI} - \frac{Wl^2}{16EI} = 0 \qquad (3)$$

$$\therefore \quad M_1 = -\frac{Wl}{8}. \qquad (4)$$

This is the actual bending-moment at either fixed end; and the bending-moment at any special section at a distance x from the origin will be

$$M + M_1,$$

where M is the bending-moment we should have at that section if the beam were merely supported, and not fixed. Hence, when it is fixed at the ends, we shall have, for the bending-moment at a distance x from O, where O is at the left-hand support,

$$M = \frac{W}{2}x - \frac{W}{8}l. \qquad (5)$$

When $x = \frac{l}{2}$, we obtain, as bending-moment at the middle,

$$M_0 = \frac{Wl}{8}; \qquad (6)$$

and, since $M_1 = -M_0$, it follows that the greatest bending-moment is

$$\frac{Wl}{8};$$

this being the magnitude of the bending-moment at the middle and also at the support.

POINTS OF INFLECTION.

The value of M becomes zero when

$$x = \frac{l}{4} \text{ and when } x = \frac{3l}{4};$$

hence it follows that at these points the beam is not bent, and that we thus have two points of inflection half-way between the middle and the supports.

SLOPE AND DEFLECTION UNDER A GIVEN LOAD.

We shall have, as before,

$$i = \int \frac{M}{EI} dx = \frac{Wx^2}{4EI} - \frac{Wlx}{8EI} + c;$$

and since, when $x = 0$, $i = 0$,

$$\therefore \quad c = 0$$

$$\therefore \quad i = \frac{dv}{dx} = \frac{W}{8EI}(2x^2 - lx) \qquad (7)$$

$$\therefore \quad v = \frac{W}{8EI}\left(\frac{2x^3}{3} - \frac{lx^2}{2}\right), \qquad (8)$$

the constant vanishing because $v = 0$ when $x = 0$. The slope becomes greatest when $x = \frac{l}{4}$, and the deflection when $x = \frac{l}{2}$. Hence for greatest slope and deflection, we have

$$i_0 = -\frac{Wl^2}{64EI}, \qquad (9)$$

$$v_0 = -\frac{Wl^3}{192EI}. \qquad (10)$$

SLOPE AND DEFLECTION UNDER THE WORKING-LOAD.

If f represent the working-strength of the material per square inch, and if W represent the centre working-load, we shall have

$$\frac{Wl}{8} = \frac{fI}{y}$$

$$\therefore \quad W = \frac{8fI}{ly} \qquad (11)$$

$$\therefore \quad i_0 = -\frac{1}{8}\frac{fl}{Ey} \quad (12) \qquad\qquad v_0 = -\frac{1}{24}\frac{fl^2}{Ey}. \quad (13)$$

CASE II. — *Uniform Section, Load uniformly Distributed.* — Pursuing a method entirely similar to that adopted in the former case, we have —

1°. Slope at end, on the supposition of supported ends, is

$$i_1 = -\frac{Wl^2}{24EI}. \qquad (1)$$

2°. Slope at end under uniform bending-moment M_1 is

$$-\frac{M_1 l}{2EI}. \qquad (2)$$

Hence, since their sum equals zero,

$$M_1 = -\frac{Wl}{12}, \qquad (3)$$

which is the bending-moment over either support.

The bending-moment at distance x from one end is

$$M = \frac{W}{2l}(lx - x^2) - \frac{Wl}{12}. \qquad (4)$$

This is greatest when $x = 0$, and is then $-\frac{Wl}{12}$. Hence greatest bending-moment is, in magnitude,

$$\frac{Wl}{12}. \qquad (5)$$

POINTS OF INFLECTION.

$$M \text{ becomes zero when } x = \frac{l}{2} \pm \frac{l}{2}\frac{1}{\sqrt{3}}. \qquad (6)$$

Hence the two points of inflection are situated at a distance $\frac{l}{2\sqrt{3}}$ on either side of the middle.

SLOPE AND DEFLECTION.

$$i = \int \frac{M}{EI}dx = \frac{W}{12lEI}(3lx^2 - 2x^3 - l^2x), \qquad (7)$$

the constant vanishing because $i = 0$ when $x = 0$.

$$v = \frac{W}{12lEI}\left\{lx^3 - \frac{x^4}{2} - \frac{l^2x^2}{2}\right\}, \qquad (8)$$

the constant vanishing because $v = 0$ when $x = 0$. Hence for greatest slope and deflection we have, i is greatest when $x = \frac{l}{2}\left(1 \pm \frac{1}{\sqrt{3}}\right)$, and v is greatest when $x = \frac{l}{2}$;

$$\therefore \quad i_0 = -\frac{Wl^2}{72\sqrt{3}EI}, \qquad (9)$$

$$v_0 = -\frac{Wl^3}{384EI}. \qquad (10)$$

SLOPE AND DEFLECTION UNDER WORKING-LOAD.

For working-load we have

$$\frac{Wl}{12} = \frac{fI}{y} \qquad (11)$$

$$\therefore \quad W = \frac{12fI}{ly} \qquad (12)$$

$$\therefore \quad i_0 = -\frac{fl}{6\sqrt{3}Ey}, \qquad (13)$$

$$v_0 = -\frac{fl^2}{32Ey}. \qquad (14)$$

EXAMPLES.

1. Given a 4-inch by 12-inch yellow-pine beam, span 20 feet, fixed at the ends; find its safe centre load, its safe uniformly distributed load, and its deflection under each load. Assume a modulus of rupture 5000 lbs. per square inch, and factor of safety 4. Modulus of elasticity, 1200000.

2. Find the depth necessary that a 4-inch wide yellow-pine beam, 20 feet span, fixed at the ends, may not deflect more than one four-hundredth of the span under a load of 5000 lbs. centre load.

§ 204. **Variation of Bending-Moment with Shearing-Force.** — *If, in any loaded beam whatever,* M *represent the bending-moment, and* F *the shearing-force at a distance* x *from the origin, then will*

$$F = \frac{dM}{dx}. \qquad (1)$$

Proof (*a*). — In the case of a cantilever (Fig. 230), assume the origin at the fixed end; then, if M represent the bending-moment at a distance x from the origin, and $M + \Delta M$ that at a distance $x + \Delta x$ from the origin, we shall have the following equations: —

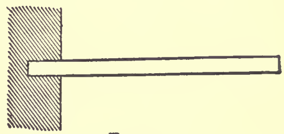


Fig. 230.

$$M = -\sum_{x=x}^{x=l} W(a - x),$$

$$M + \Delta M = -\sum_{x=x}^{x=l} W(a - x - \Delta x) \text{ nearly.}$$

a being the co-ordinate of the point of application of W,

$$\Delta M = \Delta x \sum_{x=x}^{x=l} W \text{ nearly}$$

$$\therefore \quad \frac{\Delta M}{\Delta x} = \sum_{x=x}^{x=l} W;$$

and, if we pass to the limit, and observe that

$$F = \sum_{x=x}^{x=l} W,$$

we shall obtain

$$\frac{dM}{dx} = F. \tag{2}$$

(*b*) In the case of a beam supported at the ends (Fig. 231), assume the origin at the left-hand end, and let the left-hand supporting-force be S; then, if a represent the distance from the origin to the point of application of W, we shall have the equations

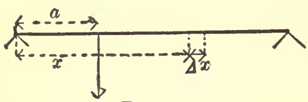


FIG. 231.

$$M = Sx - \sum_{x=0}^{x=x} W(x - a),$$

$$M + \Delta M = S(x + \Delta x) - \sum_{x=0}^{x=x} W(x - a + \Delta x) \text{ nearly.}$$

Hence, by subtraction,

$$\Delta M = S \,.\, \Delta x - \sum_{x=0}^{x=x} W\Delta x \text{ nearly}$$

$$\therefore \quad \frac{\Delta M}{\Delta x} = S - \sum_{x=0}^{x=x} W \text{ nearly};$$

and, if we pass to the limit, and observe that

$$F = S - \sum_{x=0}^{x=x} W,$$

we shall obtain

$$\frac{dM}{dx} = F, \tag{3}$$

as before.

§ 205. **Longitudinal Shearing of Beams.** — The resistance of a beam to longitudinal shearing sometimes becomes a matter of importance, especially in timber, where the resistance to shearing along the grain is very small. We will therefore proceed to ascertain how to compute the intensity of the longitudinal shear at any point of the beam, under any given load; as this should not be allowed to exceed a certain safe limit, to be determined experimentally. Assume a section AC (Fig. 232) at a distance x from the origin, and let the bending-moment at that section be M. Let the section BD be at a distance $x + \Delta x$ from the origin, and let the bending-moment at that section be $M + \Delta M$.

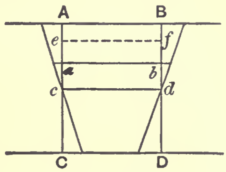


FIG. 232.

Let y_0 be the distance of the outside fibre from the neutral axis; and let $ca = y_1$ be the distance of a, the point at which the shearing-force is required, from the neutral axis.

Consider the forces acting on the portion $ABba$, and we shall have —

1°. Intensity of direct stress at $A = \dfrac{My_0}{I}$.

2°. Intensity of direct stress at a unit's distance from neutral axis $= \dfrac{M}{I}$.

3°. Intensity of direct stress at e, where $ce = y$, is $\dfrac{My}{I}$.

So, likewise, intensity of direct stress at f is $\dfrac{(M + \Delta M)y}{I}$.

Therefore, if z represent the width of the beam at the point e, we shall have —

$$\text{Total stress on face } Aa = \frac{M}{I}\int_{y_1}^{y_0} yzdy,$$

$$\text{Total stress on face } Bb = \frac{M + \Delta M}{I}\int_{y_1}^{y_0} yzdy;$$

$$\therefore \text{ Difference} = \frac{\Delta M}{I}\int_{y_1}^{y_0} yzdy:$$

and this is the total horizontal force tending to slide the piece *AabB* on the face *ab*.

Area of face *ab*, if z_1 is its width, is

$$z_1\Delta x;$$

therefore intensity of shear at *a* is approximately

$$\frac{\dfrac{\Delta M}{I}\displaystyle\int_{y_1}^{y_0} yzdy}{z_1\Delta x},$$

or exactly (by passing to the limit)

$$\frac{\left(\dfrac{dM}{dx}\right)}{z_1 I}\int_{y_1}^{y_0} yzdy.$$

And, observing that $F = \dfrac{dM}{dx}$, this intensity reduces to

$$\frac{F}{z_1 I}\int_{y_1}^{y_0} yzdy. \qquad (1)$$

We may reduce this expression to another form by observing, that, if y_2 represent the distance from c to the centre of gravity of area Aa, and A represent its area, we have

$$\int_{y_1}^{y_0} yzdy = y_2 A;$$

therefore intensity of shear (at distance y_1 from neutral axis) at point a =

$$\frac{F}{z_1 I}(y_2 A). \qquad (2)$$

This may be expressed as follows:—

Divide the shearing-force at the section of the beam under consideration, by the product of the moment of inertia of the section and its width at the point where the intensity of the shearing-force is desired, and multiply the quotient by the statical moment of the portion of the cross-section between the point in question and the outer fibre; this moment being taken about the neutral axis. The result is the required intensity of shear.

The last factor is evidently greatest at the neutral axis; hence the intensity of the shearing-force is greatest at the neutral axis.

LONGITUDINAL SHEARING OF RECTANGULAR BEAMS.

For rectangular beams, we have

$$I = \frac{bh^3}{12}, \qquad z_1 = b.$$

Hence formula (2) becomes

$$\frac{12F}{b^2h^3}(y_2A). \tag{3}$$

For the intensity at the neutral axis, we shall have, therefore,

$$\frac{12F}{b^2h^3}\left(\frac{h}{4}\,\frac{bh}{2}\right) = \frac{3}{2}\,\frac{F}{bh}, \tag{4}$$

since for the neutral axis we have

$$y_2 = \frac{h}{4} \quad \text{and} \quad A = \frac{bh}{2}.$$

EXAMPLES.

1. What is the intensity of the tendency to shear at the neutral axis of a rectangular 4-inch by 12-inch beam, of 14 feet span, loaded at the middle with 5000 lbs.

2. What is that of the same beam at the neutral axis of the cross-section at the support, when the beam has a uniformly distributed load of 12000 lbs.

3. What is that of a 9-inch by 14-inch beam, 20 feet span, loaded with 15000 lbs. at the middle.

§ 206. **Strength of Hooks.** — The following is the method to be pursued in determining the stresses in a hook due to a given load; or, *vice versa*, the proper dimensions to use for a given load.

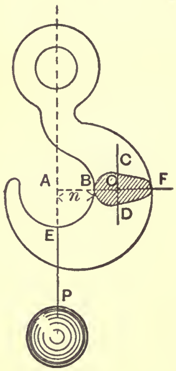


FIG. 233.

Suppose (Fig. 233) a load hung at E; the load being P, and the distances

$$AB = n;$$
$$BO = y_1;$$
$$OF = y_2.$$

O being the centre of gravity of this section, conceive two equal and opposite forces, each equal and parallel to P, acting at O.

Let A = area of section, and let I = its moment of inertia about CD ($BCDF$ represents the section revolved into the plane of the paper); then —

1°. The downward force at O causes a uniformly distributed stress over the section, whose intensity is

$$p_1 = \frac{P}{A}.$$

2°. The downward force at E and the upward force at O constitute a couple, whose moment is

$$P(n + y_1);$$

and this is resisted, just as the bending-moment in a beam, by a uniformly varying stress, producing tension on the left, and compression on the right, of CD.

If we call p_2 the greatest intensity of the tension due to this bending-moment, viz., that at B, we have

$$p_2 = \frac{P(n + y_1)y_1}{I};$$

and if p_3 denote the greatest intensity of the compression due to the bending moment, viz., that at F, we have

$$p_3 = \frac{P(n + y_1)y_2}{I};$$

therefore the actual greatest intensity of the tension is

$$p_t = p_1 + p_2 = \frac{P}{A} + \frac{P(n + y_1)y_1}{I},$$

and this must be kept within the working strength if the load is to be a safe one; and so also the actual greatest intensity of the compression, viz., that at F, is, when $p_3 > p_1$,

$$p_c = p_3 - p_1 = \frac{P(n + y_1)y_2}{I} - \frac{P}{A},$$

which must be kept within the working strength for compression.

§ 207. **Strength of Columns.**—The formulæ most commonly employed for the breaking-strength of columns subjected to a load whose resultant acts along the axis have been, until recently, the Gordon formulæ with Rankine's modifications, the so-called Euler formulæ, and the avowedly empirical formulæ of Hodgkinson. These formulæ do not give results which agree with those obtained from tests made upon such full-size columns as are used in practice.

The deductions of the first two are not logical, certain assumptions being made which are not borne out by the facts.

When a column is subjected to a load which strains any fibre beyond the elastic limit, the stresses are not proportional to the strains, and hence there can be no rational formula for the breaking-load.

Hence, all formulæ for the breaking-load are, of necessity empirical, and depend for their accuracy upon their agreement with the results of experiments upon the breaking-strength of such full-size columns as are used in practice.

Nevertheless, the ordinary so-called deductions of the Gordon, and of the so-called Euler formulæ will be given first.

§ 208. **Gordon's Formulæ for Columns.**—(*a*) *Column fixed in Direction at Both Ends.*—Let CAD be the central axis of the column, P the breaking-load, and v the greatest deflection, AB. Conceive at A two equal and opposite forces, each equal to P; then—

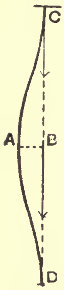


Fig. 234.

1°. The downward force at A causes a uniformly distributed stress over the section, of intensity,

$$p_1 = \frac{P}{A}.$$

2°. The downward force at C and the upward force at A constitute a bending couple whose moment is

$$M = Pv.$$

If p_2 = the greatest intensity of the compression due to this bending,

$$p_2 = \frac{(Pv)y}{I},$$

where y = distance from the neutral axis to the most strained fibre of the section at A. Then will the greatest intensity of stress at A be

$$p = p_1 + p_2 = \frac{P}{A} + \frac{Pvy}{I};$$

and, since P is the breaking-load, p must be equal to the breaking-strength for compression per square inch $= f$.

$$\therefore \quad f = \frac{P}{A}\left(1 + \frac{vy}{\rho^2}\right), \qquad (1)$$

where ρ = smallest radius of gyration of section at A.

Thus far the reasoning appears sound; but in the next step it is assumed that

$$v = \frac{1}{c}\frac{l^2}{y},$$

where c is a constant to be determined by experiment. Hence, substituting this, and solving for P,

$$P = \frac{fA}{1 + \dfrac{l^2}{c\rho^2}}, \qquad (2)$$

which is the formula for a column fixed in direction at both ends.

(*b*) *Column hinged at the Ends.*—It is assumed that the points of inflection are half-way between the middle and the ends, and hence that, by taking the middle half, we have the case of bending of a column hinged at the ends (Fig. 235). Hence, to obtain the formula suitable for this case, substitute, in (2), $2l$ for l, and we obtain

Fig. 235.

$$P = \frac{fA}{1 + \dfrac{4l^2}{c\rho^2}}. \qquad (3)$$

(*c*) *Column fixed at One End and hinged at the Other* (Fig. 236).—In this case we should, in accordance with these assumptions, take $\frac{3}{4}$ of the column fixed in direction at both ends; hence, to obtain the formula for this case, substitute, in (2), $\frac{4}{3}l$ for l, and we thus obtain

$$P = \frac{fA}{1 + \dfrac{16l^2}{9c\rho^2}}. \qquad (4)$$

Fig. 236.

Rankine gives, for values of f and c, the following, based upon Hodgkinson's experiments:

	f (lbs. per sq. in.).	c.
Wrought-iron	36000	36000
Cast-iron	80000	6400
Dry timber	7200	3000

§ 208*a*. **So-called Euler Formulæ for the Strength of Columns.**

(*a*) *Column fixed in Direction at One End only, which bends, as shown in the Figure.*

1°. Calculate the breaking-load on the assumption that the column will give way by direct compression. This will be

$$P_1 = fA, \qquad (1)$$

where f = crushing-strength per square inch, and A = area of cross-section in square inches.

2°. Calculate the load that would break the column if it were to give way by bending, by means of the following formula:

$$P_2 = \left(\frac{\pi}{2l}\right)^2 EI, \qquad (2)$$

where E = modulus of elasticity of the material, I = smallest moment of inertia of the cross-section, and l = length of column.

Then will the actual breaking-strength, according to Euler, be the smaller of these two results.

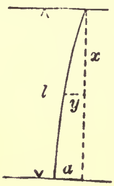


FIG. 237.

To deduce the latter formula, assume the origin at the upper end, and take x vertical and y horizontal.

Let ρ = radius of curvature at point (x, y), and let M = bending-moment at the same point.

Then we have, with compression plus and tension minus,

$$\frac{1}{\rho} = \frac{M}{EI} = \frac{Py}{EI}. \qquad (3)$$

But

$$\frac{1}{\rho} = -\frac{d^2y}{dx^2}, \text{ nearly,}$$

$$\therefore \quad -\frac{d^2y}{dx^2} = \frac{P}{EI}y,$$

$$\therefore \quad -\int \frac{dy}{dx} \cdot \frac{d^2y}{dx^2}dx = \frac{P}{EI}\int y\frac{dy}{dx}dx$$

$$\therefore \quad \left(\frac{dy}{dx}\right)^2 = -\frac{P}{EI}y^2 + c;$$

and, since for $y = a$, $\quad \frac{dy}{dx} = 0, \quad \therefore \; c = \frac{P}{EI}a^2$

$$\therefore \quad \left(\frac{dy}{dx}\right)^2 = \frac{P}{EI}(a^2 - y^2)$$

$$\therefore \quad \frac{dy}{\sqrt{a^2 - y^2}} = \sqrt{\frac{P}{EI}}dx$$

$$\therefore \quad \sin^{-1}\frac{y}{a} = \sqrt{\frac{P}{EI}}x + c.$$

And since, when $x=0$, $y=0$, $\therefore$ $c=0$, we have

$$\sin^{-1}\left(\frac{y}{a}\right) = \sqrt{\frac{P}{EI}}x. \tag{5}$$

When $y=a$, $x=l$; hence, substituting in (5), and solving for P,

$$P = \left(\frac{\pi}{2l}\right)^2 EI. \tag{6}$$

(*b*) *Column hinged at Both Ends* (Fig. 235).

1°. For the crushing-load,

$$P_1 = fA.$$

2°. For the breaking-load by bending, put $l/2$ for l in (6); hence

$$P_2 = \left(\frac{\pi}{l}\right)^2 EI. \tag{7}$$

(*c*) *Column fixed in Direction at One End, and hinged at the other* (Fig. 236).

1°. For the crushing-load,

$$P_1 = fA.$$

2°. For the breaking-load by bending, put $l/3$ for l in (6); hence

$$P_2 = \frac{9}{4}\left(\frac{\pi}{l}\right)^2 EI. \tag{8}$$

(*d*) *Column fixed in Direction at Both Ends* (Fig. 234).

1°. For the crushing-load,

$$P_1 = fA.$$

2°. For the breaking-load by bending,

$$P_2 = \left(\frac{2\pi}{l}\right)^2 EI, \tag{9}$$

this being obtained from (2) or (6) by substituting $l/4$ for l.

(*e*) In order to ascertain the length wnere incipient flexure occurs, according to this theory we should place the two results equal to each other, and from the resulting equation determine l. We should thus obtain, for the three cases respectively,

$$(\alpha) \qquad l = \pi\sqrt{\frac{EI}{fA}}, \qquad (10)$$

$$(\beta) \qquad l = \tfrac{3}{2}\pi\sqrt{\frac{EI}{fA}}, \qquad (11)$$

$$(\gamma) \qquad l = 2\pi\sqrt{\frac{EI}{fA}}. \qquad (12)$$

Hence all columns whose length is less than that given in these formulæ will, according to Euler, give way by direct crushing; and those of greater length, by bending only.

§ 209. **Hodgkinson's Rules for the Strength of Columns.** —Eaton Hodgkinson made a large number of tests of small columns, especially of cast-iron, and deduced from these tests certain empirical formulæ. The strength of pillars of the ordinary sizes used in practice has been computed by means of Hodgkinson's formulæ, and tabulated by Mr. James B. Francis; and we find in his book the following rules for the strength of solid cylindrical pillars of cast-iron, with the ends flat, i.e., "finished in planes perpendicular to the axis, the weight being uniformly distributed on these planes":

For pillars whose length exceeds thirty times their diameter,

$$W = 99318\,\frac{D^{3.55}}{l^{1.7}}, \qquad (1)$$

where D = diameter in inches, l = length in feet, W = breaking-weight in lbs.

If, on the other hand, the length does not exceed thirty times the diameter, he gives, for the breaking-weight, the following formula:

$$W' = \frac{Wc}{W + \tfrac{3}{4}c}, \qquad (2)$$

where W = breaking-weight that would be derived from the preceding formula, W' = actual breaking-weight,

$$c = 109801\left(\frac{\pi D^2}{4}\right). \qquad (3)$$

For hollow cast-iron pillars, if D=external diameter in inches, d= internal diameter in inches, we should have, in place of (1),

$$W = 99318 \frac{D^{3.55} - d^{3.55}}{l^{1.7}}, \tag{4}$$

and in place of (3),

$$c = 109801 \frac{\pi(D^2 - d^2)}{4}. \tag{5}$$

For very long wrought-iron pillars, Hodgkinson found the strength to be 1.745 times that of a cast-iron pillar of the same dimensions; but, for very short pillars, he found the strength of the wrought-iron pillar very much less than that of the cast-iron one of the same dimensions. With a length of 30 diameters and flat ends, the wrought-iron exceeded the cast-iron by about 10 per cent.

§ 210. **Breaking-load of Full-size Columns.**—The tests made upon full-size columns are not as many as would be desirable. The details will be given in Chapter VII, but a few of the empirical formulæ which represent their results will be given here.

If P=breaking-load, A=area of smallest section, l=length of column, ρ=least radius of gyration of section, and f_c=crushing-strength of the material per unit of area, it will be found that for values of l/ρ less than a certain amount, the column remains straight, and the breaking-load may be computed by means of the formula $$P = f_cA.$$

For greater values of l/ρ, the breaking-load is smaller than that given by this formula, and may be computed by means of the formula $$P = fA,$$
by using for f a value smaller than f_c, this value varying with the value of l/ρ, and being determined empirically from the results of tests of full-size columns.

(*a*) In the case of cast-iron columns no tests have been made of full-size columns of the second class, while those made upon the first class indicate that the value of f_c suitable for use in practice is from 25,000 to 30,000 lbs. per square inch.

(*b*) In the case of wrought-iron columns, the tests of the first class indicate that the value of f_c suitable for use in practice is from 30,000 to 35,000 lbs. per square inch.

(*c*) In the case of wrought-iron columns of the second class, the formula of Mr. C. L. Strobel for bridge columns with either flat or pin ends, when $l/\rho > 90$, is

$$\frac{P}{A} = 46000 - 125\frac{l}{\rho}.$$

On the other hand, those recommended by Prof. J. Sondericker, of which the first was devised by Mr. Theodore Cooper, are as follows:

(α) For Phœnix columns with flat ends $l/\rho > 80$,

$$\frac{P}{A} = \frac{36000}{1 + \frac{(l/\rho - 80)^2}{18000}}.$$

(β) For lattice columns with pin-ends and $l/\rho > 60$,

$$\frac{P}{A} = \frac{34000}{1 + \frac{(l/\rho - 60)^2}{12000}}.$$

(γ) For solid web, square, or box columns with flat ends, and $l/\rho > 80$,

$$\frac{P}{A} = \frac{33000}{1 + \frac{(l/\rho - 80)^2}{10000}}.$$

(δ) For solid web, square, or box columns with pin-ends, and $l/\rho > 60$,

$$\frac{P}{A} = \frac{31000}{1 + \frac{(l/\rho - 60)^2}{6000}}.$$

The number of tests that have been made upon full-size steel columns is very small, hence no formulæ will be given here, but the subject will be discussed in Chapter VII. The number of tests that have been made upon full-size timber columns is considerable, but this subject will also be discussed in Chapter VII.

§ 211. **Columns subjected to Loads which do not Strain any Fibre beyond the Elastic Limit.**—Under this head will be discussed, first, the mode of determining the greatest fibre stress in a straight column subjected to an eccentric load, and, secondly, the general theory of columns.

(*a*) Straight column, under eccentric load.—Let O' be the centre of gravity of the lower section, and let $A'O' = x_0$, where A' is the point of application of the resultant of the eccentric load. Conceive two equal and opposite forces at O', each equal and parallel to P. Then we have:

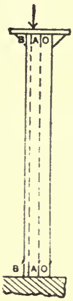


FIG. 238.

1°. Downward force along OO' causes uniform stress of intensity $p_1 = \dfrac{P}{A}$.

2°. The other two form a couple whose moment is Px_0, and the greatest intensity of the stress due to this couple is $p_2 = \dfrac{(Px_0)a}{I}$, where $a = O'B'$. Hence, the greatest intensity of the stress is

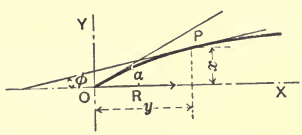
$$p = \frac{P}{A} + \frac{Px_0a}{I};$$

$$\therefore\ p = \frac{P}{A}\left(1 + \frac{x_0a}{\rho^2}\right),$$

and this should be kept within the limits of the working-strength.

(*b*) Theory of columns.—The theory of columns is that of the Inflectional Elastica, and is explained in several treatises, among which is that of A. E. H. Love on the Theory of Elasticity. It is as follows:

Let the curve OP be an elastic line, on which O is a point of inflection. It follows that there is no bending-moment at this point, and hence we may assume that at O a single force R acts. Take the origin at O, and axis of X along the line of action of the force R. Let E_1 = modulus of elasticity of the material, I = moment of inertia of section about an axis through its centre of gravity, and perpen-



dicular to the plane of the curve, ϕ=angle between OX and the tangent at any point P whose coordinates are x and y, α=value of ϕ at point O, r=radius of curvature of the curve at P, s= length of arc OP, l=length of one bay, i.e., measured from O to the next point of inflection, $a=\frac{s}{l}$, A=area of section, $\rho=\sqrt{\frac{I}{A}}$, $\sigma=\frac{R}{A}$.

Then we have for any such elastic line, when compressions are plus and tensions minus,

$$\frac{1}{\rho}=\frac{M}{EI}.$$

Moreover, since $\frac{1}{\rho}=-\frac{d\phi}{ds}$ and $M=Ry$, we have, for a column of the same cross-section throughout its length, $\frac{d\phi}{ds}=-\frac{R}{E_1I}y$, where the quantity $\frac{R}{E_1I}$ is a constant.

By differentiation we obtain

$$\frac{d^2\phi}{ds^2}=-\frac{R}{E_1I}\frac{dy}{ds}=-\frac{R}{E_1I}\sin\phi.$$

Integrating, and observing that at O, $\frac{d\phi}{ds}=0$, and $\phi=\alpha$, we obtain

$$\frac{1}{2}\left(\frac{d\phi}{ds}\right)^2=\frac{R}{E_1I}(\cos\phi-\cos\alpha). \qquad (1)$$

The integration of this equation requires the use of elliptic integrals, hence only the results will be given here.

They are:

$$x=\frac{l}{2K}\{-2aK+2[E\,\mathrm{am}(2aK)-\sin^2\tfrac{1}{2}\alpha\,\mathrm{sn}\{(2a+1)K\}\mathrm{sn}(2aK)]\} \quad (2)$$

$$y=\frac{l}{K}\{\sin\tfrac{1}{2}\alpha[-\mathrm{cn}\{(2a+1)K\}]\} \quad (3)$$

and

$$u=s\sqrt{\frac{R}{E_1I}}, \quad (4)$$

where E denotes the elliptic integral of the second kind, and K the complete elliptic integral of the first kind.

Moreover, for the determination of the load R, we obtain from equation (4)

$$K=\frac{l}{2}\sqrt{\frac{R}{E_1I}}, \quad (5)$$

and hence

$$R=\frac{4K^2}{l^2}E_1I. \quad (6)$$

From these equations, we can, by using a table of elliptic functions, deduce the following results for the coordinates of points on the inflectional elastica, for various values of α:

α	$\frac{s}{l}$	$\frac{x}{l}$	$\frac{y}{l}$
10°	0.00 0.25 0.50	0.0000 0.2476 0.4962	0.0000 0.0392 0.0554
20°	0.00 0.25 0.50	0.0000 0.2376 0.4849	0.0000 0.0773 0.1079
30°	0.00 0.25 0.50	0.0000 0.2224 0.4662	0.0000 0.1135 0.1620

Moreover, these results agree with those which we obtain by experiment, and thus we can, by making use of our calculations, compute the load required to produce a given elastica, determined by the slope at the points of inflection, which, in the case of pin-ended columns, are at the ends, and, in the case of columns fixed in direction at the ends, are half-way between the middle and the ends.

All this can be done, and can be verified by experiment, provided that the load is not so great that any fibre is strained beyond the elastic limit of the material, and provided the value of l/ρ is not so small that the curvilinear form is unstable.

For smaller values of l/ρ the only stable form is a straight line, and the column does not bend.

To ascertain the least value of l/ρ for which a curved form is stable, observe that K cannot be less than $\pi/2$, and since this corresponds to one bay, and hence to the case of a pin-ended column, we have in that case, by substituting $\pi/2$ for K in equation (6),

$$R=\frac{\pi^2}{l^2}E_1I,$$

and, since $$I=A\rho^2 \quad \text{and} \quad \frac{R}{A}=\sigma,$$

we have for the line of demarcation between the straight and curved form in a pin-ended column

$$\frac{l}{\rho}=\pi\sqrt{\frac{E_1}{\sigma}}; \tag{7}$$

and for that in the case of a column fixed in direction at the ends

$$\frac{l}{\rho}=2\pi\sqrt{\frac{E_1}{\sigma}}. \tag{8}$$

As an example, if $\sigma=10{,}000$ and $E_1=30{,}000{,}000$ we should find that a pin-ended column would not bend unless l/ρ were greater than 172, and that a column fixed in direction at the ends would not bend unless l/ρ were greater than 344. Columns with smaller values of l/ρ would remain straight when the resultant of the load acts along the axis, and no fibre is strained beyond the elastic limit.

§ 212. **Strength of Shafting.** — The usual criterion for the strength of shafting is, that it shall be sufficiently strong to resist the twisting to which it is exposed in the transmission of power.

Proceeding in this way, let EF (Fig. 239) be a shaft, AB the driving, and CD the following, pulley. Then, if two cross-sections be taken between these two pulleys, the portion of the shaft between these two cross-sections will, during the transmission of power, be in a twisted condition; and if, when the shaft is at rest, a pair of vertical parallel diameters be drawn in these sections, they will, after it is set in motion, no longer be parallel, but will be inclined to each other at an angle depending upon the power applied. Let GH be a section at a distance x from O, and let KI be another section at a distance $x + dx$ from O. Then, if di represent the angle at which the originally parallel diameters of these sections diverge from each other, and if $r =$ the radius of the shaft, we shall have, for the length of an arc passed over by a point on the outside,

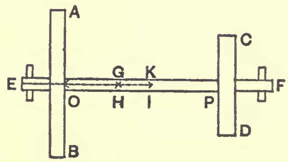


FIG. 239.

$$rdi;$$

and for the length of an arc that would be passed over if the sections were a unit's distance apart, instead of dx apart,

$$\frac{rdi}{dx} = r\frac{di}{dx}.$$

This is called the *strain* of the outer fibres of the shaft, as it is the distortion per unit of length of the shaft.

In all cases where the shaft is homogeneous and symmetrical, if i is the angle of divergence of two originally parallel diameters whose distance apart is x, we shall have the strain,

$$v = r\frac{di}{dx} = r\frac{i}{x}.$$

This also is the tangent of the angle of the helix.

A fibre whose distance from the axis of the shaft is unity, will have, for its strain,

$$\frac{di}{dx} = \frac{i}{x}.$$

A fibre whose distance from the axis of the shaft is ρ, will have, for its strain,

$$v = \rho\frac{di}{dx} = \rho\frac{i}{x}.$$

Fixing, now, our attention upon one cross-section, GH, we have that the strain of a fibre at a distance ρ from the axis (ρ varying, and being the radius of any point whatever) is

$$\rho\left(\frac{i}{x}\right),$$

where $\frac{i}{x}$ is a constant for all points of this cross-section.

Hence, assuming Hooke's law, "*Ut tensio sic vis*," we shall have, if C represent the shearing modulus of elasticity, that the *stress* of a fibre whose distance from the axis is ρ, is

$$p = Cv = C\rho\left(\frac{di}{dx}\right) = C\rho\left(\frac{i}{x}\right),$$

which quantity is proportional to ρ, or varies uniformly from the centre of the shaft.

The intensity at a unit's distance from the axis is

$$C\left(\frac{i}{x}\right);$$

and if we represent this by a, we shall have for that at a distance ρ from the axis,

$$p = a\rho.$$

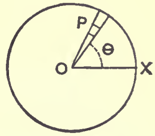


FIG. 240.

Hence we shall have (Fig, 240), that, on a small area,

$$dA = d\rho(\rho d\theta) = \rho d\rho d\theta,$$

the stress will be

$$pdA = a\rho dA = a\rho^2 d\rho d\theta.$$

The moment of this stress about the axis of the shaft is

$$\rho p dA = a\rho^2 dA = a\rho^3 d\rho d\theta,$$

and the entire moment of the stress at a cross-section is

$$a\int\rho^2 dA = a\int\int\rho^3 d\rho d\theta = aI,$$

where $I = \int\rho^2 dA$ is the moment of inertia of the section about the axis of the shaft.

This moment of the stress is evidently caused by, and hence must be balanced by, the twisting-moment due to the pull of the belt. Hence, if M represent the greatest allowable twisting-moment, and a the greatest allowable intensity of the stress at a unit's distance from the axis, we shall have

$$M = aI = \frac{p}{\rho} I.$$

If f is the safe working shearing-strength of the material per square inch, we shall have f as the greatest safe stress per square inch at the outside fibre, and hence

$$M = \frac{f}{r} I$$

will be the greatest allowable twisting-moment.

For a circle, radius r,

$$I = \frac{\pi r^4}{2} \qquad \therefore \quad M = f\frac{\pi r^3}{2} = f\frac{\pi d^3}{16}.$$

For a hollow circle, outside radius r_1, inside radius r_2,

$$I = \frac{\pi(r_1^4 - r_2^4)}{2} \qquad \therefore \quad M = f\frac{\pi}{2r_1}(r_1^4 - r_2^4).$$

Moreover, if the dimensions of a shaft are given, and the actual twisting-moment to which it is subjected, the stress at a fibre at a distance ρ from the axis will be found by means of the formula

$$p = \frac{M\rho}{I}.$$

The more usual data are the horse-power transmitted and the speed, rather than the twisting-moment.

If we let $P =$ force applied in pounds and $R =$ its leverage in inches, as, for instance, when $P =$ difference of tensions of belt, and $R =$ radius of pulley, we have

$$M = P \,.\, R\,;$$

and if $HP =$ number of horses-power transmitted, and $N =$ number of turns per minute, then

$$HP = \frac{P(2\pi RN)}{12 \times 33000};$$

$$\therefore \quad PR = \frac{12 \times 33000 HP}{2\pi N} = M.$$

EXAMPLE.

Given working-strength for shearing of wrought-iron as 10000 lbs. per square inch; find proper diameter of shaft to transmit 20-horse power, making 100 turns per minute.

Angle of Torsion.—From the formula, page 336, $p = \dfrac{M\rho}{I}$, combined with

$$p = a\rho = C\rho\frac{i}{x},$$

we have

$$C\rho\frac{i}{x} = \frac{M\rho}{I}$$

$$\therefore \quad i = \frac{Mx}{CI},$$

which gives the circular measure of the angle of divergence of two originally parallel diameters whose distance apart is x; the twisting-moment being M, and the modulus of shearing elasticity of the material, C.

EXAMPLES.

1. Find the angle of twist of the shaft given in example 1, § 212, when the length is 10 feet, and $C = 8500000$.

2. What must be the diameter of a shaft to carry 80 horses-power, with a speed of 300 revolutions per minute, and factor of safety 6, breaking shearing-strength of the iron per square inch being 50000 lbs.

§ 213. **Transverse Deflection of Shafts.** — In determining the proper diameter of shaft to be used in any given case, we ought not merely to consider the resistance to twisting, but also the deflection under the transverse load of the belt-pulls, weights of pulleys, etc. This deflection should not be allowed to exceed $\frac{1}{100}$ of an inch per foot of length. Hence the deflection should be determined in each case.

The formulæ for computing this deflection will not be given here, as the methods to be pursued are just the same as in the case of a beam, and can be obtained from the discussions on that subject.

§ 214. **Combined Twisting and Bending.** — The most common case of a shaft is for it to be subjected to combined twisting and bending. The discussion of this case involves the theory of elasticity, and will not be treated here; but the formulæ commonly given will be stated, without attempt to prove them until a later period. These formulæ are as follows: —

Let M_1 = greatest bending-moment,

M_2 = greatest twisting-moment,

r = external radius of shaft,

I = moment of inertia of section about a diameter,

for a solid shaft $I = \frac{\pi r^4}{4}$,

f = working-strength of the material = greatest allowable stress at outside fibre;

then

1°. According to Grashof,

$$f = \frac{r}{I}\left\{\tfrac{3}{8}M_1 + \tfrac{5}{8}\sqrt{M_1^2 + M_2^2}\right\}. \qquad (1)$$

2°. According to Rankine,

$$f = \frac{r}{2I}\left\{M_1 + \sqrt{M_1^2 + M_2^2}\right\}. \qquad (2)$$

§ 215. **Springs.**—The object of this discussion is to enable us to answer the following three questions: (*a*) Given a spring, to determine the load that it can bear without producing in the metal a maximum fibre stress greater than a given amount. (*b*) Given a spring, to determine its displacement (elongation, compression, or deflection) under any given load. (*c*) Given a load P and a displacement δ_1; a spring is to be made of a given material such that the load P shall produce the displacement δ_1, and that the metal shall not, in that case, be subjected to more than a given maximum fibre stress. Determine the proper dimensions of the spring.

There are practically only two cases to be considered as far as the manner of resisting the load is concerned. In the first, the spring is subjected to transverse stress, and is to be calculated by the ordinary rules for beams. In the second, the spring is subjected to torsion, and the ordinary rules for resistance to torsion apply. It is true that in most cases where the spring is subjected to torsion there is also a small amount of transverse stress in addition to the torsion; but in a well-made spring this transverse stress is of very small amount, and we may neglect it without much error.

We will begin with those cases where the spring is subjected to torsion, and for all cases we shall adopt the following notation:

$P =$ load on spring producing maximum fibre stress f;
$f =$ greatest allowable maximum fibre stress for shearing;
$C =$ shearing modulus of elasticity;
$x =$ length of wire forming the spring;
$M_1 =$ greatest twisting moment under load P;
$L =$ any load less than the limit of elasticity;
$M =$ twisting moment under this load;
$p =$ maximum fibre stress under load L;
$\rho =$ distance from axis of wire to most strained fibre;
$I =$ moment of inertia of section about axis of wire;
$i_1 =$ angle of twist of wire under load P;
$i =$ angle of twist of wire under load L;
$V =$ volume of spring;
$\delta_1 =$ displacement of point where load is applied when load is P;
$\delta =$ displacement of point where load is applied when load is L.

Then from pages 335 and 337 we obtain the following four formulæ:

$$M = \frac{p}{\rho} I, \tag{1}$$

$$i = \frac{Mx}{CI}, \tag{2}$$

$$M_1 = \frac{f}{\rho} I, \tag{3}$$

$$i_1 = \frac{M_1 x}{CI}. \tag{4}$$

These four formulæ will enable us to solve all the cases of springs subjected to torsion only. Moreover, in the cases which we shall discuss under this head, the wire will have either a circular or a rectangular section: in the former case we will denote its diameter by d, and we shall then have

$$I = \frac{\pi d^4}{32} \qquad \text{and} \qquad \rho = \frac{d}{2};$$

while in the latter case we will denote the two dimensions of the rectangle by b and h, respectively, and we shall then have

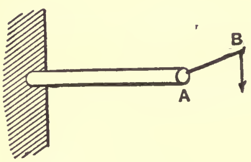
$$I = \frac{bh}{12}(b^2 + h^2), \qquad \rho = \tfrac{1}{2}\sqrt{b^2 + h^2}.$$

We will now proceed to determine the values of P, δ, δ_1, and V in each of the following four cases, all of which are cases of torsion:

CASE 1. *Simple round torsion wire.*—Let AB, the leverage of the load about the axis, be R; then we shall have

$$M = LR, \qquad M_1 = PR;$$

and we readily obtain from the formulæ (1), (2), (3), and (4)



$$P = f\frac{\pi d^3}{16R}, \tag{5}$$

$$\delta = Ri = \frac{32R^2 x}{\pi d^4}\frac{L}{C}, \tag{6}$$

$$\delta_1 = Ri_1 = 2\frac{Rx}{d}\frac{f}{C}; \tag{7}$$

and from these we readily obtain

$$P\delta_1 = \tfrac{1}{2}\frac{f^2}{C}\left(\frac{\pi d^2 x}{4}\right) = \tfrac{1}{2}\frac{f^2}{C}V;$$

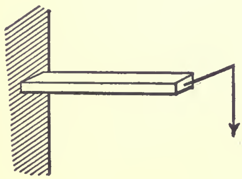
$$\therefore V = \frac{2C}{f^2}(P\delta_1). \qquad (8)$$

CASE 2. *Simple rectangular torsion wire.*—In this case we readily obtain

$$P = \tfrac{1}{6}\frac{f}{R}bh\sqrt{b^2+h^2}, \qquad (9)$$

$$\delta = Ri = \frac{12R^2x}{bh(b^2+h^2)}\frac{L}{C}, \qquad (10)$$

$$\delta_1 = Ri_1 = \frac{2Rx}{\sqrt{b^2+h^2}}\frac{f}{C}; \qquad (11)$$



$$\therefore P\delta_1 = \tfrac{1}{3}\frac{f^2}{C}(bhx) = \tfrac{1}{3}\frac{f^2}{C}V;$$

$$\therefore V = 3\frac{C}{f^2}(P\delta_1). \qquad (12)$$

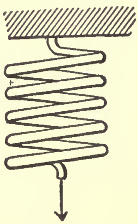
CASES 3 and 4. *Helical springs made of round and of rectangular wire respectively.*—A helical spring may be used either in tension or in compression. In either case it is important that the ends should be so guided that the pair of equal and opposite forces acting at the ends of the spring should act exactly along the axis of the spring.

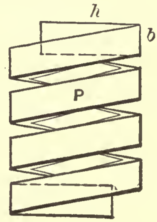
This is of especial importance when the spring is used for making accurate measurements of forces, as in the steam-engine indicator, in spring balances, etc.

Moreover, it is generally safer, as far as accuracy is concerned, to use a helical spring in tension rather than in compression, as it is easier to make sure that the forces act along the axis in the case of tension than in the case of compression.

Whichever way the spring is used, however, provided only the two opposing forces act along the axis of the spring, the resistance to which the spring is subjected is mainly torsion, inasmuch as the amount of bending is very slight.

This bending, however, we will neglect, and will compute the spring as a case of pure torsion, the same notation being used as before, except that we will now denote by R the radius





of the spring, and we shall have

$$M = LR, \quad M_1 = PR;$$

and now formulæ (5), (6), (7), and (8) become applicable to a spring made of round wire, and formulæ (9) and (10), (11) and (12), to one made of rectangular wire.

We must bear in mind, however, that x denotes the length of the wire composing the spring, and not the length of the spring. δ and δ_1 now denote the elongations or compressions of the spring.

GENERAL REMARKS.

By comparing equations (8) and (12), it will be seen that if a spring is required for a given service, its volume and hence its weight must be 50 per cent greater if made of rectangular than if made of round wire. Again, it is evident that when the kind of spring required is given, and the values of C and f for the material of which it is to be made are known, the volume and hence the weight of the spring depends only on the product $P\delta_1$, and that as soon as P and δ_1 are given, the weight of the spring is fixed independently of its special dimensions. If, however, we fix any one dimension arbitrarily, the others must be so fixed as to satisfy the equations already given. Next, as to the values to be used for f and C, these will depend upon the nature of the special material of which the spring is made, and these can only be determined by experiment. Confining ourselves now to the case of steel springs, it is plain that f and C should be values corresponding to tempered steel.

As an example, suppose we require the weight of a helical spring, which is to bear a safe load of 10000 lbs. with a deflection of one inch, assuming $C = 12600000$ and $f = 80000$ lbs. per sq. in., and as the weight of the steel 0.28 lb. per cubic inch.

From formula (8) we obtain

$$V = \frac{2 \times 12600000 \times 10000 \times 1}{80000 \times 80000} = 39.4 \text{ cu. in.}$$

Hence the weight of the spring must be (39.4) (0.28) = 11 lbs.

We may use either a single spring weighing 11 lbs., or else two or more springs either side by side or in a nest, whose combined weight is 11 lbs. Of course in the latter case they must all deflect the same amount under the portion of the load which each one is expected to bear, and this fact must be taken into account in proportioning the separate springs that compose the nest.

FLAT SPRINGS.

Let P, L, V, δ, and δ_1 have the same meanings as before, and let

$f =$ greatest allowable fibre stress for tension or compression;
$E =$ modulus of elasticity for tension or compression;
$l =$ length of spring;
$M_1 =$ maximum bending-moment under load P;
$M =$ maximum bending-moment under load L.

Moreover, the sections to be considered are all rectangular, and we will let $b =$ breadth and $h =$ depth at the section where the greatest bending-moment acts, the depth being measured parallel to the load.

Then if I denote the moment of inertia of the section of greatest bending-moment about its neutral axis, we shall have

$$I = \frac{bh^3}{12}.$$

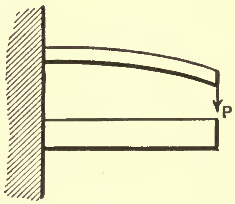
We will now consider six cases of flat springs, and will determine P, δ, δ_1, and V for each case, and for this purpose we only need to apply the ordinary rules for the strength and deflection of beams.

CASE 1. *Simple rectangular spring, fixed at one end and loaded at the other.*

$$P = \tfrac{1}{6} f \frac{bh^2}{l}, \qquad (23)$$

$$\delta = 4 \frac{l^3}{bh^3} \frac{L}{E}, \qquad (24)$$

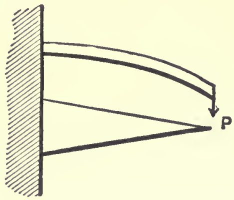
$$\delta_1 = \tfrac{2}{3} \frac{l^2}{h} \frac{f}{E}; \qquad (25)$$



$$\therefore \quad P\delta_1 = \tfrac{1}{9} \frac{f^2}{E} (bhl) = \tfrac{1}{9} \frac{f^2}{E} V$$

$$\therefore \quad V = 9 \frac{E}{f^2} P\delta_1. \qquad (26)$$

CASE 2. *Spring of uniform depth and uniform strength, triangular in plan, fixed at one end and loaded at the other.*



$$P = \tfrac{1}{6} f \frac{bh^2}{l}, \qquad (27)$$

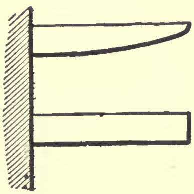
$$\delta = 6 \frac{l^3}{bh^3} \frac{L}{E}, \qquad (28)$$

$$\delta_1 = \frac{l^2}{h} \frac{f}{E}; \qquad (29)$$

$$\therefore P\delta_1 = \tfrac{1}{3} \frac{f^2}{E} \left(\frac{bhl}{2}\right) = \tfrac{1}{3} \frac{f^2}{E} V;$$

$$\therefore V = 3 \frac{E}{f^2} (P\delta_1). \qquad (30)$$

CASE 3. *Spring of uniform breadth and uniform strength, parabolic in elevation, fixed at one end and loaded at the other.*



$$P = \tfrac{1}{6} f \frac{bh^2}{l}, \qquad (31)$$

$$\delta = 8 \frac{l^3}{bh^3} \frac{L}{E}, \qquad (32)$$

$$\delta_1 = \tfrac{4}{3} \frac{l^2}{h} \frac{f}{E}; \qquad (33)$$

$$\therefore P\delta_1 = \tfrac{1}{3} \frac{f^2}{E} (\tfrac{2}{3} bhl) = \tfrac{1}{2} \frac{f^2}{E} V;$$

$$\therefore V = 3 \frac{E}{f^2} (P\delta_1). \qquad (34)$$

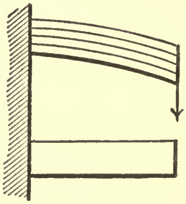
CASE 4. *Compound wagon spring, made up of n simple rectangular springs laid one above the other, fixed at one end and loaded at the other.*

Let the breadth be b, and the depth of each separate layer be h. Then

$$P = \frac{n}{6} f \frac{bh^2}{l}, \qquad (35)$$

$$\delta = \frac{4}{n} \frac{l^3}{bh^3} \frac{L}{E}, \qquad (36)$$

$$\delta_1 = \frac{2}{3} \frac{l^2}{h} \frac{f}{E}; \qquad (37)$$



$$\therefore\ P\delta_1 = \frac{1}{9} \frac{f^2}{E}(nbhl) = \frac{1}{9} \frac{f^2}{E} V;$$

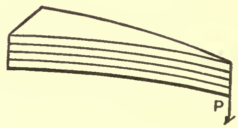
$$\therefore\ V = 9 \frac{E}{f^2}(P\delta_1). \qquad (38)$$

CASE 5. *Compound spring composed of n triangular springs laid one above the other, fixed at one end and loaded at the other.*

$$P = \frac{n}{6} f \frac{bh^2}{l}, \qquad (39)$$

$$\delta = \frac{6}{n} \frac{l^3}{bh^3} \frac{L}{E}, \qquad (40)$$

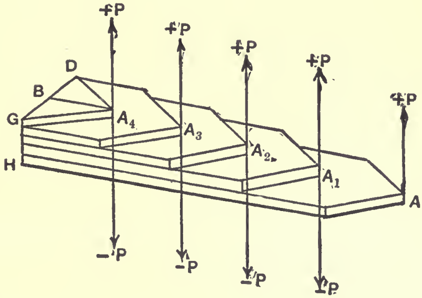
$$\delta_1 = \frac{l^2}{h} \frac{f}{E}; \qquad (41)$$



$$\therefore\ P\delta_1 = \frac{1}{3} \frac{f^2}{E}\left(\frac{nbhl}{2}\right) = \frac{1}{3} \frac{f^2}{E} V;$$

$$\therefore\ V = 3 \frac{E}{f^2}(P\delta_1). \qquad (42)$$

CASE 6. This case differs from the last in that in order to economize material we superpose springs of different lengths, and make them of such a shape that by the action of a single force at the free end they are bent in arcs of circles of nearly or exactly the same radius.



The force P bends the lowest triangular piece AA in the arc of a circle. The length of this piece is $\frac{l}{n}$.

In order that the remaining parallelopipedical portion may bend into an arc of the same circle it is necessary that it should have acting on it a uniform bending-moment throughout, and this is attained if it exerts a pressure at A_1 upon the succeeding spring equal to the force P, and following this out we should find that the entire spring would bend in an arc of a circle.

The values of P, δ, δ_1, and V are correctly expressed for this case by (39), (40), (41), and (42).

For any flat springs which are supported at the ends and loaded at the middle, or where two springs are fastened together, it is easy to compute, by means of the formulæ already developed, by making the necessary alterations, the quantities P, δ δ_1, and V, and this will be left to the student.

COILED SPRINGS SUBJECTED TO TRANSVERSE STRESS.

Three cases of coiled springs will now be given as shown in the figures, and the values of P, δ, δ_1, and V will be determined for each.

In each of these cases let R be the leverage of the load, and let $\omega =$ angle turned through under the load. Then we may observe that all the three cases are cases of beams subjected to a uniform bending-moment throughout their length, this bending-moment being LR for load L and PR for load P.

CASES 1 and 2. *Coiled spring, rectangular in section.*

$$P = \tfrac{1}{6} f \frac{bh^2}{R}, \qquad (43)$$

$$\delta = 12 \frac{lR^2}{bh^3} \frac{L}{E}, \qquad (44)$$

$$\delta_1 = 2 \frac{f}{E} \frac{Rl}{h}; \qquad (45)$$

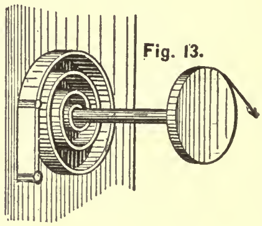


Fig. 13.

$$\therefore P\delta_1 = \tfrac{1}{3} \frac{f^2}{E} (bhl) = \tfrac{1}{3} \frac{f^2}{E} V;$$

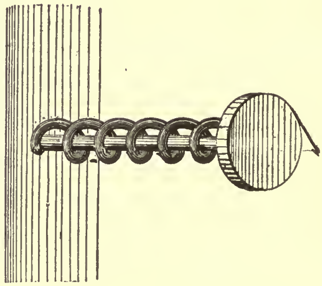
$$\therefore V = 3 \frac{E}{f^2} (P\delta_1). \qquad (46)$$

CASE 3. *Coiled spring, circular in section.*

$$P = \frac{\pi}{32} f \frac{d^3}{R}, \qquad (47)$$

$$\delta = \frac{64}{\pi} \frac{lR^2}{d^4} \frac{L}{E}, \qquad (48)$$

$$\delta = 2 \frac{lR}{d} \frac{f}{E}; \qquad (49)$$



$$\therefore P\delta_1 = \tfrac{1}{4} \frac{f^2}{E} \left(\frac{\pi d^2 l}{4}\right) = \tfrac{1}{4} \frac{f^2}{E} V;$$

$$\therefore V = 4 \frac{E}{f^2} (P\delta_1) \qquad (50)$$

TIME OF OSCILLATION OF A SPRING.

Since in any spring the load producing any displacement is proportional to the displacement, it follows that when a spring oscillates, its motion is harmonious.

Suppose the load on the spring to be P, and hence its normal displacement to be δ_1. Now let the extreme displacements on the two sides of δ_1 be δ_0, and the force producing it p, so that the actual displacement varies from $\delta_1 + \delta_0$ to $\delta_1 - \delta_0$, and the force acting varies from $P+p$ to $P-p$.

Now, from the properties of the spring we must have

$$\frac{p}{\delta_0} = \frac{P}{\delta_1}; \quad \therefore \delta_0 = \frac{p}{P}\delta_1. \tag{51}$$

Moreover, in the case of harmonic motion the maximum value of the force acting is $\dfrac{W\alpha^2 r}{g}$ (see p. 104). But the load oscillating is P instead of W, and the extreme displacement is δ_0 instead of r.

Hence we have

$$p = \frac{P}{g}\alpha^2\delta_0 = \frac{p}{g}\alpha^2\delta_1; \tag{52}$$

$$\therefore \alpha^2 = \frac{g}{\delta_1};$$

$$\therefore \alpha = \sqrt{\frac{g}{\delta_1}}. \tag{53}$$

Hence the time of a double oscillation

$$t = \frac{2\pi}{\alpha} = 2\pi\sqrt{\frac{\delta_1}{g}}. \tag{54}$$

CHAPTER VII.

STRENGTH OF MATERIALS AS DETERMINED BY EXPERIMENT.

§ 216. Whatever computations are made to determine the form and dimensions of pieces that are to resist stress and strain should be based upon experiments made upon the materials themselves.

The most valuable experiments are those made upon pieces of the same quality, size, and form as those to which the results are to be applied, and under conditions entirely similar to those to which the pieces are subjected in actual practice.

From such experiments the engineer can learn upon what he can rely in designing any structure or machine, and this class of tests must be the final arbiter in deciding upon the quality of material best suited for a given service. An attempt will be made in this chapter to give an account of the most important results of experiments on the strength of materials, and to explain the modes of using the results.

While the importance of making tests upon full-size pieces, and of introducing into the experiments the conditions of practice, is pretty generally recognized to-day, nevertheless there are some who have not yet learned to recognize the fact that attempts to infer the behavior of full-size pieces under practical conditions from the results of tests on small models, made under conditions which are, as a rule, necessarily, quite different from those of practice, are very liable to lead to conclusions that are entirely erroneous.

Such a proceeding is in direct violation of a principle that the physicist is careful to observe throughout his work, viz.: not to apply the results to cases where the conditions are essentially different from those of the experiments.

When the quality of the material suited for a given service is known, tests of the material furnished must be made to determine its quality. Such tests, made upon small samples, should be of such a kind that there may be a clear understanding, as to the quality desired, between the maker of the specifications and the producer. Whenever possible, standard forms of specimens and standard methods of tests should be used.

The determination of standards is occupying the attention of the Int. Assoc. for Testing Materials, the British Standards Committee, the Am. Soc. for Testing Materials, and others.

To ascertain the quality of the material tensile tests are most frequently employed, their objects being to determine the tensile strength per square inch, the limit of elasticity, the yield-point, the ultimate contraction of area per cent, the ultimate elongation per cent in a certain gauged length, and sometimes the modulus of elasticity.

While the standard forms and dimensions will be given later, the following general classification of the forms in use will be given here, viz.:

1°. The specimen may be provided with a shoulder at each end, having a larger sectional area than the main body of the specimen, the section of this being uniform throughout as shown in Fig. *a*, the latter being of so great a length in proportion to the diameter that the stretch of the specimen is not essentially different from what it would

Fig. *a*.

be if the section were uniform throughout. The shoulders are, of course, the portions of the specimen where the holders (or clamps) of the testing-machine are attached.

2°. In the case of a round specimen of that kind there may be a screw-thread on the shoulders as shown in Fig. *b*.

Fig. *b*.

In the case of a brittle material, as cast-iron or hard steel, it is desirable to use a holder with a ball-joint, and to screw the specimen into the holder.

3°. The specimen may be provided with a shoulder at each end, the main body of the specimen being, however, so short in proportion to the diameter that the stretch is essentially modified. Such a form is shown in Fig. *c*.

Fig. *c*.

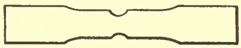


Fig. *d*.

4°. The specimen may be a grooved specimen as shown in Fig. *d*, where the length of the smallest section is zero.

5°. The section of the specimen may be uniform throughout, the length between the holders being so great in proportion to the diameter that the stretching of the fibres is not interfered with. This form of specimen is shown in Fig. *e*.

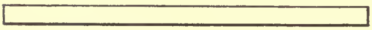


Fig. *e*.

Assume a specimen of ductile material, as mild steel or wrought-iron, of the 1st or the 5th shape, subjected to stress in the testing-machine, or else by direct weight, and suppose that we mark off upon the main body, i.e., the parallel section of the specimen, a gauged length of 8 or 10 inches (preferably 8 inches), and measure, by means of some form of extensometer, the elongations in the gauged length, corresponding to the stresses applied; then plot a stress-strain diagram as shown in Fig. *f*, having stresses per square inch for abscissæ, and the corresponding strains for ordinates.

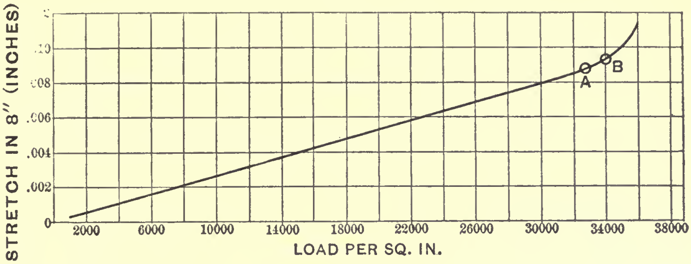


Fig. *f*.

We shall find that the strains begin by being proportional to the stresses, but when a certain stress is reached, called the "limit of elasticity" or "elastic limit," shown at A, the strains increase more rapidly than the stresses, but the rate of increase in the ratio of the strain to the stress is not large until a stress is reached called the "yield-point" or "stretch-limit," shown at B, which is usually a little larger than the elastic limit; and then the rate of increase of the ratio of strain to stress becomes much larger.

Observe, also, that if a small load be applied to the piece under test, and then removed, the deformation or distortion caused by the application of the load apparently vanishes, and the piece resumes its original form and dimensions on the removal of the load; in other words, no permanent set takes place. When the load, however, is increased beyond a certain point, the piece under test does not return entirely to its original dimensions on the removal of the load, but retains a certain permanent set. While permanent set that is easily determined begins at or near the elastic limit, and while the permanent sets corresponding to stresses greater than the elastic limit are much greater than the corresponding recoils, and hence form the greater part of the strains corresponding to such stresses, nevertheless experiments show that even a very small load will often produce a permanent set, and that the apparent return of the piece to its original dimensions is, in a number of cases, only due to the want of delicacy in the measuring-instruments at our command.

After the elastic limit and the yield-point have been passed, the ratio of the strain to the stress is much greater than before, the stretch becomes local, with a local contraction of area, this being due to the plasticity of the metal.

Finally, when the maximum stress is applied, or, in other words, the breaking-stress, the behavior is apparently somewhat different when the piece is subjected to dead weight from what it is when in a testing-machine. In the former case, when the maximum load is reached, the specimen continues to stretch rapidly, without increase in the load, until the specimen breaks.

In the case of the testing-machine, however, the application of the maximum load causes, of course, the specimen to stretch, but this stretch naturally reduces the load applied, and the actual load under which the specimen separates into two parts is less, and often very considerably less, than the maximum or breaking stress.

Observe that the terms "breaking-load" and "breaking-stress" are always used to mean the "maximum load" and "maximum stress" respectively, and are never used to denote the load or the stress under which the specimen separates into two parts when the latter differs from the former.

If the stretch of the specimen, as described above, is in any way interfered with, the behavior of the specimen will not be a proper criterion of the properties of the material; the percentage contraction of area at fracture will vary with the amount of interference with the stretch, and hence with the proportions of the specimen; and the maximum or breaking strength will be greater than the real maximum or breaking strength per square inch of the material. Hence it follows that the 3d and 4th forms of specimen do not indicate correctly the quality of the material, furnishing, as they do, erroneous values for both breaking-strength and ductility.

The quantities sought in such tests as those described above (with specimens of the 1st, 2d or 5th forms) are, as already stated:

1°. The breaking-strength per square inch of the material;

2°. The limit of elasticity of the material;

3°. The yield-point or stretch-limit of the material;

4°. The ultimate contraction of area per cent;

5°. The ultimate elongation per cent in a given gauged length;

6°. The modulus of elasticity.

The first gives, of course, the tensile strength of the material; the second and third ought both to be determined, but many content themselves with the third alone, since it is much easier to obtain. While they are commonly not far apart, it is a fact that certain kinds of stress to which the piece may be subjected may cause them to become very different from each other. The fourth and fifth are the usual ways of measuring the ductility of the metal; and while the fourth is the most definite, the fifth is very much employed, and finds favor with most iron and steel manufacturers. The sixth is not often determined for commercial work, but it is one of the important properties of the metal.

Of these six properties the two most universally insisted upon in specifications for material to be used in the construction of structures or of machines are ductility, which is universally recognized as an all-important matter, and a suitable breaking-strength per square inch, both a lower and an upper limit being generally prescribed for this last.

On the other hand, although cast-iron and hard steel are brittle metals when compared with wrought-iron and mild steel, nevertheless it is true that the third and fourth forms of specimen will show too high results for tensile strength even in these materials on account of the interference with the stretch of the metal.

§ 217. **Cast-Iron.**—Cast-iron is a combination of iron with carbon, the most usual quantity being from 3 to 4 per cent. The large amount of carbon which it contains is its distinguishing feature, and determines its behavior in most respects. Besides carbon, cast-iron contains such substances as silicon, phosphorus, sulphur, manganese, and others. A considerable amount (more than 1.37 per cent as stated by Prof. Howe) of silicon forces carbon out of combination and into the graphitic form, thus lowering the strength.

Pig-Iron is the result of the first smelting, being obtained directly from the blast-furnace. The ore and fuel (usually coke, though anthracite coal is used to some extent, and sometimes charcoal) are put into the furnace, together with a flux, which is usually limestone, in suitable proportions. The mass is brought to a high heat, a strong blast of heated air being introduced. The mass is thus melted, the fluid metal settling to the bottom, while slag, which is the result of the combination of the flux with impurities of the ore and fuel, rises to the top. The iron is drawn off in the liquid state and run into moulds, the result being pig-iron.

The metal usually contains from 3 to 4 per cent of carbon, a part being chemically combined with the iron, and a part in the form of graphite. The larger the proportion of combined carbon, the whiter the fracture, and the harder and more brittle the product, while the larger the proportion of graphite, the darker the fracture, and the softer and less brittle the product. That which has most of its carbon in combination is called white iron, while that which contains a large proportion of graphite is called gray cast-iron.

Pig-iron also contains silicon, sulphur, phosphorus, etc. The quantity of the first two can, to a certain extent, be controlled in the furnace, but not that of the last, so that if low phosphorus is desired, the ore and the fuel used must both be low in phosphorus.

Gray cast-iron has been, and is sometimes classified in various ways, according to the proportions of the combined carbon, and of the graphite, but the most modern practice is to sell, buy, and specify the iron by means of its chemical composition, and not by brands.

That which contains the largest amount of carbon in mechanical mixture is, as a rule, soft and fusible, and hence suitable for making castings where precision of form is the chief desideratum, as its fusibility causes it to fill the mould well. For general use in construction, where strength and toughness are all-important considerations, those grades are required which are neither extremely soft nor extremely hard.

As to the adaptability of cast-iron to construction, it presents certain advantages and certain disadvantages. It is the cheapest form of iron. It is easy to give it any desired form. It resists oxidation better than either wrought-iron or steel. Its compressive strength is comparatively high when the castings are small and perfect. On the other hand, its tensile strength is much less than that of wrought-iron, or that of steel, averaging in common varieties from 16000 or 17000 to about 26000 pounds per square inch. It cannot be riveted or welded. It is a brittle and not a ductile material, it does not give much warning before fracture, and, while the stretch under any given load per square inch is decidedly larger than that of wrought-iron or steel, its total stretch before fracture is small when compared with wrought iron and steel. One of the difficulties in the use of cast-iron in construction is its liability to initial strains from inequality in cooling. Thus if one part of the casting is very thin and another very thick, the thin part cools first, and the other parts, in cooling afterwards, cause stresses in the thin part.

The fracture of good cast-iron should be of a bluish-gray color and close-grained texture.

At one time cast-iron was extensively used for all sorts of structural work, but it was soon superseded by wrought-iron, and later by steel.

Thus it is no longer used in bridgework, nor for floor-beams of a building, though it is still used to a considerable extent for the columns of buildings ; and for this purpose it has in its favor the fact that it resists the action of a fire better than wrought iron or steel. Thus, in the present day, when the steel skeleton construction of buildings is so extensively employed, it is very necessary to protect the steel beams and columns by covering them with some non-conducting material, as, otherwise, they would be liable to collapse in case of fire.

It is used in cases where the form of the piece is of more importance than strength, and also where, on account of its form, it would be difficult or expensive to forge ; thus hangers, pulleys, gear-wheels, and various other parts of machinery of a similar character are usually made of cast-iron, as well as a great many other pieces used in construction. It is also used where mass and hence weight is an important consideration, as in the bed-plates and the frames of machines, etc.

Malleable Iron.—When a casting, in which toughness is required is to be made of a rather intricate form, it is frequently the custom to malleableize the cast-iron, i.e., to remove a part of its carbon, and the result is—provided the casting is small —a product that can be hammered into any desired shape when cold, but is brittle when hot.

A list of references to some of the principal experimental works on the strength and elasticity of cast iron will be given.

1°. Eaton Hodgkinson : (*a*) Report of the Commissioners on the Application of Iron to Railway Structures.

(*b*) London Philosophical Transactions. 1840.

(*c*) Experimental Researches on the Strength and other Properties of Cast-Iron. 1846.

2°. W. H. Barlow : Barlow's Strength of Materials.

3°. Sir William Fairbairn : On the Application of Cast and Wrought Iron to Building Purposes.

4°. Major Wade (U.S.A.) : Report of the Ordnance Department on the Experiments on Metals for Cannon. 1856.

5°. Capt. T. J. Rodman : Experiments on Metals for Cannon.

6°. Col. Rosset: Resistenza dei Principali Metalli da Bocchi di Fuoco.

7°. Tests of Metals made on the Government Testing Machine at Watertown Arsenal, 1887, 1888, 1889, 1890, 1891, 1892, 1893, 1894, 1896, 1897, 1898.

8°. Transactions Am. Soc. Mechl. Engrs. for 1889, p. 187 *et seq.*

9°. W. J. Keep: (*a*) Transverse Strength of Cast-iron. Trans. Am. Soc. Mechl. Engrs., 1893.
(*b*) Relative Tests of Cast-iron. Trans. Am. Soc. Mechl. Engrs., 1895.
(*c*) Transverse Strength of Cast-iron. Trans. Am. Soc. Mechl. Engrs., 1895.
(*d*) Keep's Cooling Curves. Trans. Am. Soc. Mechl. Engrs., 1895.
(*e*) Strength of Cast-iron. Trans. Am. Soc. Mechl. Engrs., 1896.

10°. Bauschinger: Mittheilungen aus dem Mech. Tech. Lab. München. Heft 12, 1885; Heft 15, 1887; Heft 27, 1902; Heft 28. 1902.

11°. Tetmajer: Mittheilungen der Materialprüfungsanstalt Zürich. Heft 3, 1886; Heft 4, 1890; Hefte 5 and 9, 1896.

12°. Technology Quarterly. October 1888, page 12 *et seq.*

13°. Technology Quarterly. Vol. 7, No. 2; Vol. 10. No. 3.

14°. Transactions of the American Foundrymen's Association.

15°. Transactions of the American Society for Testing Materials.

§ 218. **Tensile Strength of Cast-iron.**—As the use of cast-iron to resist tension has been almost entirely superseded by that of wrought-iron and steel, results of tests of full-size pieces of cast-iron in tension are not available. Tensile tests, however, have been extensively employed to determine the quality; especially so when cast-iron cannon were in use; and tensile tests of cast-iron are still made, to a certain extent, for the determination of quality. For such tests standard specimens should be used, and attempts are being made to reduce their number.

As the strength that should be attained in such specimens will become evident from the Standard Specifications of the Am. Soc. for Testing Materials, on page 385 *et seq.*, only a few tensile tests will be quoted here, and those, for the purpose of acquainting the reader with the results of some tensile tests of cast-iron.

About 1840 Eaton Hodgkinson made a few experiments to determine the laws of extension of cast-iron, and for this purpose used rods 10 feet long and 1 square inch in section. The tables of average results are given below.

These tables show that the ratio of the stress to the strain of cast-iron varies with the load, growing gradually smaller as the load increases, that with moderate loads the ratio of stress to

RESULTS OF NINE TENSILE TESTS.

Weights Laid on in Pounds.	Strains in Fractions of the Length.	Ratio of Stress to Total Strain.
1053.77	0.00c07	14050320
1580.65	0.00011	13815720
2107.54	0.00016	13597080
3161.31	0.00024	13218000
4215.08	0.00033	12936360
5268.85	0.00042	12645240
6322.62	0.00051	12377040
7376.39	0.00061	12059520
8430.16	0.00072	11776680
9483.94	0.00083	11437920
10537.71	0.00095	11314440
11591.48	0.00107	10841640
12645.25	0.00121	10479480
13699.83	0.00139	9855960
14793.10	0.00155	9549120

RESULTS OF EIGHT COMPRESSIVE TESTS.

Weights Laid on in Pounds.	Strains in Fractions of the Length.	Ratio of Stress to Total Strain.
2064.75	ocoo.16	13214400
4129.49	0.00032	12778200
6194.24	0.00050	12434040
8258.98	0.00066	12578760
10323.73	0.00083	12458280
12388.48	0.00100	12357600
14453.22	0.00188	12245880
16517.97	0.00136	12132240
18582.71	0.00154	12050400
20647.46	0.00172	12013680
24776.95	0.00208	11911560
28906.45	0.00247	11679720
33030.80	0.00295	11215560

strain for tension of cast-iron does not differ materially from that for compression, and that the difference increases as the load becomes greater. The agreement is even closer in the case of wrought-iron and steel.

The gradual decrease of the ratio of stress to strain with the increase of load shows that Hooke's law, "*Ut tensio sic vis*" (the stress is proportional to the strain), does not hold true in cast-iron. Hence, strictly speaking, cast-iron has no elastic limit and no modulus of elasticity, nevertheless we are accustomed to call the ratio of the stress to the strain under moderate loads the modulus of elasticity of the cast-iron.

In making specifications intended to secure a good quality of cast-iron it is very common to call for a transverse test. Indeed the resolutions of the international conferences relative to uniform methods of testing recommend, in the case of cast-iron:

(*a*) Test-pieces to be of the shape of prismatic bars 110 cm. standard length (43″) and to have a section of 3 cm. square (1″.18), one having an addition on one end, from which cubes can be cut for compression tests.

(*b*) Three such specimens to be tested for transverse strength.

(*c*) The tensile strength to be determined from turned test-pieces 20 mm. (0″.785) diameter and 200 mm. (7″.85) long, cut from the two ends of the test-pieces broken by flexure.

(*d*) The compressive strength to be determined from cubes 3 cm. (1″.18) on a side cut from the first specimens, pressure to be applied in the direction of the axis of the original bar.

These requirements, while calling for transverse tests, call also for tensile and compressive tests.

T e Standard Specifications of the Am. Soc. for Testing Materials will be found on page 385 *et seq.*

Inasmuch as the tensile strength has been, and is also made the basis of specifications for cast-iron, it is important to consider what should be attained in this regard.

For this purpose a few tables of comparatively modern tests will be given here, and it will be seen that in the ordinary varieties of cast-iron it is easy to secure tensile strengths from 16,000 to 25,000 pounds per square inch, and that more can be secured by taking proper precautions in the manufacture.

Indeed cast-iron which, when tested in the form of a grooved specimen, shows a tensile strength of at least 30,000 pounds per square inch is called *gun-iron*, this having been a requirement of the United States Government, in the days of cast-iron cannon, for all cast-iron that was to be used in their manufacture.

The following table is taken from a paper on the Strength of Cast-Iron, by Mr. W. J. Keep, published in the Transactions of the American Society of Mechanical Engineers for 1896, and it gives the averages of the tensile strengths of the fifteen different series of tests recorded in the paper. This table is given here merely as an example of the results that can be obtained by tension tests upon usual varieties of cast-iron. The table is as follows:

AVERAGES OF TENSION TESTS OF ROUND BARS.

No. of Series.	Area of Section, 0.375 Sq. In.	Area of Section 1.12 Sq In.	No. of Series.	Area of Section 0.375 Sq. In.	Area of Section, 1.12 Sq. In.
	Breaking Load per Sq. Inch.	Breaking Load per Sq. Inch		Breaking Load per Sq Inch.	Breaking Load per Sq. Inch.
1	20000	15700	9	....	14800
2	20580	22500	10	....	....
3	25050	20450	11	....	17000
4	21850	19350	12	17700	17500
5	22425	19750	13	14000	21300
6	25550	17200	14	24400	20300
7	18950	17700	15	23525	20500
8	17700	15350			

The following table of results of tension tests of ordinary cast-iron from another source will also be given for the same purpose as Mr. Keep's results:

CAST-IRON TENSION.

Dimensions.	Original Section. (Sq. In.)	Maximum Load. (Lbs. per sq. in.)	Modulus of Elasticity.	Dimensions.	Original Section. (Sq. In.)	Maximum Load. (Lbs. per sq in.)	Modulus of Elasticity.
1.02 × 1.04	1.06	19340	14857000	1.00 × 1.00	1.00	17100	13333000
1.03 × 1.02	1.05	23910	15481000	1.00 × 1.02	1.02	19068	13680000
1.00 × .98	.98	21180	15238000	1.00 × 1.00	1.00	18000	13333000
1.00 × .97	.97	23227	15881000	1.00 × 1.02	1.02	19299	12057000
1.02 × 1.06	1.08	19830	14539000	1.06 × .98	1.03	17488	13249000
1.[illegible] × 1.03	1.03	20413	17632000	1.00 × .98	.98	19500	13250000
.93 × 1.00	.93	16774	14337000	1.02 × 1.02	1.03	20747	14543000
1.00 × 1.00	1.00	18600	15383000	1.03 × 1.03	1.06	18620	13434000
1.00 × 1.00	1.00	18000	16666000	1.00 × 1.00	1.00	18910	13043000
1.00 × 1.00	1.00	19400	17911000				
1.00 × 1.00	1.00	20950	15789000	1.00 × 1.00	1.00	19900	15000000
1.00 × 1.00	1.00	22900	15000000	1.00 × 1.02	1.02	19594	13373000
1.00 × 1.00	1.00	22400	15564000	1.01 × 1.03	1.04	16341	13108000
1.00 × 1.00	1.00	21300	15384000	1.01 × 1.03	1.04	13844	13640000
1.00 × 1.02	1.02	19692	15966000	1.02 × 1.08	1.01	13798	11840000
1.02 × 1.03	1.05	21005	15075000	1.00 × 1.02	1.02	17647	12787000
1.08 × 1.23	1.33	20600	11900000	1.03 × 1.03	1.06	14025	12568000
1.00 × 1.03	1.03	17067	12676000	1.04 × 1.02	1.06	15083	13466000
1.00 × 1.03	1.03	19900	12929000	1.02 × 1.04	1.06	16874	9751900
.08 × 1.03	1.02	16404	12577000	1.00 × 1.00	1.00	20000	13043000
1.00 × 1.02	1.02	16450	12570000				

Colonel Rosset, of the Arsenal at Turin, made a series of experiments upon the influence of the shape of the specimen upon the tensile strength. For this purpose he used specimens with shoulders; and, among other tests, he compared the strength of the same iron by using specimens the lengths of whose smallest parts were respectively 1 metre, 30 millimetres, and 0 millimetres, with the following results:—

Length of Specimen.	Tensile Strength, in lbs., per Square Inch.		
	1st Cannon.	2d Cannon.	3d Cannon.
1 metre . .	31291	25601	28019
30 millimetres .	32571	34562	30011
0 millimetres .	33993	36411	30011

It will thus be seen that, before we can decide upon the quality of cast-iron as affected by the tensile strength, it is necessary to know the length of that part of the specimen which has the smallest area. Colonel Rosset's tests of cast-iron were almost entirely confined to high-grade irons, suitable to use in cannons.

He deduced, for mean value of the modulus of elasticity of the specimens 1 metre in length, 20419658 lbs. per square inch: this, of course, is a modulus only adapted to these high grades, and is not applicable to common cast-iron.

§ 219. **Cast-Iron Columns.**—In consequence of the high compressive strength shown by cast-iron when tested in small pieces, and in pieces free from imperfections, it was once considered a very suitable material for all kinds of columns. Nevertheless, its use for the compression members of bridge and roof trusses has been abandoned; cast-iron having been displaced first by wrought-iron and subsequently by steel, which is the substance now in use for these purposes.

The principal reasons for the change are the lack of ductility, and the consequent brittleness of cast-iron, that it cannot be riveted, and that if it breaks it cannot be easily repaired. Cast-iron is, however, used to a very considerable extent for the columns of buildings.

The Gordon, the so-called Euler, and the Hodgkinson formulæ for the breaking-strength of cast-iron columns, have all been given in paragraphs 208, 208*a*, and 209. They are, however, all based upon tests made upon very small columns, and do not give results agreeing with the tests of such full-size columns as are used in practice. We will next consider, therefore, the tests that have been made upon full-size cast-iron columns, and the conclusions that are warranted in the light of these tests.

Two sets of tests of cast-iron mill columns have been made on the Government testing-machine at Watertown Arsenal; an account of these sets of tests is published in their reports of 1887 and of 1888.

The first lot consisted of eleven old cast-iron columns, which had been removed from the Pacific Mills at Lawrence, Mass., during repairs and alterations.

The second lot consisted of five new cast-iron columns cast along with a lot that was to be used in a new mill.

Of these five, the strength of two was greater than the capacity of the testing-machine, hence only three were broken; while in the case of the other two the test was discontinued when a load of 800000 lbs. was reached. All the columns contained a good deal of spongy metal, which of course rendered their strength less than it would otherwise have been; nevertheless, inasmuch as this is just what is met with in building, it is believed that these tests furnish reliable information as to what we should expect in practice, and that this information is much more reliable than any that can be derived from testing small columns.

In all the tests the compressions were measured under a large number of loads less than the ultimate strength; but inasmuch as it is not possible, in the case of cast-iron, to fix any limits within which the stress is proportional to the strain, no attempt will here be made to compute the modulus of elasticity. Hence there will be given here a table showing the dimensions of the columns tested, their ultimate strengths, and, in those cases where they were measured, the horizontal and vertical components of their deflections, measured at the time when their ultimate strengths were reached, as the Government machine is a horizontal machine. A glance at the table will make it evident that we cannot, in the case of such columns, rely upon a crushing strength any greater than 25000 or 30000 lbs. per square inch of area of section. Hence it would seem to the writer that, in order to proportion a cast-iron column to bear a certain load in a building, we should determine the outside diameter in such a way as to avoid an excessive ratio of length to diameter; if this ratio is not much in excess of twenty, the extra stress produced by any eccentricity of the load due to the deflection of the column will be very slight. At the same time see that the thickness of metal is sufficient to insure a good sound casting.

Now, having figured the column in this way, compute the outside fibre stress (using the method of § 207) that would occur with the loading of the floors assumed to be such as to give as great an eccentricity as it is possible to bring upon the column. If this distribution of the load is one that is likely to occur, then the maximum fibre stress in the column due to it ought not to be greatly in excess of 5000 lbs. per square inch; but if it is one which there is scarcely a chance of realizing, then the maximum fibre stress under it might be allowed to reach 10000 lbs. per square inch. If by adopting the dimensions already chosen these results can be obtained, we may adopt them; but if it is necessary to increase the sectional area in order to accomplish them, we should increase it.

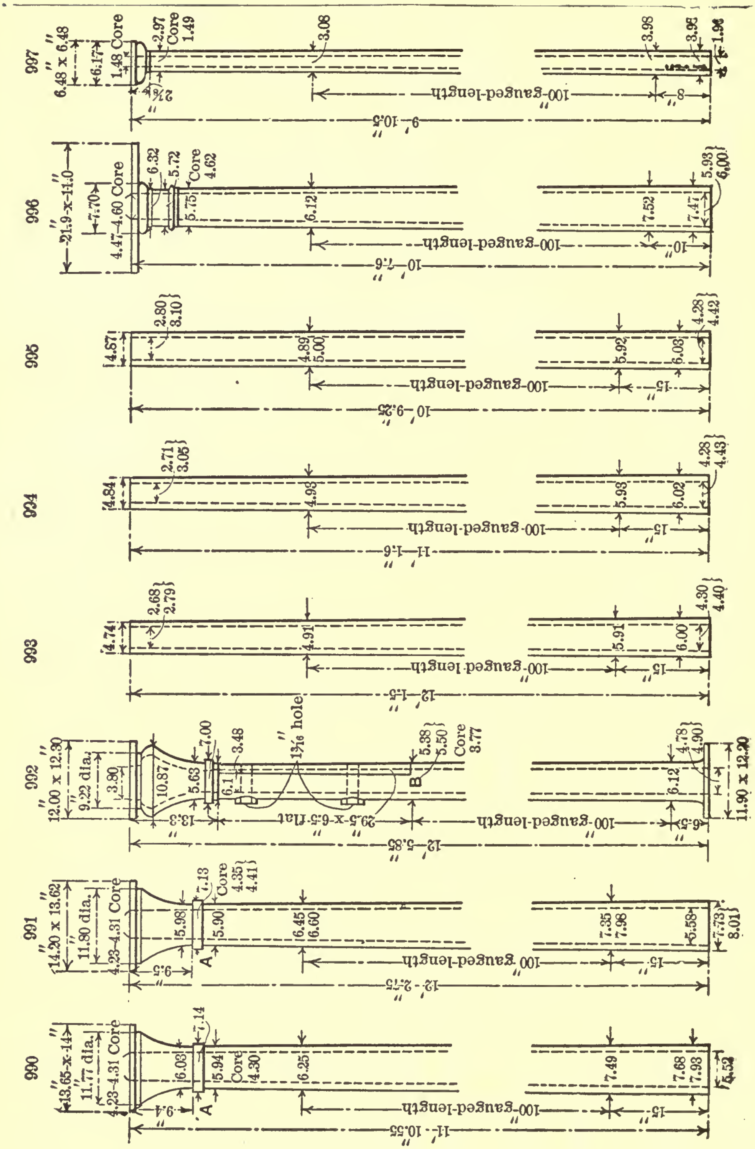
Another matter that should be referred to here is the fact that a long cap on a column is more conducive to the production of an eccentric loading than a short one; hence, that a long cap is a source of weakness in a column.

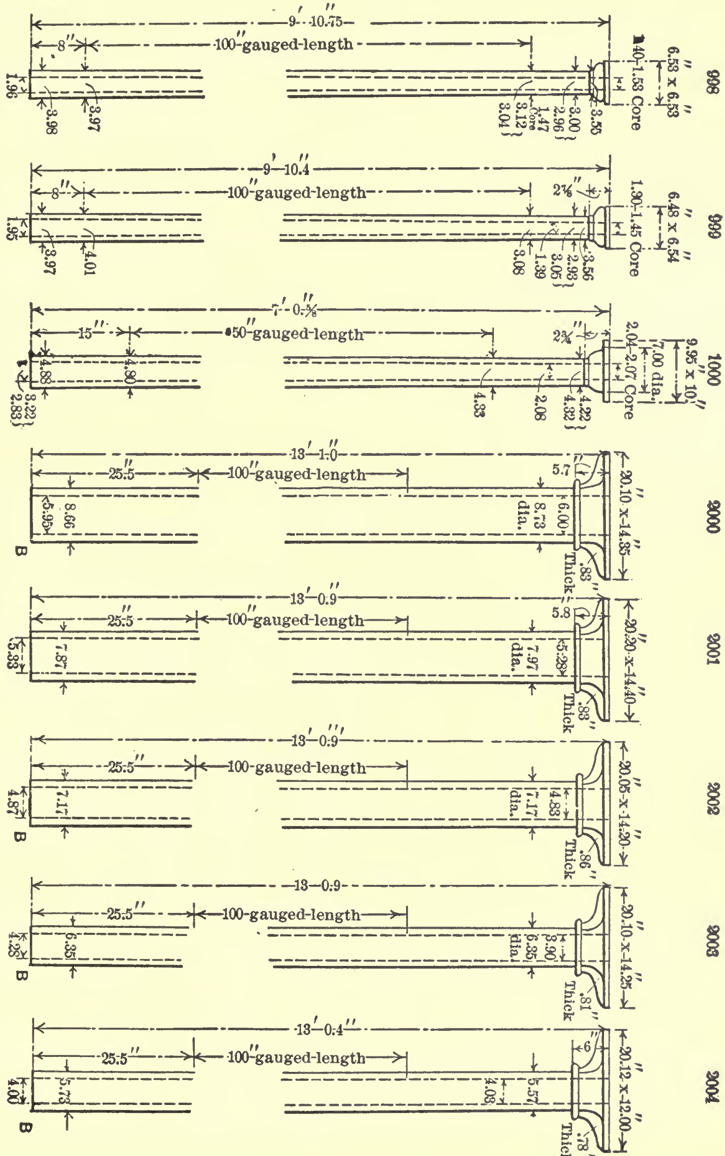
Other sources of weakness in cast-iron columns are spongy places in the casting (which correspond in a certain way with knots in wood), and also an inequality in the thickness of the two sides of the column, the result of this being the same as that of eccentric loading; and it is especially liable to occur in consequence of the fact that it is the common practice to cast columns on their side, and not on end. The engineer should, however, inspect all columns to be used in a building, and reject any that have the thickness of the shell differing in different parts by more than a very small amount.

A series of tests of full-size cast-iron columns was made by the Department of Buildings of New York City, under the direction of Mr. W. W. Ewing, in December, 1897, upon the

TESTS OF CAST-IRON COLUMNS.—(*See pages 368 and 369 for cuts.*)

	Arsenal Number.	Weight lbs.	Minimum Sectional Area, sq. in.	$\frac{l}{\rho}$	Total Ultimate Strength, lbs.	Ultimate Strength per Sq. In., lbs.	Deflections at Ultimate Strength.		Resultant Deflection.	Remarks.
							Hor.	Vert.		
Old columns.	990	715	13.19 at *A*.	77.55	512600	38880				Failed by opening cracks at *A*, and deflecting downward between the middle and *A*.
	991	733	12.27 at *A*.	79.89	531900	43350				Failed by opening cracks at *A*, and deflecting upwards near middle.
	992	639	12.08 at *B*.	90.59	404700	33500				Failed by opening cracks at *A*, and deflecting up at middle.
	993	474	11.75	106.30	315400	26840				Failed by triple flexure.
	994	439	11.89	94.86	361100	30370				Failed by triple flexure.
	995	431	11.80	90.82	352000	29830				Failed by triple flexure.
	996	598	8.94	69.20	566000	63310				Failed by triple flexure.
	997	240	5.19	142.70	165300	31850	0.68	0.42	0.80	Failed by triple flexure.
	998	240	5.27	143.00	158040	29990	0.40	1.00	1.08	Failed by triple flexure.
	999	242	5.50	144.10	183460	33350	0.25	0.50	0.56	Failed by triple flexure.
	1000	271	10.92	71.27	350900	32130				Failed by triple flexure.
New columns.	2002	959	21.75	72.59	554000	25470	1.30	0.45	1.38	Failed by triple flexure. Post concave $\frac{1}{4}''$ at middle before testing.
	2003	779	17.28	81.94	470200	27210	0.88	0.73	1.14	Failed by triple flexure.
	2004	600	13.22	90.99	231780	25100	0.38	0.85	0.93	Failed by deflecting.


997
996
995
994
993
992
991
990
100-gauged-length
Core
1 3/16" hole
29.5"-x-6.5"-flat

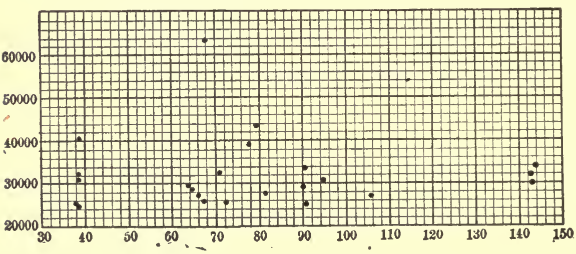

998
9′–10.75″
100″ gauged-length
999
9′–10.4″
100″ gauged-length
1000
7′–0⅝″
50″ gauged-length
2000
13′–1.0″
100″ gauged-length
2001
13′–0.9″
100″ gauged-length
2002
13′–0.9″
100″ gauged-length
2003
13–0.9
100-gauged-length
2004
13′–0.4″
100″ gauged-length

hydraulic press of the Phœnix Bridge Works. This press weighs the load on the specimen plus the friction of the piston, the latter being, of course, a variable quantity. Nevertheless great pains were taken to determine this friction, and hence the results are doubtless substantially correct.

The results are, it will be seen, similar to those obtained in the Watertown tests. The table of results is given below, and no farther comments are needed. Subsequently tests were made to determine the strength of the brackets. For this, however, the reader is referred to the Report itself, or to *Engineering News* of January 20, 1898, and for further details of the tests of the columns, to the Report itself, or to *Engineering News* of January 13, 1898.

Column Number.	Length, Inches.	Outside Diameter, Inches.	Average Thickness, Inches.	Breaking Load, Lbs.	Average Area Section, Sq. In.	ρ Inches	$\frac{l}{\rho}$	Breaking Load per sq. in., lbs.
I	190.25	15	1	1356000	43.98	4.96	38.36	30832
II	190.25	15	$1\frac{1}{8}$	1330000	49.03	4.92	38.67	27126
B_2	190.25	15	$1\frac{1}{8}$	1198000	49.03	4.92	38.67	24434
B_4	190.25	$15\frac{1}{8}$	$1\frac{1}{8}$	1246000	49.48	4.98	38.20	25181
5	190.25	15	$1\frac{11}{64}$	1632000	50.91	4.91	38.75	32057
6	190.25	15	$1\frac{3}{16}$	over 2082000	51.52	4.90	38.73	over 40411
XVI	160	$8\frac{1}{4}$ to $7\frac{3}{4}$	1	651000	21.99	2.50	64.00	29604
XVII	160	8	$1\frac{3}{64}$	645600	22.87	2.48	64.52	28229
7	120	$6\frac{1}{16}$	$1\frac{9}{64}$	455200	17.64	1.78	67.41	25805
8	120	$6\frac{3}{32}$	$1\frac{7}{64}$	474100	17.37	1.80	66.67	27236

CAST-IRON COLUMNS.



Abscissæ = length divided by radius of gyration of smallest section.

Ordinates = breaking strengths per square inch of smallest section.

The cut on page 370 shows a graphical representation of the preceding tests of full-size cast-iron columns.

In Heft VIII (1896) of the Mitt. d. Materialprüfungsanstalt in Zurich is an account of 296 cast-iron struts tested by Prof. Tetmajer; 46 being 3 cm. (1″.18) square will not be mentioned farther. The other 250 were hollow circular, the inside diameters being 10 cm. (3″.94), 12 cm. (4″.72), or 15 cm. (5″.91); the thicknesses being 1 cm. (0″.39) or 0.8 cm. (0″.31). The lengths varied from 4 m. (13′.12) to 20 cm. (7″.9). They are not the most usual thicknesses of columns for buildings, though used to a considerable extent. They might be called cast-iron pipe columns. The following table contains all those 250 cm. (8′.2) long and over, and 1 cm. thick, and one set of those 0.8 cm. thick. This will exhibit the character of the results for such columns of usual lengths. In computing $\frac{l}{\rho}$ the actual

Thickness 0″.39.					Thickness 0″.31.				
No. of Test.	Length, Feet.	Outside Diameter, Inches.	$\frac{l}{\rho}$	Ultimate Strength, Pounds per sq. in.	No. of Test.	Length, Feet.	Outside Diameter, Inches,	$\frac{l}{\rho}$	Ultimate Strength, Pounds per sq. in.
55	9.84	4.73	77.1	18481	207	13.12	4.62	103.9	11518
56	9.84	4.76	76.7	20761	208	13.12	4.61	103.9	11660
57	8.20	4.76	63.4	28156	209	11.48	4.58	91.1	16922
58	8.20	4.78	63.9	29862	210	11.48	4.59	91.9	20577
69	9.84	5.63	64.4	24174	211	9.84	4.56	78.9	19422
70	9.84	5.61	64.4	32564	212	9.84	4.60	77.7	19482
71	8.20	5.65	53.3	36546	213	8.20	4.56	65.4	31843
72	8.20	5.63	53.4	47353	214	8.20	4.61	64.7	33133
86	9.84	6.69	53.2	32564	225	13.12	5.41	87.8	15216
87	9.84	6.67	53.3	34270	226	13.12	5.43	87.5	17623
88	8.20	6.73	43.9	44224	227	11.48	5.41	76.7	22326
89	8.20	6.69	44.1	46642	228	11.48	5.39	77.3	213[illegible]
					229	9.84	5.39	66.4	23748
					230	9.84	5.41	66.1	23463
					231	8.20	5.41	54.8	38110
					232	8.20	5.41	54.5	36688
					243	13.12	6.56	71.8	22041
					244	13.12	6.54	71.9	24885
					245	11.48	6.56	62.6	27729
					246	11.48	6.56	62.5	28156
					247	9.84	6.53	53.9	35520
					248	9.84	6.54	53.9	31853
					249	8.20	6.56	44.8	41949
					250	8.20	6.56	44.8	45362

length of the strut has been used, whereas Tetmajer adds to this 9''.84, the thickness of the platforms of the machines, as they bore on knife-edges.

Prof. Bauschinger of Munich made two series of tests of full-size cast- and of wrought-iron columns to determine the effect of heating them red-hot and sprinkling them with water while under load. They were loaded in his testing-machine with their estimated safe load as calculated from the formulas.

For cast-iron,

$$P = \frac{19912A}{1 + 0.0006\frac{l^2}{\rho^2}};$$

For wrought-iron,

$$P = \frac{11378A}{1 + 0.00009\frac{l^2}{\rho^2}},$$

where $P =$ safe load (factor of safety five), $A =$ area of section, $l =$ length, $\rho =$ least radius of gyration, pounds and inches being the units.

A fire was made in a U-shaped receptacle under the post, so arranged that the flames enveloped the post. The temperature was determined from time to time by means of alloys of different melting-points; and the horizontal and vertical components of the deflections were read off on a dial as indicated by a hand attached to the post by a long wire. The post was also examined for cracks or fractures.

In the 1884 series he tested six cast-iron posts of various styles, and three wrought-iron posts, one of them being made of channel-irons and plates put together with screw-bolts, one of I irons and plates also put together with screw-bolts, and one hollow circular.

The details of the tests will not be given here, but only Bauschinger's conclusions. He said:

That wrought-iron columns, even under the most favorable adjustment of their ends and of the manner of loading, bend so much that they cannot hold their load, sometimes with a temperature less than 600° Centigrade, and always when they are at a red heat; and this bending is accelerated by sprinkling on the opposite side, even when only the ends of the post are sprinkled.

That under similar circumstances cast-iron posts bend, and this bending is increased by sprinkling; but it does not exceed certain limits, even when the post is red for its entire length and the stream of water is directed against the middle, and the post does not cease to bear its load even when cracks are developed by the sprinkling. Only when both ends of a cast-iron post are free to change their directions does sprinkling them at the middle of the opposite side when they are red make them break, but such an unfavorable case of fastening the ends hardly ever occurs in practice.

That the cracks in the columns tested occurred in the smooth parts, and not at corners or projections.

That the result of these tests warns us to be much more prudent in regard to the use of wrought-iron in building. If posts which are subjected to a longitudinal pressure bend so badly when subjected to heat on one side that they lose the power of bearing their load, how much more must this be the case with wrought-iron beams; and he urges the importance of making more experiments.

In Heft XV of the Mittheilungen he says that the results were criticised in two ways, viz.: Möller claiming that he should have used different constants, and Gerber that the wrought-iron posts were not properly made.

Bauschinger therefore concluded to make a new set of tests, and for this purpose he had made two cast-iron and five wrought-iron columns—the former being carefully cast, but on the side, while the wrought-iron ones were made by a bridge company of very good reputation, and four of them were similar to those made at the time for a new warehouse in Hamburg.

The tests were made just as before, and the following are his conclusions:

That when wrought-iron posts are as well constructed as the two referred to, they resist fire and sprinkling tolerably well, though not as well as cast-iron; but that posts constructed like the other three, even with the fire alone, and before the sprinkling begins, get so bent that they can no longer hold their load. Good construction requires that the rows of rivets shall extend through the entire length of the post, and the rivets should be quite near each other; but the tests are not extensive enough to show what are the necessary requirements to make wrought-iron posts able to stand fire and sprinkling; in order to know this more experiments are needed.

In Dingler's Polytechnisches Journal for 1889, page 259 *et seq.*, is an article by Professor A. Martens, of Berlin, upon the behavior of cast- and wrought-iron in fires, considering especially the burning of a large warehouse in Berlin, and advocating the protection of iron-work by covering it with cement. He says that there are two series of tests upon this subject, one of which is the tests of Bauschinger already explained, and the other a set of tests made by Möller and Luhmann.

No detailed account of these tests will be given here, but only Möller's conclusions, as stated by Prof. Martens, which are as follows:

1°. With ten cast-iron posts he could not get any cracks by sprinkling at a red heat; but it is to be noted that his were new posts, while those used in Bauschinger's first series were old ones, and that those in Bauschinger's second series, which were new and very carefully cast, did not show cracks either.

2°. He claims that while the cracks would allow the post still to bear a centre load, it could not bear an eccentric load or a transverse load.

3°. He claims that the load on a cast-iron post should be limited to one which shall not produce sufficient bending to bring about a tensile stress anywhere when the post is bent by the heat and sprinkling.

4°. He claims that in either cast- or wrought-iron posts, if the ends are not fixed, the ratio of length to diameter should not exceed 10, whereas if they are it should not exceed 17; also, that there is no such thing as absolute safety from fire with iron.

5°. A covering of cement delays the action of the fire, and that therefore such a covering is a protection to the post against excessive one-sided heating and cooling.

6°. Cast-iron is more likely to have at any one section a collection of hidden flaws than wrought-iron.

§ 220. **Transverse Strength of Cast-Iron.**—At one time cast-iron was very largely used for beams and girders in buildings to support a transverse load. Its use for this purpose has now been almost entirely abandoned, as it has been superseded by wrought-iron and steel.

A great many experiments have been made on the transverse strength of cast-iron; the specimens used in some cases being small, and in others large. The records of a great many experiments of this kind are to be found in the first four books of the list already enumerated in § 217. The details of these tests will not be considered here, but an outline will be given of some of the main difficulties that arise in applying the results and in using the beams.

Cast-iron is treacherous and liable to hidden flaws; it is brittle. It is also a fact that in casting any piece where the thickness varies in different parts, the unequal cooling is liable to establish initial strains in the metal, and that therefore those parts where such strains have been established have their breaking-strength diminished in proportion to the amount of these strains.

In the case of cast-iron also, the ratio of the stress to the strain is not constant, even with small loads, and is far from constant with larger loads; also, inasmuch as the compressive strength is far greater than the tensile, it follows that, in a transversely loaded beam which is symmetrical above and below the middle, the fibres subjected to tension approach their full tensile strength long before those subjected to compression are anywhere near their compressive strength. The result of all this is, that if a cast-iron beam be broken transversely, and the modulus of rupture be computed by using the ordinary formula,

$$f = \frac{My}{I},$$

we shall find, as a rule, a very considerable disagreement between the modulus of rupture so calculated and either the tensile or compressive strength of the same iron. Indeed, Rankine used to give, as the modulus of rupture for rectangular cast-iron beams, 40000 lbs. per square inch, and for open-work beams 17000 lbs. per square inch, which latter is about the tensile strength of fairly good common cast-iron.

A great deal has been said and written, and a good many experiments have been made, to explain this seeming disagreement between the modulus of rupture as thus computed, and the tensile strength of the iron. Barlow proposed a theory based upon the assumption of the existence of certain stresses in addition to those taken account of in the ordinary theory of beams, but his theory has no evidence in its favor.

Rankine claimed that the fact that the outer skin is harder than the rest of the metal would serve to explain matters, but this would not explain the fact that the discrepancy exists in the case of planed specimens also.

Neither Barlow nor Rankine seems to have attempted to find the explanation in the fact that the formula

$$f = \frac{My}{I}$$

assumes the proportionality of the stress to the strain, and hence that is less and less applicable the greater the load, and hence the nearer the load is to the breaking load. An article by Mr. Sondericker in the Technology Quarterly of October, 1888, gives an account of some experiments made by him to test the theory that "the direct stress, tension, or compression, at any point of a given cross-section of a beam, is the same function of the accompanying strain, as in the case of the corresponding stress when uniformly distributed," and the results bear out the theory very well; hence it follows that, if we use the common theory of beams, determining the stresses as such multiples of the strains as they show themselves to be in direct tensile and compressive tests, the discrepancies largely vanish, and those that are left can probably be accounted for by initial stresses due to unequal rate of cooling, and by the skin, or by lack of homogeneity. In the same article he quotes the results of other tests bearing more or less on the matter, and there will be quoted here the table on page 378.

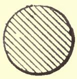
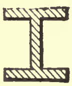
If, therefore, we wish to make use of the formula

$$M = f\frac{I}{y}$$

in calculating the strength of cast-iron beams, we cannot use one fixed value of f for all beams made of one given quality of cast-iron, but we shall have to use a very varying modulus of rupture, varying especially with the form, and also with the size of the beam under consideration. Now, in order to do this, and obtain reasonably correct results, we need, wherever possible, to use values of f that have been deduced from experiments upon pieces like those which we are to use in practice, and under, as nearly as possible, like conditions.

There are not very many records of such experiments available, and, in cases where we cannot obtain them, it will probably be best to use a value of f no greater than the tensile strength for complicated forms, and forms having thin webs. For pieces of rectangular or circular section we might probably use, for good fair cast-iron, 25000 to 30000 lbs. per square inch.

A few tests of the character referred to have been made in the engineering laboratories of the Massachusetts Institute of

Form of Beam Section.	Tensile Strength, lbs. per Sq. In.	Modulus of Rupture $f = \frac{My}{I}$, lbs. per Sq. In.	Ratio.	Condition of Specimen.	Experimenter.
	19850	41320	2.08	Turned	C. Bach.*
	16070	35500	2.21	Turned	Considère.†
	34420	63330	1.84	Turned	Considère.
	24770	54390	2.19	Turned	Robinson and Segundo. ‡
	25040	46280	1.85	Rough	Robinson and Segundo.
	Mean.		2.03		
	16070	29250	1.82	Planed	Considère.
	36270	58760	1.62	Planed	Considère.
	19090	33740	1.77	Planed	C. Bach.
	Mean.		1.74		
	19470	34000	1.75	Planed	C. Bach.
	31430	49030	1.56	Planed	Considère.
	19880	33860	1.70	Planed	Sondericker.
	24770	42340	1.71	Planed	Robinson and Segundo.
	25040	42110	1.68	Rough	Robinson and Segundo.
	Mean.		1.68		
	19470	28150	1.45	Planed	C. Bach.
	16070	22500	1.40	Planed	Considère.
	31860	36640	1.15	Planed	Considère.
	25040	31310	1.25	Rough	Robinson and Segundo.
	Mean.		1.31		
	16070	23780	1.48	Planed	Considère.
	31290	34730	1.11	Planed	Considère.
	18050	24550	1.36	Planed	Sondericker.
	22470	26150	1.16	Rough	Burgess and Vielé. §
	Mean.		1.28		

* See Zeitschrift des Vereines Deutscher Ingenieure, Mar. 3d and 10th, 1888.
† See Annales des Ponts et Chaussées, 1885.
‡ See Proceedings Institute of Civil Engineers, Vol 86.
§ See Proceedings Am. Soc. Mechl. Engrs. 1889, pp. 187 et seq.

Technology, and a brief statement of them will be given here. The first that will be referred to here is a series of experiments made by two students of the Institute, an account of which is given in the Proceedings of the American Society of Mechanical Engineers for 1889, pp. 187 *et seq.*

The object of this investigation was to determine the transverse strength of cast-iron in the form of window lintels, and also the deflections under moderate loads, and from the latter to deduce the modulus of elasticity of the cast-iron, and to compare it with the modulus of elasticity of the same iron, as determined from tensile experiments; also the tensile strength and limit of elasticity of specimens taken from different parts of the lintel were determined.

The iron used was of two qualities, marked P and S respectively.

The tensile specimens were cast at the same time, and from the same run as the lintels.

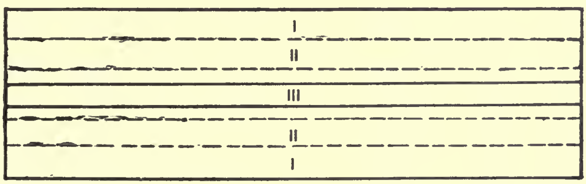
Besides this, one of each kind of window lintels was cut up into tensile specimens, and the specimens were so marked as to show from what part of the lintel they were cut.

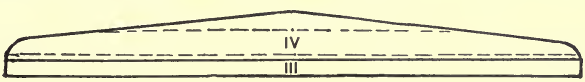
The tables of tests will now be given, and the following explanation of the symbolism employed.

P and S are used, as already stated, to denote the quality of the iron.

A and B are used to denote, respectively, that the specimen was unplaned or planed.

1, 2, 3, etc., denote the number of the test made on that particular kind and condition.





I., II., III., denote that the piece has been taken from a lintel, and also from what part, as will easily be seen by the sketch on page 379.

Thus *P. B.* 3 would signify that the specimen was of quality *P*, had been planed, and was the third test of this class.

On the other hand, *P. B.* 3 II., would signify in addition that it had been taken from a lintel, and was a piece of one of the strips marked II. in the sketch.

The following is a summary of the breaking-weights per square inch of the specimens not cut from the lintels:

P. A. 1 23757	S. A. 1 24204
P. A. 2 21423	S. A. 2 25258
P. A. 3 18938	S. A. 3 24706
P. A. 4 21409	
4)85527	3)74168
21382	24723

P. B. 1 21756	S. B. 1 29574
P. B. 3 25207	S. B. 2 23201
2)46963	2)52775
23482	26388

The following are the breaking-weights per square inch of the specimens cut from the lintels:—

P. B.	5	I	19651	
	6	I	20715	
	9	I	21076	
	10	I	21483	
	4	II	19016	
	7	II	19376	
	11	II	22146	
	12	II	20552	
	2	III	10594	(Broke at a flaw.)
	13	III	16141	
	8	IV	19616	

S. B.	6	I	29124
	7	I	28372
	8	I	25425
	3	II	24704
	4	II	29414
	5	II	23610
	9	III	27523
	10	III	18301
	4	IV	19616

All the window lintels tested were of the form shown in the figure, and all were supported at the ends and loaded at the middle, the span in every case being 52″. From the cut it will be seen that the web varied in height, being 4 inches high above the flange in the centre, and decreasing to 2.5 inches at the ends over the supports.

The following are the results of the separate tests, where tensile modulus of rupture means the outside fibre stress per square inch on the tension side, and compressive modulus of rupture that on the compression side, both being calculated from the actual breaking load by the formula

$$f = \frac{My}{I}.$$

Mark on Lintel.	Breaking-Load, Lbs.	Tensile Modulus of Rupture, lbs. per Sq. In.	Compressive Modulus of Rupture, lbs. per Sq. In.
P. 1	27220	26648	81578
P. 2	30520	29879	91467
P. 3	27200	26659	81608
S. 1	26750	26198	80164
S. 2	19850	19433	59490
S. 3	28670	28068	85924
S. 4	25120	24592	75285

The second series of experiments was made by two other students, and an account of the work is given in the same article as the former one.

The object was to determine the constants suitable to use in the formulæ for determining the strength of the arms of cast-iron pulleys; and also, incidentally, to determine the holding power of keys and set-screws.

Some old pulleys with curved arms, which had been in use at the shops, were employed for these tests. They were all about fifteen inches in diameter, and were bored for a shaft $1\frac{3}{16}$ inches in diameter.

Inasmuch as this size of shaft would not bear the strain necessary to break the arms, the hubs were bored out to a diameter of $1\frac{11}{16}$ inches diameter, and key-seated for a key one-half an inch square.

In order to strengthen the hubs sufficiently, two wrought-iron rings were shrunk on them, so as to make it a test of the arms and not of the hub.

The pulley under test is keyed to a shaft which, in its turn, is keyed to a pair of castings supported by two wrought-iron I-beams, resting upon a pair of jack-screws, by means of which the load is applied. A wire rope is wound around the rim of the pulley, and leaves it in a tangential direction vertically. This rope is connected with the weighing lever of the machine, and weighs the load applied.

In a number of the experiments one arm gave way first, and then the unsupported part of the rim broke.

The breaking-load of the separate pulleys was, of course, determined, and then it was sought to compute from this the value of f from the formula

$$f = \frac{Pxy}{nI},$$

which is the one most commonly given for the strength of pulley arms, and which is based upon several erroneous assumptions, one of which is that the bending-moment is equally divided among the several arms. In this formula

$I =$ moment of inertia of section,

$n =$ number of arms,

$y =$ half depth of each arm $=$ distance from neutral axis to outside fibre,

$x =$ length of each arm in a radial direction,

$P =$ breaking-load determined by experiment.

The results are given in the following table, the units being inches and pounds:

Number of Test.	Diam. of Pulley.	Face.	Thickness of Rim.	Width of Hub.	Thickness of Hub.	Length of Arms.	Number of Arms.	Dimensions of Arms, all elliptic. At Rim.	Dimensions of Arms, all elliptic. At Hub.	Breaking Weight.	$f=\frac{Pxy}{nl}$	Place and Manner of Fracture.	Remarks.
1	14	4	$\frac{1}{4}$	$4\frac{1}{4}$	$\frac{3}{4}$	$4\frac{1}{4}$	5	$1\frac{1}{8}\times\frac{9}{16}$	$1\frac{7}{8}\times\frac{21}{32}$	5600		Hub cracked.	Not a test of the arms.
2	15	$3\frac{1}{2}$	$\frac{5}{16}$	$3\frac{15}{16}$	$1\frac{3}{16}$	5	5	$1\frac{3}{8}\times\frac{11}{16}$	$1\frac{1}{2}\times\frac{13}{16}$	5300		Hub cracked.	Not a test of the arms.
3	$12\frac{1}{2}$	$3\frac{3}{16}$	$\frac{5}{16}$	$3\frac{1}{4}$	$\frac{3}{4}$	$4\frac{1}{4}$	6	$2\times\frac{7}{16}$	$2\times\frac{7}{16}$	2200	24425	All the arms broke at the hub.	
4	$12\frac{1}{2}$	$3\frac{3}{16}$	$\frac{5}{16}$	$3\frac{1}{4}$	$\frac{3}{4}$	$4\frac{1}{4}$	6	$2\times\frac{7}{16}$	$2\times\frac{7}{16}$	2100	23314	All the arms broke at the hub.	
5	12	3	$\frac{1}{4}$	$3\frac{3}{8}$	$\frac{11}{16}$	$3\frac{1}{2}$	5	$1\frac{5}{16}\times\frac{1}{2}$	$1\frac{5}{8}\times\frac{9}{16}$	6700	32160	One arm broke at rim and hub.	Load subsequently increased to 8000, when the rim broke.
6	$14\frac{3}{4}$	$3\frac{1}{4}$	$\frac{5}{16}$	$3\frac{3}{4}$	$\frac{17}{32}$	5	6	$1\times\frac{9}{16}$	$1\frac{1}{4}\times\frac{5}{8}$	4400	38245	All the arms broke at the hub.	
7	15	$3\frac{7}{16}$	$\frac{1}{4}$	4	$\frac{9}{16}$	5	5	$1\frac{5}{16}\times\frac{3}{4}$	$1\frac{1}{2}\times\frac{27}{32}$	4300	23060	One arm broke at the hub.	Load subsequently increased to 5300, when the rim broke.
8	24	$3\frac{1}{16}$	$\frac{5}{16}$	$3\frac{1}{4}$	$\frac{5}{8}$	$9\frac{5}{8}$	5	$1\frac{1}{8}\times\frac{11}{16}$	$1\frac{9}{16}\times\frac{3}{4}$	2000	21430	One arm broke at the hub.	Load subsequently increased to 2200, when the rim broke.
9	14	4	$\frac{5}{16}$	$4\frac{1}{4}$	$1\frac{5}{32}$	4	5	$1\frac{1}{16}\times\frac{1}{2}$	$1\frac{7}{16}\times\frac{5}{8}$				One of the arms was broken in driving it on to the shaft, so no test was made.
10	$13\frac{1}{2}$	$4\frac{1}{4}$	$\frac{5}{16}$	$4\frac{1}{4}$	$1\frac{1}{32}$	$4\frac{1}{2}$	5	$1\frac{7}{16}\times\frac{3}{4}$	$1\frac{11}{16}\times\frac{25}{32}$				There was a bushing inside the hub keyed to shaft; pulley slipped on bushing, hence no test.
11	15	$3\frac{7}{16}$	$\frac{5}{16}$	4	$\frac{9}{16}$	5	5	$1\frac{5}{16}\times\frac{3}{4}$	$1\frac{1}{2}\times\frac{27}{32}$	4300	23060		One arm broke at the hub during test of keys.
12	$19\frac{3}{4}$	4	$\frac{1}{4}$	$4\frac{1}{4}$	$\frac{3}{4}$	$7\frac{1}{8}$	5	$1\frac{5}{8}\times\frac{25}{32}$	$1\frac{15}{16}\times\frac{27}{32}$				This pulley was the one used in testing set-screws, and also some of the keys, and one of the arms broke during a key test.
Average.											26528		

In the cases of numbers 5, 7, 8, 9, and 10 some of the arms were not broken, the rims were now broken off, and the remaining arms were tested separately, the pull being exerted by a yoke hung over the end of the arm, the lower end being attached to the link of the machine.

The arms were always placed so that the direction of the pull was tangent to the curve of the rim at the end of the arm. The actual modulus of rupture was then determined by calculation from the experimental results, and is recorded in the following table, the units being inches and pounds:—

Number of Arms.	Dimensions of Section at Fracture: all elliptical.	Bend of Arm with or against Load.	Modulus of Rupture.	Average Modulus of Rupture for each Pulley.
5 — 1	$1\frac{9}{16} \times \frac{17}{32}$	against	45396	45396
7 — 1	$1\frac{1}{2} \times \frac{7}{8}$	against	36802	
7 — 2	$1\frac{15}{32} \times \frac{7}{8}$	against	39537	
7 — 3	$1\frac{17}{32} \times \frac{7}{8}$	with	46407	40915
8 — 1	$1\frac{15}{32} \times \frac{13}{16}$	against	35503	
8 — 2	$1\frac{11}{32} \times \frac{25}{32}$	against	36091	
8 — 3	$1\frac{11}{32} \times \frac{13}{16}$	with	39939	
8 — 4	$1\frac{13}{32} \times \frac{13}{16}$	with	42469	38500
9 — 1	$1\frac{7}{16} \times \frac{5}{8}$	against	41899	
9 — 2	$1\frac{7}{16} \times \frac{21}{32}$	against	44148	
9 — 3	$1\frac{7}{16} \times \frac{5}{8}$	with	55442	47163
10 — 1	$1\frac{3}{4} \times \frac{13}{16}$	against	54743	
10 — 2	$1\frac{45}{64} \times \frac{13}{16}$	against	50943	
10 — 3	$1\frac{13}{16} \times \frac{13}{16}$	against	38605	
10 — 4	$1\frac{7}{8} \times \frac{13}{16}$	with	55229	49880

Total.............................. 663153
Average.......................... 44210

STANDARD SPECIFICATIONS FOR CAST-IRON, OF THE AMERICAN SOCIETY FOR TESTING MATERIALS.

The standard specifications for cast-iron, of the American Society for Testing Materials, contain specifications for 1° Foundry Pig-iron, 2° Gray Iron Castings, 3° Malleable Iron Castings, 4° Locomotive Cylinders, 5° Cast-iron Pipe and Special Castings, 6° Cast-iron Car-wheels. Of these, 1°, 2°, and 4° will be quoted in full, and extracts will be given from 5°. For the remainder see the proceedings of the Society.

AMERICAN SOCIETY FOR TESTING MATERIALS.
SPECIFICATIONS FOR FOUNDRY PIG-IRON.

Analysis.

It is recommended that all purchases be made by analysis.

Sampling.

In all contracts where pig-iron is sold by chemical analysis, each car load, or its equivalent, shall be considered as a unit. At least one pig shall be selected at random from each four tons of every car load, and so as to fairly represent it.

Drillings shall be taken so as to fairly represent the fracture-surface of each pig, and the sample analysed shall consist of an equal quantity of drillings from each pig, well mixed and ground before analysis.

In case of disagreement between buyer and seller, an independent analyst, to be mutually agreed upon, shall be engaged to sample and analyze the iron. In this event one pig shall be taken to represent every two tons.

The cost of this sampling and analysis shall be borne by the buyer if the shipment is proved up to specifications, and by the seller if otherwise.

Allowances and Penalties.

In all contracts, in the absence of a definite understanding to the contrary, a variation of 10 per cent in silicon, either way, and of 0.02 sulphur, above the standard, is allowed.

A deficiency of over 10 per cent and up to 20 per cent, in the silicon, subjects the shipment to a penalty of 4 per cent of the contract price.

Base Analysis of Grades.

In the absence of specifications, the following numbers, known to the trade, shall represent the appended analyses for standard grades of foundry pig-irons, irrespective of fracture, and subject to allowances and penalty as above:

Grade.	Per Cent Silicon.	Per Cent Sulphur (Volumetric).	Per Cent Sulphur (Gravimetric).
No. 1 . . .	2.75	0.035	0.045
No. 2 . . .	2.25	0.045	0.055
No. 3 . . .	1.75	0.055	0.065
No. 4 . . .	1.25	0.065	0.075

PROPOSED SPECIFICATIONS FOR GRAY IRON CASTINGS.

Process of Manufacture.

Unless furnace iron is specified, all gray castings are understood to be made by the cupola process.

Chemical Properties.

The sulphur contents to be as follows:

Light castings . . . . . . not over 0.08 per cent.
Medium castings . . . . " " 0.10 " "
Heavy castings . . . . . . " " 0.12 " "

Definition.

In dividing castings into light, medium, and heavy classes, the following standards have been adopted:

Castings having any section less than ½ of an inch thick shall be known as *light castings.*

Castings in which no section is less than 2 ins. thick shall be known as *heavy castings.*

Medium castings are those not included in the above definitions.

Physical Properties.

Transverse Test. The minimum breaking-strength of the "Arbitration Bar" under transverse load shall not be under:

Light castings . . . . 2500 lbs.
Medium castings . . . . 2900 "
Heavy castings . . . . 3300 "

In no case shall the deflection be under .10 of an inch.

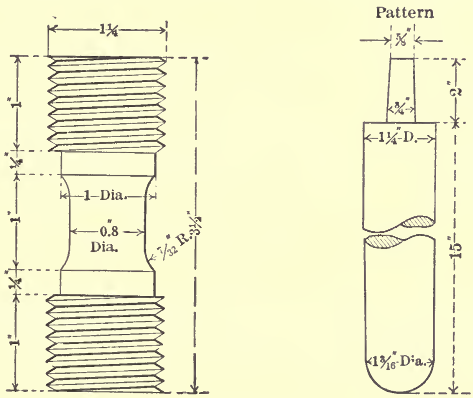
Tensile Test. Where specified, this shall not run less than:

Light castings	18000 lbs. per square inch.
Medium castings	21000 " " " "
Heavy castings.	24000 " " " "

THE "ARBITRATION BAR" AND METHODS OF TESTING.

The quality of the iron going into castings under specification shall be determined by means of the "Arbitration Bar." This is a bar $1\frac{1}{4}$ ins. in diameter and 15 ins. long. It shall be prepared as stated further on and tested transversely. The tensile test is not recommended, but in case it is called for, the bar as shown in Fig. 1, and turned up from any of the broken pieces of the transverse test, shall be used. The expense of the tensile test shall fall on the purchaser.

Two sets of two bars shall be cast from each heat, one set from the first and the other set from the last iron going into the castings. Where



the heat exceeds twenty tons, an additional set of two bars shall be cast for each twenty tons or fraction thereof above this amount. In case of a change of mixture during the heat, one set of two bars shall also be cast for every mixture other than the regular one. Each set of two bars is to go into a single mold. The bars shall not be rumbled or otherwise treated, being simply brushed off before testing.

The transverse test shall be made on all the bars cast, with supports 12 ins. apart, load applied at the middle, and the deflection at rupture noted. One bar of every two of each set made must fulfill the requirements to permit acceptance of the castings represented.

The mold for the bars is shown in Fig. 2 (not shown here). The bottom of the bar is $\frac{1}{16}$ of an inch smaller in diameter than the top, to allow for draft and for the strain of pouring. The pattern shall not be rapped before withdrawing. The flask is to be rammed up with green molding-sand, a little damper than usual, well mixed and put through a No. 8 sieve, with a mixture of one to twelve bituminous facing. The mold shall be rammed evenly and fairly hard, thoroughly dried and not cast until it is cold. The test-bar shall not be removed from the mold until cold enough to be handled.

Speed of Testing.

The rate of application of the load shall be thirty seconds for a deflection of .10 of an inch.

Samples for Chemical Analysis.

Borings from the broken pieces of the "Arbitration Bar" shall be used for the sulphur determinations. One determination for each mold made shall be required. In case of dispute, the standards of the American Foundrymen's Association shall be used for comparison.

Finish.

Castings shall be true to pattern, free from cracks, flaws, and excessive shrinkage. In other respects they shall conform to whatever points may be specially agreed upon.

Inspection.

The inspector shall have reasonable facilities afforded him by the manufacturer to satisfy him that the finished material is furnished in accordance with these specifications. All tests and inspections shall, as far as possible, be made at the place of manufacture prior to shipment.

SPECIFICATIONS FOR LOCOMOTIVE CYLINDERS.

Process of Manufacture.

Locomotive cylinders shall be made from good quality of close-grained gray iron cast in a dry sand mold.

Chemical Properties.

Drillings taken from test-pieces cast as hereafter mentioned shall conform to the following limits in chemical composition:

Silicon	from 1.25 to 1.75 per cent
Phosphorus	not over .9 " "
Sulphur	" " .10 " "

Physical Properties.

The minimum physical qualities for cylinder iron shall be as follows:

The "Arbitration Test-Bar," $1\frac{1}{4}$ ins. in diameter, with supports 12 ins. apart shall have a transverse strength not less than 3000 lbs., centrally applied, and a deflection not less than 0.10 of an inch.

Test-Pieces and Method of Testing.

The standard test shall be $1\frac{1}{4}$ ins. in diameter, about 14 ins. long, cast on end in dry sand. The drillings for analysis shall be taken from this test-piece, but in case of rejection of the manufacturer shall have option of analyzing drillings from the bore of the cylinder, upon which analysis the acceptance or rejection of the cylinder shall be based.

One test-piece for each cylinder shall be required.

Character of Castings.

Castings shall be smooth, well cleaned, free from blow-holes, shrinkage cracks, or other defects, and must finish to blue-print size.

Each cylinder shall have cast on each side of saddle manufacturer's mark, serial number, date made, and mark showing order number.

Inspector.

The inspector representing the purchaser shall have all reasonable facilities afforded to him by the manufacturer to satisfy himself that the finished material is furnished in accordance with these specifications. All tests and inspections shall be made at the place of the manufacturer.

CAST-IRON PIPE AND SPECIAL CASTINGS.

This specification is divided into the following sections, viz.: 1° Description of Pipes, 2° Allowable Variation in Diameter of Pipes and Sockets, 3° Allowable Variation in Thickness, 4° Defective Spigots may be Cut, 5° Special Castings, 6° Marking, 7° Allowable Percentage of Variation in Weight, 8° Quality of Iron, 9° Tests of Material, 10° Casting of Pipes, 11° Quality of Castings, 12° Cleaning and Inspection, 13° Coating, 14° Hydrostatic Test, 15° Weighing, 16° Contractor to Furnish Men and Materials, 17° Power of Engineer to Inspect, 18° Inspector to Report, 19° Castings to be Delivered Sound and Perfect, 20° Definition of the Word Engineer.

Of these, only sections 8° and 9° will be quoted here, as follows:

Quality of Iron.

Section 8. All pipes and special castings shall be made of cast-iron of good quality, and of such character as shall make the metal of the castings strong, tough, and of even grain, and soft enough to satisfactorily admit of drilling and cutting. The metal shall be made without any admixture of cinder-iron or other inferior metal, and shall be remelted in a cupola or air furnace.

Tests of Material.

Section 9. Specimen bars of the metal used, each being 26 inches long by 2 inches wide and 1 inch thick, shall be made without charge as often as the engineer may direct, and, in default of definite instructions, the contractor shall make and test at least one bar from each heat or run of metal. The bars, when placed flatwise upon supports 24 inches apart and loaded in the centre, shall for pipes 12 inches or less in diameter support a load of 1900 pounds and show a deflection of not less than .30 of an inch before breaking, and for pipes of sizes larger than 12 inches shall support a load of 2000 pounds and show a deflection of not less than .32 of an inch. The contractor shall have the right to make and break three bars from each heat or run of metal, and the test shall be based upon the average results of the three bars. Should the dimensions of the bars differ from those above given, a proper allowance therefor shall be made in the results of the tests.

§ 221. **Wrought-Iron.**—Wrought-iron is obtained by melting pig-iron in contact with iron ore, oxidizing, and burning out, as far as may be, the carbon, the phosphorus, and the silicon. In many cases, however, the charge consists largely of wrought-iron or steel scrap, and cast-iron borings.

The process is commonly carried on in a puddling furnace, where an oxidizing flame is passed over the melted pig-iron.

As the heat is not sufficiently intense to melt the wrought-iron produced, the metal is left in a plastic condition, full of bubbles and holes, which contain considerable slag. It is then squeezed, and rolled or hammered, to eliminate, as far as possible, the slag, and to weld the iron into a solid mass.

The result of this first rolling is known as muck-bar, and must be "piled," heated, and rolled or hammered at least once more before it is suitable for use in construction.

In making the piles, while muck-bar is sometimes used exclusively, a considerable part, and often the greater part, is made of scrap.

Wrought-iron is thus, throughout its manufacture, a series of welds. Moreover, wherever slag is present, these welds cannot be perfect. It is also subject to the impurities of the cast-iron from which it is made. Thus, the presence of sulphur makes it red-short, or brittle when hot; and the presence of phosphorus makes it cold-short, or brittle when cold.

It cannot, like cast-iron, be melted and run into moulds; but it can be easily welded by the ordinary methods

Wrought-iron is much more capable of bearing a tensile or transverse stress than cast-iron: it is tougher, it stretches more, and gives more warning before fracture. At one time cast-iron was the principal structural material, but it was soon displaced by wrought-iron, which became the principal metal used in construction, but now, since the modern methods of steel-making supply a more homogeneous product at a cheaper price, wrought-iron has been superseded by mild steel in most pieces used in construction.

Wrought-iron is also expected to withstand a great many trials that would seriously injure cast-iron: thus, two pieces of wrought-iron are generally united together by riveting; the holes for the rivets have to be punched or drilled, and then the rivets have to be hammered; the entire process tending to injure the iron. Wrought-iron has to withstand flanging, and is liable to severe shocks when in use; as, for instance, those that occur from the changes of temperature in the different parts of a steam-boiler.

The following references to a large number of tests of wrought-iron will be given:

1°. Eaton Hodgkinson: (*a*) Report of Commissioners on the Application of Iron to Railway Structures.
(*b*) London Philosophical Transactions. 1840.

2°. William H. Barlow: Barlow's Strength of Materials.

3°. Sir William Fairbairn: On the Application of Cast and Wrought Iron to Building Purposes.

4°. Franklin Institute Committee: Report of the Committee of the Franklin Institute. In the Franklin Institute Journal of 1837.

5°. L. A. Beardslee, Commander U.S.N.: Experiments on the Strength of Wrought-Iron and of Chain Cables. Revised and enlarged by William Kent, M.E., or Executive Document 98, 45th Congress, as stated below.

6°. David Kirkaldy: Experiments on Wrought-Iron and Steel.

7°. G. Bouscaren: Report on the Progress of Work on the Cincinnati Southern Railway, by Thomas D. Lovett. Nov. 1, 1875.

8°. Tests of Metals made at Watertown Arsenal. Of these the first two volumes were published before 1881, and since that time one volume has been published every year. Nearly all of them contain tests of wrought-iron and a great many of them contain tests of full-size pieces of wrought-iron.

9°. A. Wöhler: (*a*) Die Festigkeits versuche mit Eisen und Stahl.
(*b*) Strength and Determination of the Dimensions of Structures

of Iron and Steel, by Dr. Phil. Jacob J. Weyrauch. Translated by Professor Dubois.

10°. Technology Quarterly, Vol. VII. No. 2, Vol. VIII. No. 3, Vol. IX. Nos. 2 and 3, and Vol. X. No. 4.

11°. Mitt. der Materialprüfungsaustalt in Zürich.

12°. Mitt. aus dem Mech. Tech. Lab. in Berlin.

13°. Mitt. aus dem Mech. Tech. Lab. in München.

§ 222. **Tensile Strength of Wrought-Iron.**—About the year 1840 was published the report of the Commission appointed by the British Government to investigate the application of iron to railway structures. While a number of tests of iron had been previously made, this work may properly be regarded as having been the first investigation of the kind that was at all thorough. At that time cast-iron was the metal most used in construction, and hence the greater part of the work of the Commission was devoted to a study of that metal. They made, however, a number of tests of wrought-iron, which, though they were of the greatest value at the time, and still have some value, will not be quoted here.

At about that time the use of wrought-iron began to increase at a rapid rate, the necessary appliances were introduced to roll it into I beams, channel-irons, angle-irons, and other shapes, and it began to displace cast-iron for one after another purpose until it came to be the metal most extensively used in construction, both in the case of structures and machines.

At first the chief desideratum was assumed to be that it should have a high tensile strength, and scarcely any attention was paid to its ductility.

About 1865, however, engineers began to realize that ductility is an all-important property of a metal to be used in construction, and that this is not necessarily and not generally obtainable with a very high tensile strength. The most prominent advocate, at that time, of the importance of ductility was David Kirkaldy, who published a book, entitled "Experiments on Wrought Iron and Steel," containing the results of his tests down to 1866.

In the early part of his book will be found a summary of what had been done by earlier experimenters in this line.

Kirkaldy tested a large number of English irons, determining both their breaking-strengths and their ductility.

In the light of the results obtained by him, he proceeded to draw up his famous sixty-six conclusions.

These sixty-six conclusions will not be quoted here, but the following statement will be made regarding the main results of his work:

1°. He proved that the results obtained by testing grooved specimens (or specimens of such form as to interfere with the flow of the metal while under test) did not indicate correctly the quality of the metal, but that such specimens should be used as did not interfere with the flow of the metal.

2°. He advocated, with all the earnestness of which he was capable, the conclusion that it was of the greatest importance that all wrought-iron used in construction should have a good ductility, and, in his tests, he adopted five different methods of measuring ductility.

These methods are: 1°. Contraction of area at fracture per cent; 2°. Ultimate elongation per cent; 3°. Breaking-strength per square inch of fractured area; 4°. Contraction of stretched area per cent, i.e., the contraction of area attained when the maximum load is first reached; 5°. Breaking-weight per square inch of stretched area. Of these only two are used at the present time, the first and second, and they serve as measures of ductility. These two are the principal conclusions from Kirkaldy's tests, though he cites a great many more, one of the principal of them being his conclusion regarding so-called cold crystallization, which will be mentioned later.

Tests of the tensile strength of wrought-iron may be divided into two classes: 1° those made mainly for the purpose of determining the quality of the material, and 2° those made upon such full-size pieces as are used in practice to resist tension.

The tests of the first class are made upon small specimens, and, in order that the results may be comparable, the use of standard forms and dimensions is, generally, a desideratum. The specifications for wrought-iron of the American Society for Testing Materials will be given first, as they refer to the kind of wrought-iron that is in most common use, and then some other tensile tests of various kinds of wrought-iron in small pieces will be given. Subsequently tests of wrought-iron eye-bars will be quoted.

AMERICAN SOCIETY FOR TESTING MATERIALS. SPECIFICATIONS FOR WROUGHT-IRON.

Process of Manufacture.

1. Wrought-iron shall be made by the puddling process or rolled from fagots or piles made from wrought-iron scrap, alone or with muck-bar added.

Physical Properties.

2. The minimum physical qualities required in the four classes of wrought-iron shall be as follows:

	Stay-bolt Iron.	Merchant Iron. Grade "A."	Merchant Iron. Grade "B."	Merchant Iron, Grade "C."
Tensile strength, pounds per square inch . . .	46000	50000	48000	48000
Yield-point, pounds per square inch . . .	25000	25000	25000	25000
Elongation, per cent in 8 inches	28	25	20	20

3. In sections weighing less than 0.654 pound per lineal foot, the percentage of elongation required in the four classes specified in para-

graph No. 2 shall be 12 per cent., 15 per cent., 18 per cent., and 21 per cent., respectively.

4. The four classes of iron when nicked and tested as described in paragraph No. 9 shall show the following fracture:

(*a*) Stay-bolt iron, a long, clean, silky fibre, free from slag or dirt and wholly fibrous, being practically free from crystalline spots.

(*b*) Merchant iron, Grade "A," a long, clean, silky fibre, free from slag or dirt or any course crystalline spots. A few fine crystalline spots may be tolerated, provided they do not in the aggregate exceed 10 per cent of the sectional area of the bar.

(*c*) Merchant iron, Grade "B," a generally fibrous fracture, free from coarse crystalline spots. Not over 10 per cent of the fractured surface shall be granular.

(*d*) Merchant iron, Grade "C," a generally fibrous fracture, free from coarse crystalline spots. Not over 15 per cent of the fractured surface shall be granular.

5. The four classes of iron, when tested as described in paragraph No. 10, shall conform to the following bending tests:

(*e*) Stay-bolt iron, a piece of stay-bolt iron about 24 inches long, shall bend in the middle through 180° flat on itself, and then bend in the middle through 180° flat on itself in a plane at a right angle to the former direction without a fracture on outside of the bent portions. Another specimen with a thread cut over the entire length shall stand this double bending without showing deep cracks in the threads.

(*f*) Merchant iron, Grade "A," shall bend cold 180° flat on itself, without fracture on outside of the bent portion.

(*g*) Merchant iron, Grade "B," shall bend cold 180° around a diameter equal to the thickness of the tested specimen, without fracture on outside of bent portion.

(*h*) Merchant iron, Grade "C," shall bend cold 180° around a diameter equal to twice the thickness of the specimen tested, without fracture on outside of the bent portion.

6. The four classes of iron when tested as described in paragraph No. 11, shall conform to the following hot bending tests:

(*i*) Stay-olt iron, shall bend through 180° flat on itself, without showing cracks or flaws. A similar specimen heated to a yellow heat and suddenly quenched in water between 80° and 90° F. shall bend, without hammering on the bend, 180° flat on itself, without showing cracks or flaws.

(*j*) Merchant iron, Grade "A," shall bend through 180° flat on itself, without showing cracks or flaws. A similar specimen heated to a yellow heat and suddenly quenched in water between 80° and 90° F. shall bend, without hammering on the bend, 180° flat on itself, without showing cracks or flaws. A similar specimen heated to a bright-red heat shall be split at the end and each part bent back through an angle of 180°. It will also be punched and expanded by drifts until a round hole is formed whose diameter is not less than nine-tenths of the diameter of the rod or width of the bar. Any extension of the original split or indications of fracture, cracks, or flaws developed by the above tests will be sufficient cause for the rejection of the lot represented by that rod or bar.

(*k*) Merchant iron, Grade "B," shall bend through 180° flat on itself, without showing cracks or flaws.

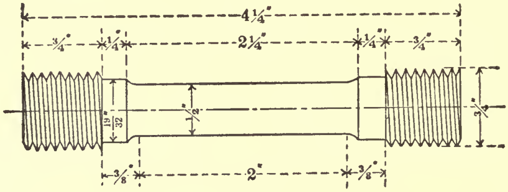
(*l*) Merchant iron, Grade "C," shall bend sharply to a right angle, without showing cracks or flaws.

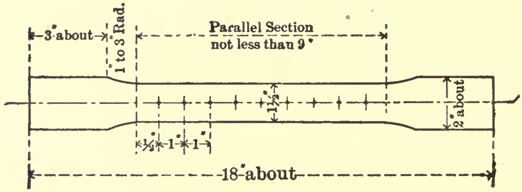
7. Stay-bolt iron shall permit of the cutting of a clean sharp thread and be rolled true to gauges desired, so as not to jam in the threading dies.

Test Pieces and Methods of Testing.

8. Whenever possible, iron shall be tested in full size as rolled, to determine the physical qualities specified in paragraphs Nos. 2 and 3, the elongation being measured on an eight inch (8″) gauged length. In flats and shapes too large to test as rolled, the standard test specimen shall be one and one-half inches ($1\frac{1}{2}''$) wide and eight inches (8″) gauged length.

In large rounds, the standard test specimen of two inches (2″) gauged length shall be used; the center of this specimen shall be half-way between the center and outside of the round. Sketches of these two standard test specimens are as follows:





PIECE TO BE OF SAME THICKNESS AS THE PLATE.

9. Nicking tests shall be made on specimens cut from the iron as rolled. The specimen shall be slightly and evenly nicked on one side and bent back at this point through an angle of 180° by a succession of light blows.

10. Cold bending tests shall be made on specimens cut from the bar as rolled. The specimen shall be bent through an angle of 180° by pressure or by a succession of light blows.

11. Hot bending tests shall be made on specimens cut from the bar as rolled. The specimens, heated to a bright red heat, shall be bent through an angle of 180° by pressure or by a succession of light blows and without hammering directly on the bend.

If desired, a similar bar of any of the four classes of iron shall be worked and welded in the ordinary manner without showing signs of red shortness.

12. The yield-point specified in paragraph No. 2 shall be determined by the careful observation of the drop of the beam or halt in the gauge of the testing-machine.

Finish.

13. All wrought-iron must be practically straight, smooth, free from cinder spots or injurious flaws, buckles, blisters or cracks.

In round iron, sizes must conform to the Standard Limit gauge as adopted by the Master Car Builders' Association in November, 1883.

Inspection.

14. Inspectors representing the purchasers shall have all reasonable facilities afforded them by the manufacturer to satisfy them that the finished material is furnished in accordance with these specifications. All tests and inspections shall be made at the place of manufacture prior to shipment.

TESTS OF COMMANDER BEARDSLEE.

One of the most valuable sets of tests of wrought-iron is that obtained by committees D, H, and M of the Board appointed by the United States Government to test iron and steel; the special duties of these committees being to test such iron as would be used in chain-cable, and the chain-cable itself. The chairman of these three committees, which were consolidated into one, was Commander L .A. Beardslee of the United States Navy. The full account of the tests is to be found in Executive Document 98, 45th Congress, second session; and an abridged account of them was published by William Kent, as has been already mentioned.

The samples of bar-iron tested were round, and varied from one inch to four inches in diameter.

Certain conclusions which they reached refer to all kinds of wrought-iron, and will be given here before giving a table of the results of the tests.

1°. Kirkaldy considers the breaking-strength per square inch of fractured area as the main criterion by which to determine the merits of a piece of iron or steel. Commander Beardslee, on the other hand, thinks that a better criterion is what he calls the "tensile limit;" i.e., the maximum load the piece sustains divided by the area of the smallest section when that load is on, i.e., just before the load ceases to increase in the testing-machine.

2°. Kirkaldy had already called attention to the fact that the tensile strength of a specimen is very much affected by its shape, and that, in a specimen where the shape is such that the length of that part which has the smallest cross-section is practically zero (as is the case when a groove is cut around the specimen), the breaking-strength is greater than it is when this portion is long; the excess being in some cases as much as 33 per cent.

Commander Beardslee undertook, by actually testing specimens whose smallest areas varied in length, to determine what must be the least length of that part of the specimen whose cross-section area is smallest, in order that the tensile strength may not be greater than with a long specimen. The conclusion reached was, that no test-piece should be less than one-half inch in diameter, and that the length should never be less than four diameters; while a length of five or six diameters is necessary with soft and ductile metal in order to insure correct results. The following results of testing steel are given in Mr. Kent's book, as confirming the same rule in the case of steel. The tests were made upon Bessemer steel by Col. Wilmot at the Woolwich arsenal.

	Tensile Strength.	Pounds per Square Inch.
By groove form	Highest	162974
	Lowest	136490
	Average	153677
By cylinder	Highest	123165
	Lowest	103255
	Average	114460

3°. Commander Beardslee also noticed that rods of certain diameters of the same kind of iron bore less in proportion than rods of other diameters; and, after searching carefully for the reason, he found it to lie in the proportion between the diameter of the rod and the size of the pile from which it is rolled. The following examples are given: —

$1\frac{1}{8}$-in.	diameter,	6.62%	of pile,	56543	lbs. per	sq. in.	tensile	strength.
$1\frac{1}{4}$	"	8.18%	"	56478	"	"	"	"
$1\frac{3}{8}$	"	9.90%	"	54277	"	"	"	"
$1\frac{1}{2}$	"	11.78%	"	53550	"	"	"	"
$1\frac{5}{8}$	"	7.68%	"	56344	"	"	"	"
$1\frac{3}{4}$	"	8.90%	"	55018	"	"	"	"
$1\frac{7}{8}$	"	10.22%	"	54034	"	"	"	"
2	"	11.63%	"	51848	"	"	"	"

He therefore claims, that, in any set of tests of round iron, it is necessary to give the diameter of the rod tested, and not merely the breaking-strength per square inch.

4°. He gives evidence to show, that if a bar is under-heated, it will have an unduly high tenacity and elastic limit; and that if it is over-heated, the reverse will be the case.

5°. The discovery was made independently by Commander Beardslee and Professor Thurston, that wrought-iron, after having been subjected to its ultimate tensile strength without breaking it, would, if relieved of its load and allowed to rest, have its breaking-strength and its limit of elasticity increased.

His experiments show that the increase is in irons of a fibrous and ductile nature, rather than in brittle and steely ones; hence the latter class would be but little benefited by the action of this law.

The most characteristic table regarding this matter is the following:—

EFFECT OF EIGHTEEN HOURS' REST ON IRONS OF WIDELY DIFFERENT CHARACTERS.

	Ultimate Strength per Square Inch.		Remarks.	
	First Strain.	Second Strain.		
Boiler iron . . .	48600	56500	Not broken.	
" " . . .	49800	57000	Broken	Average gain, 15.8%.
" " . . .	49800	58000	Broken	
" " . . .	48100	54400	Broken	
" " . . .	48150	55550	Broken	
Contract chain iron,	50200	54000	Broken	Average gain, 6.4%.
" " "	50250	53200	Not broken	
" " "	50700	55300	Not broken	
" " "	49600	52900	Not broken	
" " "	51200	52800	Not broken	
Iron K	58800	64500	Broken	Average gain, 9.4%.
" "	59000	65800	Broken	
" "	56400	60600	Broken	

§ 233. **Chain Cable.** — The most thorough set of tests of the strength of chain cable is that made by Commander Beardslee for the United-States government, an account of which may be found either in the report already referred to, or in the abridgment by William Kent.

In this report are to be found a number of conclusions, some of which are as follows: —

1°. That cables made of studded links (i.e., links with a cast-iron stud, to keep the sides apart) are weaker than open-link cables.

2°. That the welding of the links is a source of weakness; the amount of loss of strength from this cause being a very uncertain quantity, depending partly on the suitability of the iron for welding, and partly on the skill of the chain-welder.

3°. That an iron which has a high tensile strength does not necessarily make a good iron for cables. Of the irons tested, those that made the strongest cables were irons with about 51000 lbs. tensile strength.

4°. The greatest strength possible to realize in a cable per square inch of the bar from which it is made being 200 per cent of that of the bar-iron from which it was made, the cables tested varied from 155 to 185 per cent of that of the bar-iron.

5°. The Admiralty rule for proving chain cables, by which they are subjected to a load in excess of their elastic limit, is objected to, as liable to injure the cable: and the report suggests, in its place, a lower set of proving-strengths, as given in the following table; the Admiralty proving-strengths being also given in the table.

In these recommendations, account is taken of the different proportion of strength of different size bars as they come from the rolls, also no proving-stress is recommended greater than 50 per cent of the strength of the weakest link, and 45.5 per cent of the strongest; whereas in the Admiralty tests, 66.2

per cent of the strength of the weakest, and 60.3 per cent of the strongest, is sometimes used.

For the details of this investigation, see the report, Executive Document No. 98, 45th Congress, second session, or the abridgment already referred to.

Diameter of Iron, in inches.	Recommended Proving-Strains.	Admiralty Proving-Strains.	Diameter of Iron, in inches.	Recommended Proving-Strains.	Admiralty Proving-Strains.
2	121737	161280	$1\frac{7}{16}$	66138	83317
$1\frac{15}{16}$	114806	151357	$1\frac{3}{8}$	60920	76230
$1\frac{7}{8}$	108058	141750	$1\frac{5}{16}$	55903	69457
$1\frac{13}{16}$	101499	132457	$1\frac{1}{4}$	51084	63000
$1\frac{3}{4}$	95128	123480	$1\frac{3}{16}$	46468	56857
$1\frac{11}{16}$	88947	114817	$1\frac{1}{8}$	42053	51030
$1\frac{5}{8}$	82956	106470	$1\frac{1}{16}$	37820	45517
$1\frac{9}{16}$	77159	98437	1	33840	40320
$1\frac{1}{2}$	71550	90720			

While steel long ago displaced wrought-iron for boiler-plate, and while steel I beams, channel-bars, angle-irons, and other shapes, as well as eye-bars, have, of late years, displaced wrought-iron to a very great extent, nevertheless wrought-iron is still very extensively used, and for a great variety of structural purposes.

For wrought-iron to be used in construction, ductility, homogeneity, and often weldability are the great desiderata, together with as large a tensile strength as is consistent with these. As to the requirements made by different engineers for wrought-iron for structural purposes, the minimum tensile strength called for varies from about 46000 to about 50000 pounds per square inch, with ultimate elongations varying from 15% to 30% in 8 inches, according to the purpose for which it is wanted. It is also very common, when good iron is wanted, to insist that it shall not be made of scrap. The following tables of tensile tests of wrought iron of various kinds will show what results can be obtained.

Norway Iron.					Burden's Best.				
Diameter, Inches.	Maximum Load, lbs. per sq. in.	Elastic Limit, lbs. per sq. in.	Reduction of Area, per cent.	Modulus of Elasticity.	Diameter, Inches.	Maximum Load, lbs. per sq. in.	Elastic Limit, lbs. per sq. in.	Reduction of Area, per cent.	Modulus of Elasticity.
.75	48390	23620	62.6	30090000	.76	53566	27554	57.6	29175000
.75	46340	21160	62.7	30780000	.75	50023	26030	49.8	30643000
.75	48280	28030	62.6	29020000	.76	47724	25350	47.6	30310000
.77	45160	20400	68.8	27388000	.77	46772	24700	45.2	28347000
.75	46063	19240	68.6	27666000	.77	46600	22550	46.2	29528000
.77	44490	20510	67.5	28452000	.77	47395	22550	46.2	28347000
.74	43233	22079	70.5	29026000	.77	47963	22695	48.6	29475000
.75	43470	19400	75.5	26700000	.77	47860	26948	46.4	26948000
.73	38950	22030	72.3	30140000	.77	47500	26927	42.3	28435000
.74	43240	21970	75.2	27726000	.76	47610	23036	53.1	29551000
.74	44564	21970	72.8	28663000	.77	49238	22725	49.2	27470000
.74	43860	19658	75.0	18000000	.76	50037	27700	53.6	29251000
1.00	41620	15560	70.3	27295000	.76	48538	27224	48.8	29355000
.75	42215	.. ...	68.6	29292000	.76	50060	23201	53.0	31028000
.75	42033	19239	62.4	29729000	.76	49143	23240	50.4	30438000
.76	41574	14328	69.5	27450000	.76	48655	23414	49.6	30062000
.76	41574	16531	68.7	29098000	.76	47220	22880	53.4	29969000
.75	42656	19240	59.0	31785000	.76	47090	23020	54.1	33657000
.75	41875	16978	70.1	30487000	.76	49690	27480	53.3	29614000
.75	43396	19112	59.3	28000000	.76	47430	22950	51.8	29443000
.74	39210	15216	73.2	30294000	.76	47950	23000	57.0	29504000
.74	40343	12603	70.5	28810000	.76	49381	22892	45.8	28779000
.74	39896	15187	69.7	31153000	.77	49411	18420	46.6	30112000
.74	39156	16123	76.4	29807000	.76	49660	23186	51.3	30160000
.74	41030	17490	69.8	29310000	.76	48055	20940	56.7	28809000
.75	41180	18000	72.5	31073000	.77	49026	22578	40.5	27292000
.74	42320	19660	68.0	30834000	.76	47220	23060	51.2	33710000
.74	43913	19833	69.8	26970000	.76	50149	20940	41.5	27450000
.74	42102	19[illegible]81	78.3	29127000	.75	48553	23767	74.3	31124000
.74	39698	17638	70.5	30023000	.75	49350	21503	66.5	31793000
.73	43187	17846	68.6	28553000	.76	50083	20940	51.4	29097000
.73	40669	17820	73.8	30159000	.76	47019	23146	51.4	29978000
.73	39348	16593	69.3	29518000	.76	47504	20942	53.2	28527000
.73	39671	12987	78.1	28861000	.76	47747	20942	49.5	29874000
.75	39951	16886	77.2	30020000	.76	50927	23453	46.9	28350000
.74	41093	16400	74.1	28634000	.75	51269	21182	51.9	32551000
.74	40192	14053	76.3	28627000	.75	50930	23770	53.7	29293000
.73	44470	16844	73.4	31114000	.76	50083	23146	45.2	29097000
.74	41940	17523	78.5	29373000	.75	48168	23767	55.6	29879000
.74	42532	16449	70.5	30410000	.76	49500	26500	55.0	31600000
					.75	48400	27200	46.2	28700000
					.75	47600	27200	50.1	29300000
					.77	47200	23600	56.1	27800000
					.76	46700	24200	41.8	29700000
					.77	45600	23600	54.5	28200000

Refined Iron.					Wrought-iron Wire.					
Diameter, Inches.	Maximum Load, lbs. per sq. in.	Elastic Limit, lbs. per sq. in.	Reduction of Area, per cent.	Modulus of Elasticity.	Kind of Wire.	Diameter, Inches.	Maximum Load, lbs. per sq. in.	Elastic Limit, lbs. per sq. in.	Reduction in Area, per cent.	Modulus of Elasticity, lbs. per Square Inch.
.75	56270	28293	33.9	28618000	Annealed wire	.135	70400	43800	63.1	....
.77	53450	28990	22.0	26997000	Annealed wire	.136	61500	... ..	75.0	....
.76	55880	29758	33.5	27711000	Annealed wire	.135	61100	39800	49.4	23000000
.77	53850	29370	33.3	28718000	Annealed wire	.136	59500	39200	71.2	25500000
.77	52770	33722	14.8	27355000	Annealed wire	.135	45100	35800	76.8	23500000
.74	52770	28829	33.5	29273000	Annealed wire	.136	59800	34000	72.7	... .
.77	51320	29294	22 6	28082000	Annealed wire	.135	62400	......	57.5	... .
.77	53778	27138	25.4	28659000	Common wire	.110	90900	64000	52.3	30300000
.74	48882	28822	13 8	28137000	Common wire	.109	103000	... ..	64.4	27500000
.75	49240	28190	14.0	27520000	Common wire	.110	104000	60000	51.0	22900000
.75	50190	30590	17.8	26237000	Common wire	.113	93700	68200	60.5	25200000
.77	51460	29256	22.4	25680000	Common wire	.080	113000	45800	41.9	27000000
.75	47495	30387	12.2	27613000	Common wire	.080	113000	56700	51.0	26500000
.75	48352	30574	17.3	27177000	Common wire	.079	112000	54300	53.3	26600000
.76	47151	25982	75.4	21628000	Common wire	.079	120000	73600	28.1	26100000
.77	50351	35720	25.3	27477000	Common wire	.079	109000	54300	40.4	26400000
.75	48202	28521	14.7	27888000	Common wire	.080	98300	......	43.8	27100000
.75	50703	30558	13.0	23713000	Annealed wire	.081	99600	.. ...	61.9	26600000
.75	49223	30517	15.2	27126000	Annealed wire	.082	93500	50400	64.3	.....
.75	49120	29000	17.8	28290000	Annealed wire	.082	86300	50400	68.5	27100000
.75	47060	31700	15.4	....	Annealed wire	.082	89900	54000	51.7	24900000
.75	47830	29400	17.8	29290000	Annealed wire	.082	97100	57600	55.0	....
.76	51300	26000	29.1	30100000	Annealed wire	.082	93500	39600	65.7	26100000
.76	52400	35000	29.1	25400000	Annealed wire	.082	71000	50400	67.1	27000000
.75	53400	29000	24.9	28200000	Common wire	.167	57200	45100	65.6	....
.76	52100	26000	29.1	....	Annealed wire	.081	95900	55300	60.4	....
.75	54100	29000	24.9	27700000	Annealed wire	.082	93500	43200	75.0	....
.76	51500	26500	24.6	33100000	Common wire	.163	67400	40100	56.9	....
.76	52500	24200	22.3	....	Common wire	.163	61500	47300	52.8	....
.77	77300	34400	26.5	26800000	Piano wire,					
.75	53100	34000	24.9	27200000	No. 13	.031	345000	......	......	29500000
.75	52900	31700	27.2	26100000	Piano wire,					
.76	51600	28700	22.3	26000000	No. 23	.048	262500	......	......	29300000
4.00×										
1.01	40700	.... .	14.4	....						
.76	53100	28700	24.6	....						
.76	52200	33100	26.9	31000000						
.75	50100	31700	22.6	28700000						
.76	49400	26500	26.9	... ..						
1.02	50300	31800	16.9	27200000						
1.01	47000	32500	20.6	27700000						
1.01	50400	30000	25.8	28300000						
1.02	49600	31800	23.9	26500000						
1.01	50200	30000	32.5	26800000						
1.02	50500	29400	30.6	26200000						
1.01	51400	30000	29.2	28300000						
1.02	50400	......	20.4	28200000						
1.02	50200	31800	15.1	27200000						
1.01	48100	30000	32.5	27700000						
1.01	50600	30000	27.5	25800000						
.77	48600	25800	52.6	28000000						
.74	53900	27900	38.6	29700000						
.74	54000	25600	18.0	29700000						
.76	49500	26400	41.8	....						
.76	53500	30900	33.4	27600000						

In Heft IV (1890) of the Mitt. d. Materialprüfungsanstalt in Zürich is an account of a set of tensile tests of wrought-iron and mild-steel angles, tees, and channels. The following is a summary of his results for wrought-iron shapes: —

ANGLE-IRONS.

Number.	Dimensions, Inches.	Weight per Yard.	Maximum Load, Pounds per Square Inch.	Elastic Limit, Pounds per Square Inch.	Yield Point, Pounds per Square Inch.	Reduction of Area, per Cent.	Modulus of Elasticity, Pounds per Square Inch.
		Lbs.					
2	2.76 × 2.76 × 0.31	16.53	49910	25020	37680	9.5	28824000
4	2.76 × 2.76 × 0.51	27.62	49060	20190	32560	11.7	28070000
6	3.54 × 3.54 × 0.35	26.21	50620	25310	34130	15.8	28269000
8	3.15 × 3.54 × 0.55	38.51	51190	25310	32000	16.4	27786000
10	4.13 × 4.13 × 0.47	35.48	49200	28010	33130	12.0	28537000
12	4.13 × 4.13 × 0.67	55.24	46070	22750	32280	10.2	28554000
14	5.12 × 5.12 × 0.67	62.09	47780	22610	30430	12.0	27985000
16	5.12 × 5.12 × 0.87	102.61	48490	.....	31140	12.3	....

TEE-IRONS.

3	3".60 × 3".35	22.88	52470	25880	37680	14.2	27615000
4	" "	"	49200	23610	34700	15.5	27672000
7	3".94 × 3".94	31.75	51760	21610	38820	11.7	27857000
8	" "	"	54040	18630	35410	21.3	27743000
9	5".90 × 3".94	46.91	53610	23600	36970	19.0	27402000
10	" "	"	52900	22890	35980	14.5	28255000

CHANNEL-IRONS.

1	4.13 × 2.56	28.43	50630	23329	35120	15.9	27544000
2	" "	"	49200	24170	33700	12.7	27885000
4	4.13 × 2.64	31.65	54320	23040	36690	20.6	27772000
5	6.93 × 2.83	48.89	51760	24460	35550	19.9	27658000
6	" "	"	54610	23320	34270	17.5	27999000
7	6.93 × 2.91	54.43	51900	19620	35690	20.4	27487000
8	" "	"	52900	24320	30860	14.5	29663000
9	8.46 × 3.35	85.68	52050	22330	34560	20.9	27701000
10	" "	"	53470	24170	36260	17.0	28710000
12	8.46 × 3.50	92.33	52760	22040	34840	11.9	28568000

TENSILE TESTS MADE SUBSEQUENTLY AT THE WATERTOWN ARSENAL.

Here will next be given, in tabulated form, the results of a number of tensile tests made on the government machine at the Watertown Arsenal.

The following tables of results on rolled bars, from the Elmira Rolling-Mill Company (mark L) *and from the Passaic Rolling-Mills (mark* S), *are given in Executive Document 12, 47th Congress, 1st session, and in Executive Document 1, 47th Congress, 2d session.*

SINGLE REFINED BARS.

Mark on Bar.		Sectional Area, in square inches.	Elastic Limit, in lbs., per Square Inch.	Ultimate Strength, in lbs., per Square Inch.	Elongation in 80 inches, %.	Contraction of Area, %.	Appearance of Fracture.		Modulus of Elasticity at Load of 20000 Lbs. per Square Inch.	
							Fibrous, %.	Crystalline, %.		
L	1	3.06	28500	52710	18.4	33.3	95	5	26981450	
L	2	3.06	29500	53630	16.4	36.0	92	8	27826036	
L	3	3.06	29000	52090	21.4	34.6	95	5	28419182	
L	4	3.06	29000	51440	15.0	20.3	90	10	30888030	
L	5	6.46	27500	50500	14.5	27.6	95	5	27826036	
L	6	6.40	27500	50530	17.3	22.3	70	30	27118644	
L	7	6.39	27000	50200	18.0	22.5	95	5	27444253	
L	8	3.24	–	51667	22.0	36.0	70	30	28318584	Round.
L	9	3.20	–	50844	16.3	22.0	15	85	27972027	"
L	10	3.20	–	53062	21.0	40.0	95	5	28119507	"
S	11	3.08	28500	48640	13.3	24.3	100	Slightly	27586206	
S	12	3.08	28000	50390	16.9	35.1	100	0	27586206	
S	13	3.05	28500	47050	9.0	22.0	100	0	27874564	
S	15	6.40	26000	49700	17.1	19.2	85	15	29906542	
S	16	6.40	24000	49280	15.7	17.7	85	15	26490066	
S	17	6.41	24500	48740	14.3	17.3	80	20	28119507	
S	18	3.17	24600	49680	19.5	32.0	100	Slightly	27972027	Round.
S	19	3.17	25870	49338	18.3	38.0	100	0	29357798	"
S	20	3.17	24920	48864	18.4	37.0	100	Cinder at centre	27729636	"

DOUBLE REFINED BARS.

Mark on Bar.	Sectional Area, in square inches.	Elastic Limit, in lbs., per Square Inch.	Ultimate Strength, in lbs., per Square Inch.	Elongation in 80 inches, %.	Contraction of Area, %.	Appearance of Fracture.		Modulus of Elasticity at Load of 20000 lbs. per Square Inch.	
						Fibrous, %.	Crystalline, %.		
L 201	3.06	29000	53560	15.3	37.9	100	0	27633851	
L 202	3.03	30000	52650	16.2	20.6	85	15	34042553	
L 203	3.06	32500	53500	16.5	27.5	100	0	28169014	
L 204	3.06	32500	54480	15.4	24.8	100	0	29090909	
L 205	6.33	27000	51230	17.8	24.2	80	20	28119507	
L 206	6.34	27500	50500	17.6	21.1	100	Slightly	29629629	
L 207	6.34	27000	51030	21.4	31.9	100	0	27826086	
L 208	3.20	–	50156	22.7	43.0	100	Cup-shaped	28021015	Round.
L 209	3.20	–	49937	22.6	45.0	100	"	28622540	"
L 210	3.20	–	50188	19.9	43.0	100	"	28985507	"
S 211	3.05	29500	51150	22.0	31.5	100	0	32989690	
S 212	3.05	28500	51110	22.0	36.1	100	0	25559105	
S 213	3.11	29500	51860	22.5	39.2	100	0	26446280	
S 215	6.31	27500	50980	19.1	23.6	95	5	29357798	
S 216	6.38	27000	50770	20.7	29.6	100	0	28268551	
S 217	6.33	27000	51340	19.3	35.2	100	0	28070175	
S 218	3.17	24610	50631	20.4	41.0	100	0	28622540	Round.
S 219	3.17	–	50915	25.5	44.0	100	Cup-shaped	28268551	"
S 220	3.17	–	50205	23.7	44.0	100	"	28070175	"

The moduli of elasticity had not been computed in the report, but have been computed in these tables from the elongations under a load of 20000 lbs. per square inch in each case, as recorded in the details of the tests.

In these reports are also to be found tensile tests of iron from other companies, as the Detroit Bridge Company, the Phœnix Company, the Pencoyd Company, etc. Some of these tests were made to determine the effect of rest upon the bar after it had been strained to its ultimate strength, also to determine the strength after annealing. The following table shows these latter results:

	Condition of Bar when Tested.	Size of Bar when Tested.			Stresses on Original Section.		Uniform Elongation from Original Length, p. c.	Contraction of Area from Original Section, p. c.	Appearance of Fracture.	Remarks.
		Width, In.	Thickness, In.	Sectional Area, Sq. In.	Elastic Limit, lbs. per Sq. In.	Ultimate Strength, lbs. per Sq. In.				
993	Original	3.05	1.	3.05	29500	51150	21	31.5	Fine fibrous.	
	Rested four months since original test	–	–	–	–	60000	24	–	Fibrous, 90 %; granular, 10 %.	
994	Original	3.05	1.	3.05	28500	51110	21	36.1	Fine fibrous.	
	Rested four and two-thirds months	–	–	–	53440	61960	25	33.8	Fibrous, 92 %; granular, 8 %.	
981	Original	3.05	1.01	3.08	28000	50390	14	35.1	Fibrous.	
	Rested five mos. .	–	–	–	–	60390	20	21.8	Granular, 70 %; fibrous, 30 %.	
973	Original	5.05	1.28	6.46	27500	50500	13	24.2	Fibrous, 95 %; granular, 5 %.	
	Rested eight mos. .	–	–	–	56500	58980	18	17	Fibrous, 30 %; granular, 70 %.	
	Heated cherry-red, and cooled in open air	4.75	1.17	5.56	21670	36220 / 41640	33	–	Fibrous.	Broke at end of annealed section.
774	Original	3.03	1.01	3.06	29500	53630	16	36	Fibrous, 92 %; granular, 8 %.	
384	Rested ten months.	2.84	.94	2.67	53920	63130	23	21.9	Granular.	
385	Heated dull red, and cooled in open air	2.80	.92	2.58	22800	41240	32	34	Dull fibrous.	
777	Original	3.03	1.01	3.06	29000	53560	15	37.9	Fibrous.	
386	Rested ten months.	2.87	.95	2.73	53920	63070	19	19.6	Fibrous, 10 %; granular, 90 %.	
387	Heated to 370°, and cooled in open air	2.83	.92	–	57520	57520	19	34.6	Fibrous, 60 %; granular, 40 %.	
	Rested fifteen days after third test .	–	–	–	–	62420				
971	Original	3.03	1.01	3.06	32500	53500	16	27.5	Fibrous.	
388	Rested nine mos. .	2.84	.94	2.67	55800	63360	21	19.9	Fibrous, 30 %; granular, 70 %.	
389	Rested nine mos. .	2.84	.92	2.61	–	66010	21	19.9	Fine granular.	

Some tests were made to determine the values of the modulus of elasticity of the same iron for tension and for compression; and these were found experimentally to be almost identical, as was to be expected. For these tests the student is referred to the reports themselves; and only certain tests on eye-bars of the Phœnix Company will be appended here.

Arsenal Number.	Outside Length, Inches.	Gauged Length, Inches.	Sectional Area, Sq. In.	Ultimate Strength, Pounds per sq. in.	Contraction of Area at Fracture, per cent.
511	67.75	50	1.478	40600	16.8
513	67.80	50	1.940	39480	13.9
518	96.05	75	5.103	46720	8.1

Quite a number of tests of the iron of different American companies are to be found in the "Report on the Progress of Work on the Cincinnati Southern Railway," by Thomas D. Lovett, Nov. 1, 1875.

For these the student is referred to the report named.

WROUGHT-IRON PLATE.

The following table contains some tests of wrought-iron plate and bars made on the Government testing-machine at Watertown in 1883 and 1884 for the Supervising Architect at Washington, D.C.

Number of Test.	Mark on Specimen.	Width.	Thick-ness.	Sectional Area.	Elastic Limit. Total Lbs.	Elastic Limit. Lbs. per Sq. In.	Ultimate Strength. Total Lbs.	Ultimate Strength. Lbs. per Sq. In.	Elongation at Fracture.	Elongation at Fracture. Per Cent.	Contraction of Area at Fracture.	Appearance of Fracture.
		in.	in.	sq. in.								
2637	6 m	1.017	0.495	0.503	17100	34000	25300	51100	In eight inches.	21.1	25.8	Fibrous, fine lamination.
2638	7 m	1.016	0.509	0.517	17650	34140	26350	50970		20.4	26.1	" " "
2639	8 m	0.778	Diam.	0.475	16100	33890	24620	51830		17.7	40.4	Fine fibrous.
2640	9 m	1.007	0.514	0.518	16050	30980	26950	52030		20.7	27.0	Fibrous, laminated.
2641	10 m	0.968	0.514	0.498	15400	30920	26050	52310		20.5	24.9	" "
2642	11 m	1.225	0.371	0.454	14980	33000	21950	48350		13.9	*	Fine fibrous, laminated.
2643	12 m	1.208	0.376	0.454	14300	31500	21840	48110		13.0	*	" " "
2644	13 m	1.337	0.373	0.499	16200	32460	23270	46630		7.9	*	" " "
2645	14 m	1.335	0.371	0.495	16650	33640	23980	48440		10.6	*	" " "
2646	1 B	0.765	Diam.	0.460	15500	33700	23600	51300		14.4	30.0	Fibrous, seamy.
2647	2 B	1.010	0.515	0.520	17830	34290	25980	49960		19.4	*	" laminated.
2648	3 B	1.003	0.511	0.513	17800	34700	25590	49880		21.6	28.8	" slight lamination.
2649	4 B	0.990	0.502	0.497	16150	32490	24540	49380		20.5	23.9	" seamy.
2650	5 B	0.979	0.500	0.490	14900	30400	23650	48270		17.6	22.8	" laminated.
2651	6 B	0.985	0.365	0.360	12380	34390	17890	49690		16.0	28.3	" fine lamination.
2652	6 B	0.980	0.359	0.352	12260	34830	17560	49890		19.0	22.4	" " "
2653	7 B	1.005	0.396	0.398	13900	34920	19500	48990		13.6	*	"
2654	7 B	1.005	0.396	0.398	14250	35800	19100	47990		10.0	*	"
1665	C 6	1.325	0.370	0.490	14100	28770	24080	49140		17.9	22.8	" lamellar.
1666	C 6	1.325	0.373	0.494	14150	28640	24280	49150		20.9	26.5	" "
1667	L 1	Diam.	0.703	0.388	11200	28870	21050	54250		22.4	24.7	" "
1668	L 2	Diam.	0.700	0.384	11500	29950	19430	50600		24.5	31.2	" "
1669	L 3	1.010	0.505	0.510	16100	31570	25550	50100		20.6	25.9	" "
1670	L 4	1.006	0.502	0.505	15050	29800	25780	51050		17.0	21.6	" seamy.
1671	L 5	1.006	0.505	0.508	13700	26970	23150	45570		10.8	*	Spongy, stratified.
1672	L 6	1.003	0.512	0.514	15300	29770	24810	48270		14.4	18.5	Fibrous lamellar.
2655	A 1	0.705	Diam.	0.390	11980	30720	19070	48900	In ten inches.	24.5	36.9	Fine fibrous, seamy.
2656	A 3	0.705	Diam.	0.390	11280	28920	19200	49230		23.7	41.3	" " "
2657	A 4	0.700	Diam.	0.385	11250	29220	18500	48050		21.7	40.5	" " "
2658	B 5	0.988	0.506	0.500	15740	31480	25380	50760		23.6	31.2	Fibrous, laminated.
2659	B 5	1.007	0.510	0.514	15980	31090	25730	50060		17.8	22.9	" slight lamination.
2660	C 6	1.334	0.400	0.534	16900	31640	25290	47360		15.8	21.3	Fibrous.
2661	C 6	1.327	0.413	0.548	16300	29740	24690	45050		10.9	23.3	" laminated.
				Tests	made in	1884.						
1	K 246	0.994	0.510	0.507	18450	36390	28040	55310	In eight inches.	19 6	28.0	" lamellar.
2	K 246	0.994	0.493	0.490	16200	33060	26120	53310		16.0	24.7	" "
3	K 247	1.325	0.383	0.507	17800	35110	26280	51830		12 3	19.3	" "
4	K 247	1.325	0.383	0.507	18400	36290	27000	53250		17.0	21.7	" "
5		0.808	Diam.	0.513	16950	33040	28440	55440		28.5	41.1	"
6		0.802	Diam.	0.505	16120	31920	27360	54180		21.3	46.0	" surface seamy.

* Broke in neck.

WROUGHT-IRON AND STEEL EYE-BARS.

In the report of the Government tests for 1886 is given the following table of tensile tests of wrought-iron eye-bars. The wrought-iron ones were furnished by the General Manager of the Boston and Maine Railroad, and the steel ones by the Chief Engineer for the American Committee of the Statue of Liberty.

WROUGHT-IRON EYE-BARS.

Dimensions.			Elastic Limit per Square Inch.	Tensile Strength per Square Inch.	Elongation.		Contraction of Area.	Modulus of Elasticity per Square Inch.	Maximum Compression on Pin-holes per Square Inch.	Fracture.	
Length, Center to Center of Pin-holes.	Width.	Thickness.			In Gauged Length.	Center to Center of Pin-holes.				Location.	Appearance.
Ins.	Ins.	Ins.	Lbs.	Lbs.	%	%	%	Lbs.	Lbs.		
238.55	5.00	1.14	22456	45105	11.7	11.6	31.2	28037000	52763	Stem	Fibrous, traces of granulation.
238.60	5.00	1.15	22610	44540	9.4	9.4	29.6	28125000	50588	"	Fibrous, 70%. Granular, 30%.
238.57	4.99	1.14	21790	43320	7.8	8.0	26.4	27950000	48492	"	Fibrous, 70%. Granular, 30%.
238.64	5.00	1.16	22410	39550	5.1	4.8	9.8	27355000	54013	"	Fibrous, 70%. Granular, 30%.
238.62	6.05	1.44	19750	43260	12.05	12.06	24.8	28800000	43166	"	Granular, 80%. Fibrous, 20%.
238.62	6.05	1.44	22730	42020	6.5	6.6	19.2	28301000	41929	"	Granular 5% at one edge, fibrous for balance of fracture.

The gauged length of the bars was 180 inches. The moduli of elasticity computed between 5000 and 10,000 pounds per square inch.

COMPRESSIVE STRENGTH OF WROUGHT-IRON.

In regard to the compressive strength of wrought-iron, we may wish to study it with reference to—

1°. The strength of wrought-iron columns;

2°. The strength of wrought-iron beams;

3°. The effects of a crushing force upon small pieces not laterally supported;

4°. The effects of a crushing force upon small pieces laterally supported.

1°. In this case it may be said that, by reference to the tests of wrought-iron bridge columns, the compressive strength per square inch of wrought-iron in masses of such sizes is given by the tests of the shorter lengths of such columns, i.e., by those that are short enough not to acquire, when the maximum load is just reached, a deflection sufficient to throw any appreciably greater stress per square inch on any part of the column in consequence of the eccentricity of the load due to the deflection. The results thus obtained are naturally lower than we should expect to obtain in smaller masses.

2°. In this case the evidence that there is goes to show that the compressive strength is the same as in the case of 1°, and hence that it is less than the tensile strength. Indeed, the results of tests of full-size beams show a modulus of rupture greater than the compressive strength, less than the tensile strength in I sections, and greater in circular sections; all this being what would naturally be expected.

3°. If a small cylinder of ductile wrought-iron is tested without lateral support, and with flat ends, the friction of the ends against the platforms of the testing-machine comes in to interfere with the flow of the metal; and if, besides this, the ratio of length to diameter is so small as to prevent buckling, then the specimen will gradually flatten out, and it becomes impossible to find any maximum load, because the area of the central part is constantly increasing.

4°. In this case the crushing strength per square inch that causes continuous flow, and also the maximum strength per square inch, is greater than that where the specimen has no lateral support. Hence follows, that in the case of wrought-iron rivets it is entirely safe to allow a bearing pressure in the neighborhood of 90,000 or 100,000 pounds per square inch, according to circumstances.

§ 223. **Wrought-Iron Columns.**—Until after about 1880 there was but little experimental knowledge on this subject beyond the experiments of Hodgkinson, which have furnished the constants for Hodgkinson's, and also for Gordon's formula, as already given in § 208 and § 209.

These formulæ have been in very general use, and it is only of late years that we have been able to test their accuracy by tests on full-size wrought-iron columns. The disagreement of the formulæ already referred to, with the results of the tests, has led to the proposal of a large number of similar formulæ, each having its constants determined to suit a certain definite set of tests, and hence all these formulæ thus proposed, which are, of course, empirical, and can only be applied with safety within the range of the cases experimented upon.

A few of these will now be enumerated; and then will follow tables of the actual tests, which furnish the best means of determining the strength of these columns; and it would appear that it is these tables themselves which the engineer would wish to use in designing any structure.

On the 15th of June, 1881, Mr. Clark, of the firm of Clark, Reeves & Co., presented to the American Society of Civil Engineers a report of a number of tests on full-size Phœnix columns, made for them at the Watertown Arsenal, together with a comparison of the actual breaking-weights with those which would have been obtained by using the common form of Gordon's formula for wrought-iron. The table is shown on page 416.

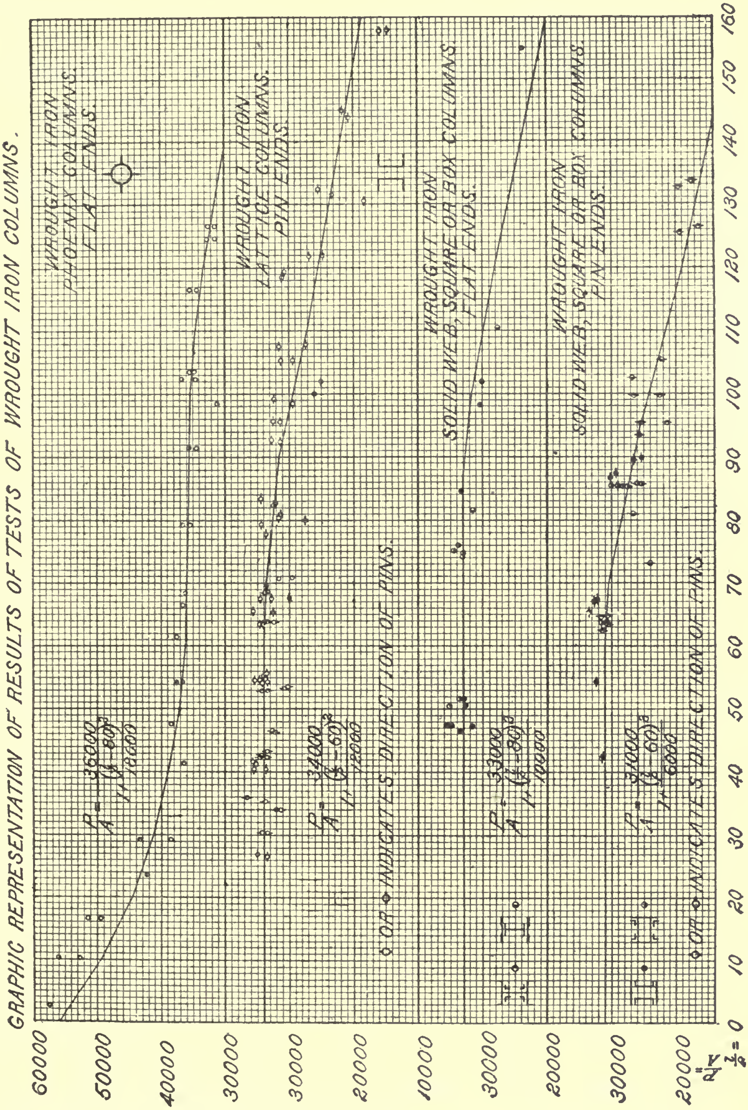
The very considerable disagreement between the breaking-loads as calculated by Gordon's formula, and the actual breaking-loads, led a number of people to propose empirical formulæ of one form or another which should represent this set of tests, and also others which should represent some other tests on full-size bridge columns, which had been previously made in other places.

Of these I shall only give those proposed by Mr. Theodore Cooper, which are as follows: —

For square-ended columns . . . $$\frac{P}{A} = \frac{f}{1 + \dfrac{\left(\dfrac{l}{\rho} - 80\right)^2}{18000}}.$$

For pin-ended columns . . . . $$\frac{P}{A} = \frac{f}{1 + \dfrac{\left(\dfrac{l}{\rho} - 33\right)^2}{18000}}.$$

No. of Experiment.	Length of Column.	Ratio of Diameter to Length.	Weight.	Sectional Area.	Total Compression under Loads.		Elastic Limit.		Ultimate Strength.		Total Ultimate Strength, in lbs., by Gordon's Formula.
					200000 lbs.	300000 lbs.	Total lbs.	Lbs. per Square Inch.	Total lbs.	Lbs. per Square Inch.	
	ft.		lbs.	sq. in.							
1	28	42	1142	12.062	0.190	–	–	–	424000	35150	330146
2	28	42	1153	12.181	0.186	–	–	–	416000	34150	333459
3	25	37½	1034	12.233	–	0.255	342000	27960	431500	35270	352013
4	25	37½	1023	12.100	0.168	0.264	–	–	424000	35040	348119
5	22	33	920	12.371	0.160	0.243	–	–	440000	35570	372837
6	22	33	–	12.311	0.152	0.236	–	–	423000	34360	371043
7	19	28½	773	12.023	–	0.198	–	–	425200	35365	377955
8	19		777	12.087	0.139	0.213	354000	29290	446000	36900	380197
9	16	24	650	12.000	0.120	–	–	–	439000	36580	391701
10	16		650	12.000	0.116	–	–	–	439000	36580	391701
11	13	19½	536	12.185	0.092	0.142	342000	28890	449000	36857	410660
12	13		531	12.009	0.091	–	–	–	449000	37200	406866
13	10	15	415	12.248	–	0.110	330000	26940	446800	36480	423886
14	10		418	12.339	–	0.109	350000	28360	449100	36397	427047
15	7	10½	291	12.265	0.054	–	360000	29350	468000	38157	433021
16	7		284	11.962	–	–	354000	29590	517000	43300	469324
17	4	6	164	12.081	0.031	–	–	–	598000	49500	432132
18	4	6	164½	12.119	0.025	0.042	340000	28050	621000	51240	433507
	in.										
19	8	1	27½	11.903	0.008	0.013	–	–	680000	57130	–
20	8	1	27½	11.903	0.007	0.011	–	–	682000	57300	–
21	25′ 2″.63	25½	1561	18.300	0.115	0.178	–	–	659000	36010	–
22	8′ 9″.50	9	544	18.300	0.045	0.067	540000	29510	772000	42180	–


GRAPHIC REPRESENTATION OF RESULTS OF TESTS OF WROUGHT IRON COLUMNS.
WROUGHT IRON PHOENIX COLUMNS. FLAT ENDS.
WROUGHT IRON LATTICE COLUMNS. PIN ENDS.
WROUGHT IRON SOLID WEB, SQUARE OR BOX COLUMNS. FLAT ENDS.
WROUGHT IRON SOLID WEB, SQUARE OR BOX COLUMNS. PIN ENDS.
◊ OR ◆ INDICATES DIRECTION OF PINS.
◊ OR ◆ INDICATES DIRECTION OF PINS.
60000
50000
40000
30000
30000
20000
10000
30000
20000
30000
20000
l/r = 0
10
20
30
40
50
60
70
80
90
100
110
120
130
140
150
160

And he gives, for the values of f,

For Phœnix columns . . . . $f = 36000$;
" American Company's columns . . . . $f = 30000$;
" box and open columns . . . . $f = 31000$.

He deduces these values of f from some tests made in 1875 by Mr. Bouscaren, combined with those, already referred to, made at the Watertown Arsenal. The box and open columns were made of channel-bars and latticing. The tables or diagrams presented to justify the formulæ proposed can be found in the "Transactions of the American Society of Civil Engineers" for 1882.

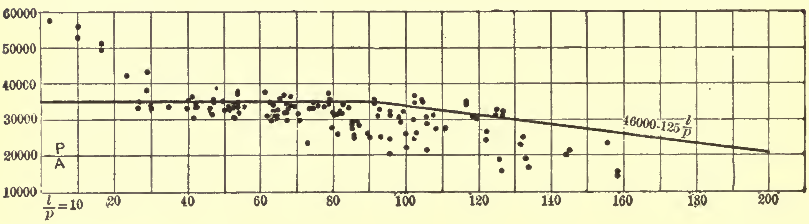
Besides the above there will be given here tables of three sets of tests of full-size wrought-iron columns, viz.:

1°. The series made at Watertown Arsenal, this being the most complete set of tests of full-size wrought-iron columns in existence.

2°. A series of tests of Z-bar columns made by Mr. C. L. Strobel.

3°. A few tests made at the Mass. Institute of Technology. Reference will also be made to the tests of Mr. G. Bouscaren, and to those made by Prof. Tetmajer, at the Materialprüfungs-anstalt in Zurich.

Graphical rep esentations, however, will first be given of the



results of those tested at Watertown Arsenal, with the corresponding curves, representing (a) the formulæ of Prof. Sondericker (see page 417), and (*b*) that of Mr. Strobel (see page 418). These diagrams will be preceded by the corresponding formulæ.

A perusal of them will show that, for values of $\frac{l}{\rho}$ less than a certain quantity, which Mr. Strobel assumes as 90, and Prof. Sondericker as 80 for flat-ended, and 60 for pin-ended columns; the value of $\frac{P}{A}$ (i.e., the breaking-load divided by the area) is constant. For greater values of $\frac{l}{\rho}$ the value of $\frac{P}{A}$ decreases, and for this portion of the curve, Prof. Sondericker's formulæ are as follows:

For flat-ended Phœnix columns he recommends Cooper's formula.

For lattice columns with pin-ends, reported in Exec. Doc. 12, 47th Congress, 1st session, and Exec. Doc. 5, 48th Congress, 1st session, he recommends the formula

$$\frac{P}{A}=\frac{34000}{1+\frac{\left(\frac{l}{\rho}-60\right)^2}{12000}}.$$

For the box and solid web columns reported in Exec. Doc. 5, 48th Congress, 1st session, and Exec. Doc. 35, 49th Congress, 1st session, taken together with Bouscaren's results on box and on American Bridge Company's columns, he recommends

For flat-ends. . . . . $\frac{P}{A}=\frac{33000}{1+\frac{\left(\frac{l}{\rho}-80\right)^2}{10000}}.$

For pin-ends. . . . . $\frac{P}{A}=\frac{31000}{1+\frac{\left(\frac{l}{\rho}-60\right)^2}{6000}}.$

In these formulæ P = breaking-load in pounds, A = sectional area in square inches, l = length in inches, and ρ = least radius of gyration of section in inches.

Moreover, the numerator in each of these formulæ is the value of $\frac{P}{A}$ corresponding to the case when $\frac{l}{\rho}$ is less than 80 in flat-ended, and less than 60 in pin-ended columns.

Instead of the above Mr. Strobel recommends for value of $\frac{P}{A}$ when $\frac{l}{\rho}$ is less than 90, 35000 pounds per square inch, and, for values of $\frac{l}{\rho}$ greater than 90, the formula

$$\frac{P}{A} = 46000 - 125\frac{l}{\rho}.$$

Moreover, if P' = safe load, in pounds, he recommends

(*a*) For $\frac{l}{\rho} < 90$, $\qquad \frac{P'}{A} = 8000;$

(*b*) For $\frac{l}{\rho} > 90$, $\qquad \frac{P'}{A} = 10600 - 30\frac{l}{\rho}.$

While Gordon's formula, or a modification of it, is still in use in many bridge specifications, quite a number of them have substituted the Strobel formula, or a modification of it.

Wrought-iron columns subjected to eccentric load.

All the formulæ given thus far for the breaking or for the safe load on wrought-iron columns are only applicable when the load is so applied that its resultant acts along the axis of the column, and either the diagrams on pages 417 and 418, or the corresponding formulæ, give us the breaking-strength per square inch, i.e., the number of pounds per square inch which, multiplied by the area in square inches, gives the breaking-load of the column; the safe load per square inch being obtained by dividing the breaking-load per square inch by a suitable factor of safety. On the other hand, whenever the resultant of the load on the column does not act along the axis of the column, we must determine the fibre-stress due to the direct load, and to this add the greatest fibre stress due to the bending-moment, the sum of the two being the actual greatest fibre stress, and the column must be so proportioned that this greatest fibre stress shall not exceed the safe strength per square inch, as determined by dividing the breaking-strength per square inch by the proper factor of safety; and this proceeding should be followed whatever be the cause of the eccentric load—whether it be due to the beams supported by the column on one side being more heavily loaded than those on the other, whether it be due to the load transmitted from the columns above being eccentric, whether it be due to the mode of connection of the column to the other parts of the structure, whether it be due to poor fitting, or to any other cause.

TESTS OF FULL-SIZE WROUGHT-IRON COLUMNS.

The tests made at the Watertown Arsenal will next be given, together with cuts showing the form of the columns; these being taken from the Tests of Metals for 1881, 1882, 1883, 1884, and 1885.

The following tables are taken from the volume for 1881.:

LATTICE-COLUMNS AND CHANNEL-BARS FROM DETROIT BRIDGE AND IRON COMPANY.

TABULATION OF EXPERIMENTS ON WROUGHT-IRON COLUMNS.

LATTICED COLUMNS.

Columns with Pin Ends tested, with Pins Vertical; Channel-Bars spaced 8 Inches apart.

No. of Test.	Kind.	Channel-Bars.	Length.		Sectional Area.	Lattice Spacing	Ultimate Strength.		Manner of Failure.
							Actual.	Per Square Inch.	
		in.	ft.	in.	sq. in.	in.	lbs.	lbs.	
1059	Flat ends	6	10	0	4.760	18	174800	36720	Channel-bars buckled.
1060	" "	6	10	0	4.670	18	165000	35330	" "
1095	One flat end, one pin end,	6	10	0	4.750	18	160000	33680	" "
1096	" " " " " "	6	10	0	4.580	18	154800	33800	" "
1107	Pin ends	6	12	0	4.600	18	159800	34740	Horizontal deflection.
1108	" "	6	12	0	4.570	18	156100	34160	" "
1	" "	6	12	6	4.560	18	163600	35880	" "
2	" "	6	12	6	4.740	18	153500	32380	" "
1231	" "	6	15	0	4.480	18	151500	33820	" "
1232	" "	6	15	0	4.560	18	157500	34540	" "
1229	" "	6	17	6	4.660	18	152600	32750	" "
1230	" "	6	17	6	4.740	18	147500	31120	" "
1117	" "	6	20	6	4.660	18	136000	29180	" "
1118	" "	6	20	0	4.630	18	143500	30990	" "
1119	" "	6	22	6	4.570	18	139800	30590	" "
1120	" "	6	22	6	4.660	18	144700	31050	" "
1121	" "	6	25	0	4.710	18	110000	23350	" "
1122	" "	6	25	0	4.630	18	117500	25360	" "
20	" "	6	27	6	4.690	18	102500	21850	" "
21	" "	6	27	6	4.670	18	97200	20810	" "
18	" "	6	30	0	4.700	18	69300	14740	" "
19	" "	6	30	0	4.730	18	75200	15900	" "
1111	" "	8	13	4	7.520	18	261800	34810	Defl. upward; chan.-bars buckled.
1112	" "	8	13	4	7.500	18	264300	35240	" horizon. " "

LATTICED COLUMNS. — *Concluded.*

No. of Test.	Kind.	Chan-nel-Bars.	Length.	Sectional Area.	Lattice Spacing	Ultimate Strength. Actual.	Ultimate Strength. Per Square Inch.	Manner of Failure.
		in.	ft. in.	sq. in.	in.	lbs.	lbs.	
1113	Pin ends	8	16 8	7.480	18	254100	33970	Defl. horizon.; chan.-bars buckled.
1114	" "	8	16 8	7.480	18	251400	33610	" " " "
1115	" "	8	20 0	7.550	18	246200	32610	" " " "
1116	" "	8	20 0	7.510	18	241400	32140	" " and vertically.
1123	" "	8	23 4	7.990	18	257500	32230	" "
1124	" "	8	23 4	7.670	18	240600	31370	" "
24	" "	8	26 8	7.780	18	243900	31350	" "
25	" "	8	26 8	7.750	18	215800	27850	" "
22	" "	8	30 0	7.810	18	194100	24850	" "
23	" "	8	30 0	7.800	18	210000	26920	" "
13	" "	10	12 6	9.680	22	344100	35550	Channel-bars buckled.
14	" "	10	12 6	9.590	22	339000	35350	" "
11	" "	10	16 8	9.550	22	323200	33840	" "
12	" "	10	16 8	9.610	22	326700	34000	" "
3	" "	10	20 10	9.740	22	330000	33880	" "
4	" "	10	20 10	9.806	22	330100	33660	Defl. diagonally.
5	" "	10	25 0	10.040	22	342700	34130	" " chan.-bars buckled.
6	" "	10	25 0	10.000	22	319300	31930	" " " "
26	" "	10	29 2	9.300	22	299300	32180	" horizontally.
27	" "	10	29 2	9.570	22	281200	29380	" "
1109	" "	12	10 0	12.150	22	406000	33420	Channel-bars buckled.
17	" "	12	10 0	12.060	22	423000	35070	" "
15	" "	12	15 0	12.120	22	410000	33830	" "
16	" "	12	15 0	12.470	22	442600	35490	" "
9	" "	12	20 0	11.980	22	411600	34360	" "
10	" "	12	20 0	12.340	22	414800	33610	" "
7	" "	12	25 0	12.144	22	400000	32940	" "
8	" "	12	25 0	11.910	22	407800	34240	" "
28	" "	12	30 0	12.180	22	385000	31610	" "
29	" "	12	30 0	12.540	22	393000	31340	Deflected horizontally.

COMPRESSION TESTS OF CHANNEL-BARS; SIZES USED IN THE PRECEDING LATTICE-COLUMNS.

No. of Test.	Kind.	Size of Bars.	Length.	Sectional Area.	Ultimate Strength.		Manner of Failure.
					Actual.	Per Square Inch.	
		in.	in.	sq. in.	lbs.	lbs.	
1049	Flat ends . .	6	6.00	2.33	98530	42290	Flanges buckled inward.
1050	" " . .	6	6.00	2.33	99840	42695	" " "
1071	" " . .	6	17.58	2.37	86900	36670	Deflection.
1072	" " . .	6	17.70	2.23	82500	37000	"
1069	" " . .	6	23.83	2.23	78400	35160	"
1070	" " . .	6	23.90	2.37	77400	32660	"
1064	" " . .	6	48.00	2.38	66980	28140	"
1051	" " . .	8	8.00	3.85	164700	42780	Flanges buckled inward.
1052	" " . .	8	8.00	3.85	168690	43810	Web buckled outward, flanges inward.
1068	" " . .	8	17.90	3.73	131600	35280	Flanges bent outward.
1065	" " . .	8	23.85	3.73	136300	36540	Deflection.
1066	" " . .	8	23.85	3.73	132100	35410	"
1067	" " . .	8	29.90	3.73	124600	33400	"
1063	" " . .	8	48.00	3.73	114200	30620	"
1053	" " . .	10	10.00	4.78	166400	34810	Flanges buckled inward.
1054	" " . .	10	10.00	4.78	169000	35350	" " outward.
1074	" " . .	10	17.85	4.76	161000	33820	" " inward.
1075	" " . .	10	23.90	5.04	176800	35080	" " outward.
1076	" " . .	10	23.87	5.04	169500	33630	" " inward.
1073	" " . .	10	29.90	4.76	162100	34050	" " "
1062	" " . .	10	48.00	4.76	162250	34080	Deflection.
1055	" " . .	12	12.00	5.97	222300	37240	Web bent inward, flanges outward.
1056	" " . .	12	12.00	5.97	222300	37240	" " " " "
1079	" " . .	12	17.84	5.95	217700	36590	" " " " "
1077	" " . .	12	23.92	6.02	218800	36350	" " outward, " inward.
1078	" " . .	12	23.87	6.02	223000	37040	" " " " "
1080	" " . .	12	29.90	5.96	209500	35150	" " " " "
1061	" " . .	12	48.00	6.19	223100	36040	" " inward, " outward.

LATTICED COLUMNS BUILT BY THE DETROIT BRIDGE AND IRON COMPANY.

Columns tested with Pins Vertical; Channel-Bars spaced 6 Inches apart.

No. of Test.	Kind.	Channel-Bars.	Length.		Sectional Area.	Lattice Spacing.	Ultimate Strength.		Manner of Failure.
							Actual.	Per Square Inch.	
		in.	ft.	in.	sq. in.	in.	lbs.	lbs.	
463	Pin ends . . .	6	20	0	4.68	16	117000	25000	Deflected horizontally.
464	" " . . .	6	25	0	4.68	16	71400	15260	" "
465	" " . . .	8	20	0	7.75	16	215100	27750	" "
466	" " . . .	8	25	0	7.75	16	201500	26000	" "
467	" " . . .	10	20	0	9.19	18	275500	29980	" upward.
468	" " . . .	10	25	0	9.19	18	294080	32000	" horizontally.
469	" " . . .	12	20	0	12.95	–	375200	28970	" upward.
470	" " . . .	12	25	0	12.95	–	388500	30000	" horizontally.

CIRCULAR COLUMNS, 4 SEGMENTS, BUILT BY THE PHŒNIX IRON COMPANY.

Tested with Diameter through Flanges 45° from Vertical ✠.

No. of Test.	Kind.	Diameter.	Length.		Sectional Area.	Ultimate Strength.		Manner of Failure.
						Actual.	Per Square Inch.	
		in.	ft.	in.	sq. in.	lbs.	lbs.	
325	Flat ends . .	8.04	0	30.000	11.610	651000	56070	Web bulged near end.
327	" " . .	8.04	0	29.940	11.902	628500	52800	" " " middle of column.
326	" " . .	8.00	11	10.625	12.181	466000	38256	Deflected downward.
31	" " . .	8.00	31	0.000	11.430	356000	31150	" upward and horizontally.
32	" " . .	8.00	31	0.000	11.310	370500	32760	" " " "
33	" " . .	8.04	31	6.000	11.660	363600	31180	" " " "
34	" " . .	8.04	31	6.000	11.510	373100	32220	" " " "

AMERICAN BRIDGE COMPANY'S STEEL COLUMNS.

Column with Pin Ends tested, with Pins Vertical.

No. of Test.	Kind.	Size.	Length.	Sectional Area.	Ultimate Strength.		Manner of Failure.
					Actual.	Per Square Inch.	
		in.	ft. in.	sq. in.	lbs.	lbs.	
1058	Flat ends	10.27 × 10.30	0 43.10	15.28	719000	47055	Flanges buckled.
30	Pin ends	10.27 × 10.35	30 0.44	15.28	290000	18980	Deflected horizontally.

LATTICED COLUMN BUILT BY THE KELLOGG BRIDGE COMPANY.

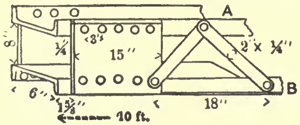
Column tested with Pin Vertical.

No. of Test.	Kind.	Channel-Bars.	Length.	Sectional Area.	Ultimate Strength.		Manner of Failure.
					Actual.	Per Square Inch.	
		in.	ft. in.	sq. in.	lbs.	lbs.	
492	One flat end, one pin end,	9.92	21 8	17.65	581000	32920	Deflected horizontally and upward.

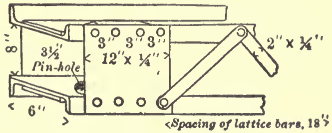
The following are the figures showing the columns of which the tests were recorded in the tables given above: —

1-2-1059-1060.

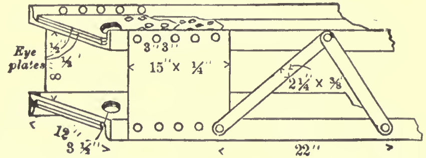
Latticing staggered ¾ inch rivets



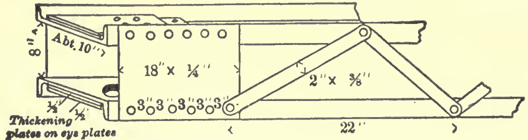
1095-1096-1107-1108-1117-1118-1119-1120
1121-1122-1229-1230-1231-1232.

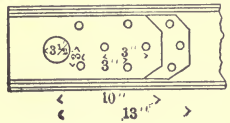


1109-17.



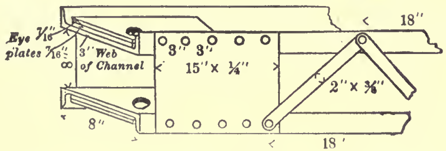
3-4-5-6-11-12-13-14.



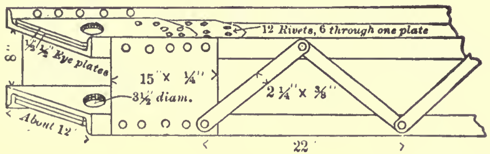


Sketch showing riveting of eye plates.

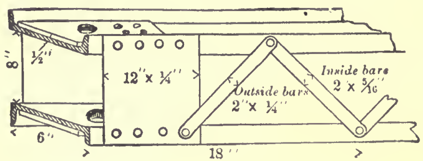
1111-1112-1113-1114-1115-1116-1123-1124.

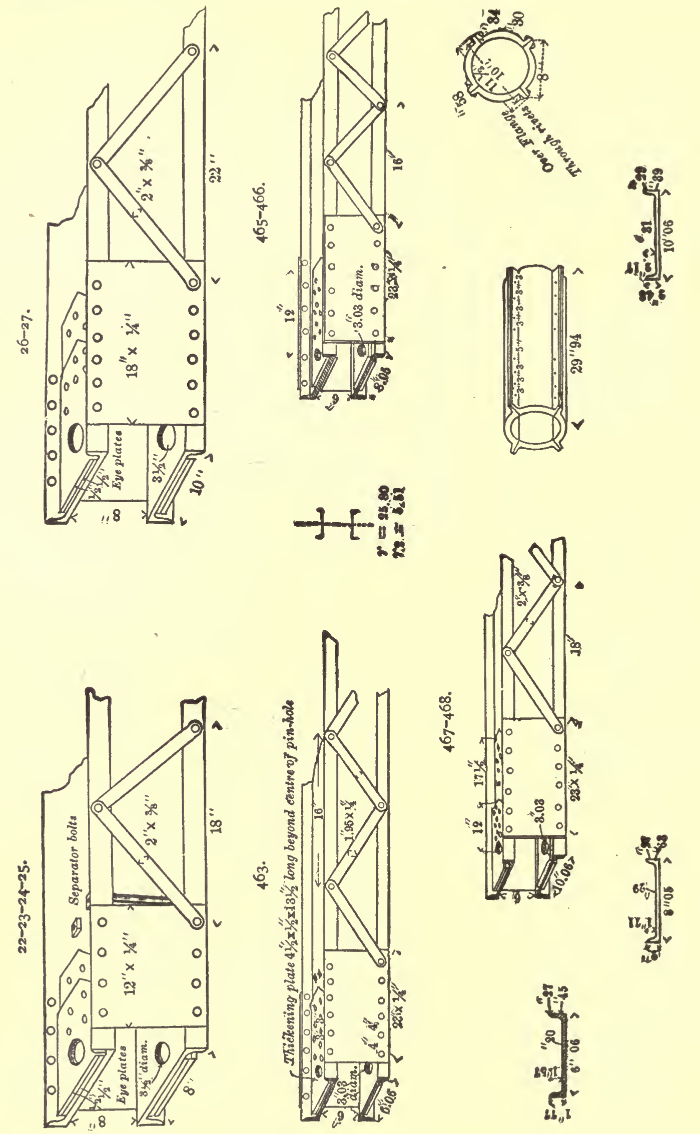


7-8-9-10-15-16-28-29.



18-19-20-21.




26–27.
18" x ¼"
2" x ⅜"
22"
Eye plates
3½"
10"
8"
465–466.
12"
3.03 diam.
23"x ¼"
16"
22–23–24–25.
Separator bolts
12" x ¼"
2" x ⅜"
18"
Eye plates
3½" diam.
8"
463.
Thickening plate 4½"x½"x13½" long beyond centre of pin-hole
16"
1.95 x ¼
22"x ¼"
3.03 diam.
6.06
467–468.
17½"
12"
3.03
23"x ¼"
2"x ⅜"
18"
6.06
Over flange
Through rivets
29"94
10"06
8"05
6"06

The next table taken from the volume for 1882 mentioned above contains the results of some compressive tests of wrought-iron **I**-beams placed in the machine with the ends vertical and tested with flat-ends; also of some tensile specimens cut off from two of them.

TESTS OF I-BEAMS BY COMPRESSION.

	Length. In.	Width of Flange. In.	Thickness of Web. In.	Total Depth. In.	Weight. Lbs.	Sectional Area. Sq. In.	Ultimate Strength.	
							Actual. Lbs.	Per Sq. In Lbs.
1	57.06	5.45	0.64	9.00	228	14.40	545100	37854
2	155.45	4.40	0.40	10.52	443	10.26	207000	20170
3	191.90	3.56	0.40	9.08	365	6.85	85380	12460
4	191.90	3.59	0.43	9.09	381	7.15	85200	11916
5	119.85	2.98	0.28	6.11	139	4.18	101200	24210
6	180.33	3.60	0.42	6.96	303	6.05	84650	13990
7	192.04	3.58	0.45	7.94	355	6.65	83400	12540
8	192.90	3.60	0.44	7.98	353	6.59	92300	14010
9	215.88	4.28	0.40	10.52	561	9.30	149000	16020
10	264.08	4.49	0.48	10.53	747	10.19	113100	11100
11	264.08	4.43	0.50	10.51	767	10.46	107800	10306
12	264.00	4.90	0.53	15.15	1085	14.80	184700	12400
13	263.95	4.84	0.53	14.74	1081	14.74	187000	12686

TESTS OF SPECIMENS FROM NOS. 1 AND 2 BY TENSION.

	Cut from Flange or Web.	Width. In.	Depth. In.	Sectional Area. Sq. In.	Ultimate Strength.		Contraction of Area. Per Cent.
					Actual. Lbs.	Per Sq. In. Lbs.	
1	Web.	3.00	0.65	1.95	103300	52970	10
	Web.	3.00	0.50	1.51	65400	43340	3.9
	Flange.	4.00	0.75	3.01	146400	48640	19.6
	Flange.	4.00	0.76	3.02	147100	48640	15.9
2	Flange.	3.00	0.51	1.53	55400	36210	11.1
	Web.	3.00	0.40	1.19	52900	44640	16.5

Next will be given the set of tests which is reported in the volumes for 1883 and 1884.

The following is quoted from the first of the two:

"COMPRESSION TESTS OF WROUGHT-IRON COLUMNS, LATTICED, BOX, AND SOLID WEB.

"This series of tests comprises seventy-four columns, forty of the number having been tested, the results of which are herewith presented.

"The columns were made by the Detroit Bridge and Iron Company.

"The styles of posts represented are those composed of—

"Channel-bars with solid webs;

"Channel-bars and plates;

"Plates and angles;

"Channel-bars latticed, with straight and swelled sides;

"Channel-bars, latticed on one side, and with continuous plate on one side.

"All the posts were tested with $3\frac{1}{2}$-inch pins placed in the centre of gravity of cross-section; except two posts of set *N*, which had the pins in the centre of gravity of the channel-bars.

"This gave an eccentric loading for these columns, on account of the continuous plate on one side of the channel-bars.

"The pins were used in a vertical position, unless otherwise stated in the details of the tests.

"In the testing-machine the posts occupied a horizontal position.

"They were counterweighted at the middle.

"Cast-iron bolsters for pin-seats were used between the ends of the columns and the flat compression platforms of the testing-machine.

"The sectional areas were obtained from the weights of the channel-bars, angles, and plates, which were weighed before any holes were punched, calling the sectional area, in square inches, one-tenth the weight in pounds per yard of the iron.

"Compressions and sets were measured within the gauged length by a screw micrometer.

"The gauged length covered the middle portion of the post, and was taken along the centre line of the upper channel-bar or plate, always using a length shorter than the distance between the eye-plates, to obtain gaugings unaffected by the concentration of the load at those points.

"The deflections were measured at the middle of the post. The pointer, moving over the face of a dial, indicated the amount and direction of the deflection.

"Loads were gradually applied, measuring the compressions and deflections after each increment; returning at intervals to the initial load to determine the sets.

"The maximum load the column was capable of sustaining was recorded as the ultimate strength, although, previous to reaching this load, considerable distortion may have been produced.

"Observations were made on the behavior of the posts after passing the maximum load, while the pressure was falling, showing, in some cases, a tendency to deflect with a sudden spring, accompanied by serious loss of strength.

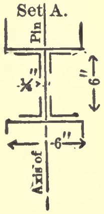
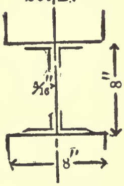
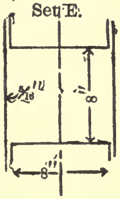
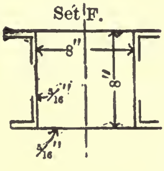
"The slips of the eye-plates along the continuous plates and channel-bars during the tests were measured for certain posts in sets *F*, *G*, *H*, and *I*. The measurements of slip were taken in a length of 10 inches or 20 inches, one end of the micrometer being secured to the eye-plate, and one end to the channel-bar. The readings include both the compression movement of the material and the slip of the plates.

"Columns *H*, *I*, *L*, and *M* were provided with pin-holes for placing the pins either parallel or perpendicular to the webs of the channel-bars.

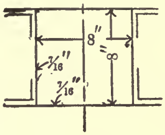
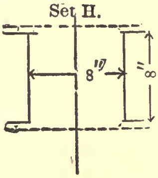
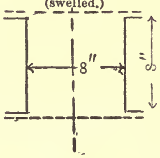
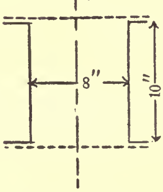
"After the ultimate strength had been determined with the pins in their first position, a supplementary test was made, if the condition of the column justified it, with the pins at right angles to their former position; thus changing the moment of inertia of the cross-section, taken about the pin as an axis.

"The experiments with columns *N* show how much strength is saved by employing pins in the centre of gravity of the cross-section. Where such was not the case, the columns showed the effect of the eccentric loading by deflections perpendicular to the axis of the pins, from the initial loads, which resulted in their early failure."

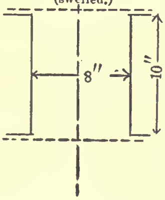
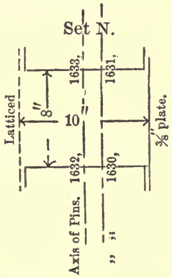
TABULATION OF EXPERIMENTS ON WROUGHT-IRON COLUMNS WITH $3\frac{1}{2}$-INCH PIN-ENDS.

No. of Test.	Style of Column.	Length, Centre to Centre of Pins. In.	Sectional Area. Sq. In.	Ultimate Strength. Total, Lbs.	Ultimate Strength. Lbs. per Sq. In.	Manner of Failure.
752	Set A. 	126.20	9.831	297100	30220	Deflected perpendicular to axis of pins.
757		120.07	10.199	320000	31380	Sheared rivets in eye-plates.
755		180.00	9.977	251000	25160	Deflected perpendicular to axis of pins.
756		180.00	9.977	210000	21050	Do. do.
753		240.00	9.732	188600	19380	Do. do.
754		240.10	9.762	158300	16220	Do. do.
751	Set D. 	240.00	16.077	425000	26430	Deflected perpendicular to axis of pins.
1642		240.00	16.281	367000	22540	Do. do.
1646		320.00	16.179	318800	19700	Do. do.
1647		320.10	16.141	283600	17570	Do. do.
1653	Set E. 	320.00	17.898	474000	26480	Deflected perpendicular to axis of pins.
1654		320.00	19.417	491000	25290	Do. do.
1645	Set F. 	319.95	16.168	453000	28020	Deflected parallel to axis of pins.
1650		320.00	16.267	454000	27910	Deflected perpendicular to axis of pins.

TABULATION OF EXPERIMENTS ON WROUGHT-IRON COLUMNS WITH $3\frac{1}{2}$-INCH PIN-ENDS.

No. of Test.	Style of Column.	Length, Centre to Centre of Pins. In.	Sectional Area. Sq. In.	Ultimate Strength. Total, Lbs.	Ultimate Strength. Lbs. per Sq. In.	Manner of Failure.
	Set G. 					
1651		320.00	20.954	540000	25770	Deflected in diagonal direction.
1652		320.10	20.613	535000	25950	Sheared rivets in eye-plates.
746	Set H. 	159.20	7.628	258700	33910	Deflected perpendicular to axis of pins.
747		159.27	8.056	294700	36580	Do. do.
748		239.60	7.621	260000	34120	Do. do.
749		239.60	7.621	254600	33410	Deflected in diagonal direction.
1648		319.90	7.705	243600	31610	Deflected parallel to axis of pins.
1649		319.85	7.673	229200	29870	Deflected in diagonal direction.
740	Set I. (swelled.) 	159.90	7.645	262500	34340	Deflected perpendicular to axis of pins.
741		159.90	7.624	255650	33530	Do. do.
739		239.70	7.517	251000	33390	Deflected parallel to axis of pins.
750		239.70	7.531	259000	34390	Deflected perpendicular to axis of pins.
1643		319.80	7.691	237200	30840	Deflected parallel to axis of pins.
1644		319.92	7.702	237000	30770	Deflected in diagonal direction.
	Set L. 					
1640		199.84	11.944	403000	33740	Deflected perpendicular to axis of pins.
1641		200.00	12.302	426500	34670	Deflected in diagonal direction.
1634		300.00	12.148	408000	33630	Deflected perpendicular to axis of pins.
1635		300.00	12.175	395000	32440	Do. do.

TABULATION OF EXPERIMENTS ON WROUGHT-IRON COLUMNS WITH $3\frac{1}{2}$-INCH PIN-ENDS.

No. of Test.	Style of Column.	Length, Centre to Centre of Pins. In.	Sectional Area. Sq. In.	Ultimate Strength. Total, Lbs.	Ultimate Strength. Lbs. per Sq. In.	Manner of Failure.
1638	Set M. (swelled.) 	199.25	12.366	385000	31130	Deflected perpendicular to axis of pins.
1639		199.50	12.659	405000	31990	Do. do.
1636		300.20	11.920	391400	32830	Deflected in diagonal direction.
1637		300.15	11.932	390700	32740	Do. do.
1630	Set N. 	300.00	17.622	461500	26190	Deflected perpendicular to axis of pins.
1631		300.00	17.231	485000	28150	Do. do.
1632		300.00	17.570	306000	17420	Do. do.
1633		300.00	17.721	307000	17270	Do. do.

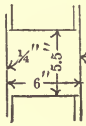
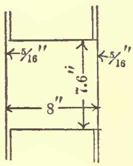
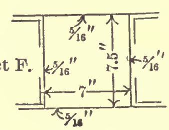
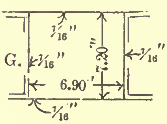
The remainder of the tests of this series of seventy-four columns is reported in the volume for 1884.

The only portion of the description that it is worth while to quote is the following, as the tests were made in a similar way to what has been already described:

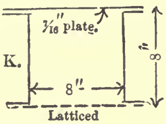
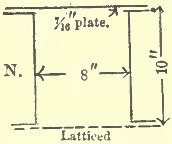
"Sixteen posts were tested with flat ends; eighteen were tested with $3\frac{1}{2}$-inch pin-ends.

"The pins were placed in the centre of gravity of cross-section, except two posts of set *K*, which had the pins in the centre of gravity of the channel-bars, giving an eccentric bearing to these columns, on account of the continuous plate on one side of the channel-bars."

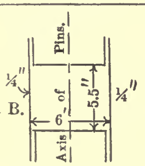
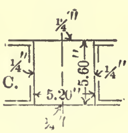
TABULATION OF EXPERIMENTS ON WROUGHT-IRON COLUMNS WITH FLAT ENDS.

No. of Test.	Style of Column.	Total Length. Ft. In.	Sectional Area. Sq. In.	Ultimate Strength. Total, Lbs.	Ultimate Strength. Per Sq. In., Lbs.	Number of Failure.
377	Set B. 	10 7.90	12.08	383200	31722	Buckling-plate D between the riveting.
378		10 7.90	11.11	372900	33564	Buckling-plates.
379	Set E. 	13 11.80	17.01	594500	34950	Buckling-plates between the riveting.
380		13 11.80	17.80	633600	35595	Triple flexure.
346	Set F. 	13 11.9	15.74	517000	32846	Buckling-plates.
347		13 11.65	15.84	555200	35050	Do. do.
342		20 7.63	15.68	517500	33003	Deflecting upward.
344		20 7.80	15.56	536900	34505	Buckling-plates.
348	Set G. 	13 11.75	21.02	708000	33682	Buckling-plates.
349		13 11.75	21.46	709500	33061	Triple flexure.
341		20 7.60	21.20	700000	33019	Deflecting upward.
343		20 7.63	21.49	729450	33943	Deflecting downward.

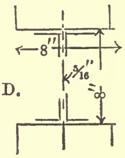
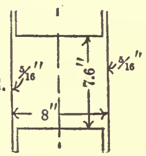
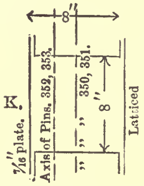
TABULATION OF EXPERIMENTS ON WROUGHT-IRON COLUMNS WITH FLAT ENDS.

No. of Test.	Style of Column.	Total Length. Ft.	In.	Sectional Area. Sq. In.	Ultimate Strength. Total, Lbs.	Ultimate Strength. Per Sq. In., lbs.	Manner of Failure.
339	Set K. 	20	7.94	12.64	412900	32666	Deflecting upward.
340		20	7.94	12.74	431400	33862	Do. do.
337	Set N. 	25	7.75	16.99	582400	34279	Deflecting downward and sideways.
338		25	7.88	17.40	580000	33333	Deflecting diagonally channel B and latticing on the concave side.

TABULATION OF EXPERIMENTS ON WROUGHT-IRON COLUMNS WITH $3\frac{1}{2}$-INCH PIN-ENDS.

No. of Test.	Style of Column.	Length, Centre to Centre of Pins. Ft.	In.	Sectional Area. Sq. In.	Ultimate Strength. Total, Lbs.	Ultimate Strength. Per Sq. In., lbs.	Manner of Failure.
368	Set B. 	15	0.1	11.42	379200	33205	Hor. deflection perpendic. to plane of pins.
367		15	0.0	11.42	369200	32329	Hor. deflection.
356		20	0.0	11.42	342000	29947	Do. do.
357		20	0.0	11.31	330100	29186	Do. do.
371	Set C. 	9	11.9	9.14	286100	31302	Buckling-plates between rivets.
372		10	0.0	10.07	319200	31698	Do. do.
370		15	0.0	9.21	291500	31650	Hor. deflec. and buckling between rivets.
369		15	0.0	9.44	290000	30720	Do. do.
354		20	0.0	9.24	267500	28950	Triple flexure.
365		20	0.0	9.36	279700	29879	Hor. deflection.

TABULATION OF EXPERIMENTS ON WROUGHT-IRON COLUMNS WITH 3½-INCH PIN-ENDS.

No. of Test.	Style of Column.	Length, Centre to Centre of Pins. Ft. In.	Sec-tional Area. Sq. In.	Ultimate Strength. Total, Lbs.	Ultimate Strength. Per Sq. In., Lbs.	Manner of Failure.
360		13 4.13	15.34	475000	30965	Deflecting upward in plane of pins.
361	Set D.	13 4.00	15.40	485000	31494	Hor. deflection perpendicular to plane of pins.
358	 Set E.	20 0.0	17.77	570000	32077	Hor. deflection perpendicular to plane of pins.
359		20 0.0	17.22	555400	32253	Do. do.
350		20 0.25	12.48	202700	16242	Hor. deflection, concave on lattice side.
351	Set K.	20 0.00	10.84	208200	19207	Do. do.
352		20 0.25	12.65	350000	27668	Do. do.
353		20 0.25	12.76	390400	30596	Hor. deflection perpendicular to plane of pins, convex on lattice side.

Besides the above, there are four tests of lattice columns reported in Exec. Doc. 36, 49th Congress, 1st session, but as these columns were rather poorly constructed and form rather special cases they will not be quoted here.

In determining the strength of a bridge column made of channel-bars and latticing, these results of tests on full-size columns furnish us the best data upon which to base our conclusions.

In the Trans. Am. Soc. C. E. for April, 1888, Mr. C. L. Strobel gives an account of his tests on wrought-iron Z-bar columns, from which the following is condensed, viz.: The Z-irons used in making the columns were $2\frac{1}{2} \times 3 \times 2\frac{1}{2}$ inches in size, and $\frac{5}{16}$ inch thick.

Two columns were about 11 ft. long, two 15 ft., two 19 ft., three 22 ft., three 25 ft., and three 28 ft., a total of fifteen columns. The table of results follows:

Length, Inches.	Sectional Area, Sq. Ins.	Ultimate Strength by Tests per Sq. In. Lbs.	$\frac{l}{\rho}$.	Ultimate Strength, by Strobel's Formula per Sq. In. Lbs.
$131\frac{1}{4}$	9.435	36800	64	—
$131\frac{1}{4}$	9.984	34600	64	—
180	9.480	34600	88	35000
180	9.280	36600	88	35000
$228\frac{3}{4}$	9.241	33800	112	32200
$228\frac{3}{4}$	10.104	33700	112	32200
264	9.286	30700	129	29900
264	9.286	29500	129	29900
264	9.286	30700	129	29900
300	9.156	28100	146	27750
300	9.456	28000	146	27750
300	9.516	28400	146	27750
336	9.375	27700	164	25500
336	9.643	28000	164	25500
336	9.375	27600	164	25500

The following table shows the results of compression tests made in the engineering laboratories of the Massachusetts Institute of Technology upon some wrought-iron pipe columns. They were tested with the ordinary cast-iron flange-coupling screwed on to the ends, bearing against the platforms of the testing-machine, which were adjustable, inasmuch as they were provided with spherical joints.

The tests of full-size wrought-iron columns made by Mr. G. Bouscaren, are given in the Report of the Progress of Work on the Cincinnati Southern Railway, by Thos. D. Lovett, Nov. 1, 1875.

In Heft IV (1890) of the Mittheilungen der Materialprüfungsanstalt in Zurich is given an account of a large number of tests of wrought-iron and steel columns of the following forms, viz.: 1°. Angle-irons; 2°. Tee iron; 3°. Channel-bars; 4°. Two angle-irons riveted together; 5°. Four angle-irons riveted together; 6°. Two channel-bars riveted together; 7°. Two tee irons riveted together; also quite a number of tests of columns of some of these forms subjected to eccentric loads, the eccentricity of the load being, in some cases, as much as 8 cm. ($3''.15$). The columns tested were of a variety of lengths, the longest ones being 560 cm. (18.37) feet long.

In Heft VIII (1896) of the same Mittheilungen is an account of another set of tests of columns of the above-described forms. The results of these valuable tests will not be quoted here, but for them the reader is referred to the Mittheilungen themselves.

Nominal Size of Pipe.	Inside Diameter.	Outside Diameter.	Diameter of Flanges.	Length of Inside of Flanges.	Gauge Length.	Maximum Load.	Area of Cross-section.	Maximum Load per Sq. In.	Compression, Modulus of Elasticity.	$\frac{l}{\rho}$, or Ratio of Length to Radius of Gyration.
In.	In.	In.	In.	In.	In.	Lbs.	Sq. In.	Lbs.		
2	2.06	2.37	7	$69\frac{15}{16}$	51	30000	1.08	27809	24300000	88.8
2	2.04	2.39	7	$69\frac{15}{16}$	51	29800	1.22	24500	22200000	89.1
$2\frac{1}{2}$	2.50	2.89	8	$93\frac{11}{16}$	86	34500	1.65	20900	25200000	98.1
$2\frac{1}{2}$	2.48	2.88	8	$93\frac{1}{2}$	86	37000	1.68	22000	25900000	98.4
3	3.06	3.44	$8\frac{5}{8}$	$93\frac{13}{16}$	86	45500	1.94	23500	27700000	81.4
3	3.09	3.48	$8\frac{5}{8}$	$93\frac{5}{8}$	86	51000	2.01	25300	25100000	80.5
$3\frac{1}{2}$	3.60	4.00	$9\frac{1}{8}$	$105\frac{1}{4}$	100.5	55000	2.39	23000	25200000	78.2
$3\frac{1}{2}$	3.59	3.99	$9\frac{1}{8}$	$105\frac{3}{8}$	100.5	65000	2.39	27200	24600000	78.5
4	4.07	4.53	$9\frac{1}{8}$	$117\frac{7}{16}$	100.5	80000	3.11	25700	25800000	77.1
4	4.09	4.50	$9\frac{1}{8}$	$117\frac{9}{16}$	100.5	69000	2.76	25000	24900000	77.3

§ 224. **Transverse Strength of Wrought-Iron.**—Wrought-iron owes its extensive introduction into construction as much or more to the efforts of Sir William Fairbairn than to any one else; and while he was furnishing the means to Eaton Hodgkinson to make extensive experiments on cast-iron columns, and while he made experiments himself on cast-iron beams, which were in use at that time, he also carried on a large number of tests on beams built of wrought-iron, more especially those of tubular form, and those having an **I** or a **T** section, and made of pieces riveted together. In his book on the "Application of Cast and Wrought Iron to Building Purposes" he gives an account of a large number of these experiments, including those made for the purpose of designing the Britannia and Conway tubular bridges, a fuller account of which will be found in his book entitled "An Account of the Construction of the Britannia and Conway Tubular Bridges." In the first-named treatise he urges very strongly the use of wrought-iron, instead of cast-iron, to bear a transverse load.

Fairbairn tested a number of wrought-iron built-up beams, but they were of small dimensions and are hardly comparable with those used in practice.

In the light of the tests made upon wrought-iron columns, it is evident that the compressive strength of wrought-iron is less than the tensile strength. Hence we should naturally expect that the modulus of rupture would be, in all cases, greater than the compressive strength, and that it might or might not be greater than the tensile strength of the iron. Of course the modulus of rupture varies very much with the shape of the cross-section, for the same reasons as were explained in the paragraph 191, i.e., that the formula $M = f\frac{I}{y}$ assumes Hooke's law, "the stress is proportional to the strain," to hold, and that this is not true near the breaking-point.

The value of the modulus of rupture is also influenced by the reduction in the rolls, and hence somewhat by the size of the beam.

Small round or rectangular bars tested for transverse strength show a modulus of rupture very much in excess of the compressive strength per square inch of the iron, and exceeding very considerably even the tensile strength.

While a great many tests of such specimens have been made, none will be quoted here, but the last five tests of the table on page 542 show that for a wrought-iron having a tensile strength per square inch from 58700 to 60250 pounds, moduli of rupture were obtained from 80000 to 90000 pounds, as, the number of turns of these rotating shafts being comparatively small, the breaking-loads were not far below the quiescent breaking loads. On the other hand the moduli of rupture of I beams and other shapes used in building have very much lower values, but for these, tests will be cited.

As to experiments on large beams, we have:

1°. Some tests made by Mr. William Sooy Smith and by Col. Laidley at the Watertown Arsenal.

2°. Some tests made in Holland on iron and steel beams, an account of which is given in the Proceedings of the British Institute of Civil Engineers for 1886, vol. lxxxiv. p. 412 *et seq.*

3°. Some tests made in the laboratory of Applied Mechanics of the Massachusetts Institute of Technology, on iron and steel I beams.

4°. Tests made by the different iron companies upon beams of their own manufacture, and recorded in their respective hand-books.

Mr. Smith's tests are recorded in Executive Document 23, 46th Congress, second session.

5°. In Heft IV (1890) of the Mittheilungen der Materialprüfungsanstalt in Zurich will be found accounts of tests made by Prof. Tetmajer upon the transverse strength of I beams, of deck-beams, and of plate girders.

The results of these tests will be given in the table on top of page 443.

Specimens cut from the flanges, and also from the webs of the last seven of these beams, were tested for tension. In the case of those cut from the flanges, the tensile strength varied

Depth. (Inches.)	Moment of Inertia. (Inches)^4.	Span. (Inches.)	Modulus of Rupture. (Lbs. per Sq. In.)	Modulus of Elasticity. (Lbs. per Sq. In.)
7.87	52.04	62.96	51190	27501500
7.87	52.04	62.96	56453	28937700
3.93	4.13	31.44	62852	28767100
5.91	17.85	47.28	56453	28212500
7.87	51.95	62.96	53894	28226700
9.45	103.04	75.60	51619	27373500
11.81	....	94.48	53183	26463400
13.39	....	107.12	53894	27700600
15.75	....	126.00	52472	27679500

from 50200 in the 15''.75 beam to 57300 pounds per square inch in the 3''.93 beam. On the other hand, in the case of the specimens cut from the web, the tensile strengths varied from 44900 in the 11''.81 beam to 54400 pounds per square inch in the 3''.93 beam, the contraction of area per cent varying from 23.6 to 32.1 per cent in the flanges, and from 12.5 to 15.9 per cent in the web.

The results obtained with the deck-beams are as follows:

Depth. (Inches.)	Moment of Inertia. (Inches)^4.	Span. (Inches.)	Modulus of Rupture. (Lbs. per Sq. In.)	Modulus of Elasticity. (Lbs. per Sq. In.)
4.93	19.88	70.86	56170	25112500
4.26	9.38	59.06	48920	25823500
3.52	5.33	47.24	55320	25596000
3.48	4.71	39.37	54180	26804700
2.36	1.30	31.50	52760	24202400
1.93	0.60	23.62	58160	....

Tensile tests of specimens cut from these deck-beams showed tensile strengths of from 47540 in the 1''.93 beam to 54750 pounds per square inch in the 2''.36 beam, and contractions of area of from 14.1 per cent to 18.4 per cent.

The results obtained with the plate girders are as follows, viz.:

Depth of Web. (Inches.)	Modulus of Rupture. (Lbs. per Sq. In.)	Modulus of Elasticity. (Lbs. per Sq. In.)	Depth of Web. (Inches.)	Modulus of Rupture. (Lbs. per Sq. In.)	Modulus of Elasticity. (Lbs. per Sq. In.)
15.75	51480	26449200	23.62	52760	26321200
15.75	53180	25539100	23.62	48490	26548700
19.69	51476	24813900	27.56	47780	25667100
19.69	52610	25695500	27.56	46500	26776300

The tensile strength of the material of the webs varied from 29860 to 41240 pounds per square inch, while the contraction of area was only 0.4 per cent. The tensile strength of the material of the flange-plates was 51050 pounds per square inch, with a contraction of area of 17 per cent. The tensile strength of the angle-irons was 46357 pounds per square inch, with a contraction of area of 14 per cent.

The following table gives the results that have been obtained in the tests that have been made upon wrought-iron I beams in the laboratory of Applied Mechanics of the Massachusetts Institute of Technology. This table will give a fair idea of the strength and elasticity of such beams.

TESTS OF WROUGHT-IRON BEAMS MADE IN THE LABORATORY OF APPLIED MECHANICS OF THE MASSACHUSETTS INSTITUTE OF TECHNOLOGY, ALL LOADED AT THE CENTER.

No. of Test.	Depth. Ins.	Moment of Inertia.	Span. Ft.	Span. Ins.	Breaking Load. Lbs.	Moduli of Rupture. Lbs. per Sq. In.	Moduli of Elasticity. Lbs. per Sq. In.	Remarks.
121	6	24.41	12	0	9500	42386	26679000	From Phœnix Co.
124	7	43.50	14	0	14100	48082	28457000	" " "
126	5	12.47	12	0	6450	46624	29549000	" " "
209	7	44.93	13	8	12200	39670	31057000	" " "
211	8	67.32	13	8	17000	42000	28532000	" " "
215	9	110.78	14	8	23000	41680	27165000	" " "
227	8	61.20	13	6	18300	49640	27397000	From Belgium.
230	9	86.41	13	8	21300	45850	27365000	" "
235	9	40.91	13	8	13800	49140	27923000	" "
253	7	43.05	14	8	11319	40660	28045000	From Phœnix Co.
256	8	66.56	14	8	14547	38460	28187000	" " "
263	9	108.67	14	8	19694	36160	27050000	" " "
291	7	45.96	14	6	10700	36340	26790000	" " "
292	8	66.39	14	6	14300	38200	27380000	" " "
294	9	92.89	14	6	19200	41470	27050000	" " "
338	6	25.92	14	7	7200	37800	27860000	N. J. Steel & Iron Co.
341	7	46.73	12	11	13600	40600	27410000	" " " " "
345	8	71.25	14	7	15400	38400	26940000	" " " " "
379	7	48.84	12	10	15500	44300	26170000	" " " " "

§ 225. **Steel.**—While steel is a malleable compound of iron, with less than 2 per cent of carbon and with other substances, the definition recommended by an international committee of metallurgists in 1876, and used to some extent in German and Scandanavian countries, is different from that in general use in English-speaking countries, and in France.

The definition recommended by the international committee may be found in the Trans. Am. Inst. Min. Engrs. for October, 1876, and is in the following:

1°. That all malleable compounds of iron with its ordinary ingredients, which are aggregated from pasty masses, or from piles, or from any form of iron not in a fluid state, and which will not sensibly harden and temper, and which generally resemble what is called "wrought-iron," shall be called weld-iron (Schweisseisen).

2°. That such compounds, when they will, from any cause, harden and temper, and which resemble what is now called puddled steel, shall be called weld-steel (Schweissstahl).

3°. That all compounds of iron, with its ordinary ingredients which have been cast from a fluid state into malleable masses, and which will not sensibly harden by being quenched in water while at a red heat, shall be called ingot-iron (Flusseisen).

4°. That all such compounds, when they will, from any cause, so harden, shall be called ingot-steel (Flussstahl).

On the other hand, in English-speaking countries, those compounds which have been aggregated from a pasty mass, usually in the puddling-furnace, and which contain slag, are generally called wrought-iron, while those which have been cast from a molten state into a malleable mass are generally called steel.

While this classification is not perfect, it states the most common practice in a general way. Exceptions, two of which are that it does not include the cases of cementation steel and of puddled steel, will not be discussed here.

In view of the above, it will be plain that what is commonly called mild steel in America, would be called ingot-iron under the definition of the international committee. Steel is usually made by one of three processes, viz.: the crucible process, the Bessemer process, or the open-hearth process.

While other processes, as the cementation process and others, are sometimes used, the three enumerated above are in most common use at the present time.

Crucible Steel.—This is very commonly made by re-melting blister-steel in crucibles; the blister-steel being made by the cementation process, in which bars of very pure wrought-iron, especially low in phosphorus, are heated in contact with charcoal until they have absorbed the necessary amount of carbon.

A cheaper process, and one much used at the present day, is to melt a mixture of charcoal and crude bar-iron in a crucible.

Crucible steel, which is always high-carbon steel, is used for the finest cutlery, tools, etc., and wherever a very pure and homogeneous quality of steel is required.

Bessemer Steel.—In the Bessemer process, a blast of air is blown into melted cast-iron, removing the greater part of its carbon and burning out more or less of the other ingredients. The process is conducted in a converter, which is usually so arranged that, when the operation is complete, it can be rotated around a horizontal axis to such an extent that the tuyeres are above the surface of the molten steel, and the blast is shut off.

In the acid Bessemer process, the lining of the converter is made of some silicious substance, the burning of silicon being relied upon to develop a sufficiently high temperature to keep the metal fluid.

In the basic Bessemer process, the lining of the converter is of such a nature as to resist the action of basic slags. It is usually made of dolomite, or of some kind of limestone. Burned lime is added to the charge to seize the silicon and phosphorus, the latter serving to develop a sufficiently high temperature.

In the latter part of the operation, the phosphorus is largely burned out, whereas in the acid process, in order to produce a steel that is low in phosphorus, it is necessary to use a pig-iron that is low in phosphorus.

Open-hearth Steel.—In the open-hearth process, a charge of pig-iron and scrap is placed on the bed of a regenerative furnace, and exposed to the action of the flame, and is thus converted into steel.

In the acid open-hearth process, the lining of the furnace is of a silicious nature, and is covered with sand, while in the basic it is usually of dolomite, or of some kind of limestone.

Bessemer and open-hearth steel contain more impurities than crucible steel, but they are very much cheaper, and are just as suitable for many purposes. It is only in consequence of their introduction that steel can be extensively used on the large scale, as crucible steel would be too expensive for many purposes.

Steel, unlike wrought-iron, is fusible; unlike cast-iron, it can be forged; and, with the exception of the harder grades, it can be welded by heating and hammering, the welding of high-carbon steel in large masses being a very uncertain operation, though small masses can be welded by taking proper care.

The special characteristic, however, is, that, with the exception of the milder grades, when raised to a red heat and suddenly cooled, it becomes hard and brittle, and that, by subsequent heating and cooling, the hardness may be reduced to any desired degree. The first process is called hardening and the second tempering.

The principal element in the steels that are ordinarily used is carbon; nevertheless, both Bessemer and open-hearth steel contain also silicon, manganese, sulphur, phosphorus, etc., which have more or less effect upon the resisting properties of the metal. Sulphur, silicon, and phosphorus usually come from the ore, the fuel, and the flux, while manganese, which is added, operates, among other things, to render the steel ductile while hot, and therefore workable, and to absorb oxygen from the melted mass.

Sulphur is injurious by causing brittleness when hot, and phosphorus by causing brittleness when cold. Phosphorus is the most harmful ingredient in steel, so that when steel is to be used for structural purposes, it is important to have as little phosphorus as possible, and any excess of phosphorus is not to be tolerated.

The injury done to steel plates by punching is greater than that done to iron plates: this injury can, however, be removed by annealing. Steel requires greater care in working it than iron, whether in punching, flanging, riveting, or other methods of working; otherwise it may, if overheated, burn, or receive other injury from careless workmanship.

The chemical composition of steel is one important element in its resisting properties; but, on the other hand, the mode of working also has a great influence on the quality.

The introduction of the Bessemer process was quickly followed by the general use of steel rails, and later, as this and the other processes for making steel for structural purposes have been developed, there has been a constant increase in the purposes for which steel has been used.

One of the earlier applications was to the construction of steam-boilers, steel boiler-plate displacing almost entirely wrought-iron boiler-plate. Of late years the development of the steel manufacture has so perfected, and at the same time cheapened, structural steel that it is now used in most cases where wrought-iron was formerly employed. Thus the eye-bars and the struts of bridges are almost exclusively made of steel, also such shapes as angle-irons, channel-bars, Z bars, tee iron, I beams, etc., are almost exclusively made of steel, and while steel has long been used for many parts of machinery, nevertheless it is now generally used in many cases where a considerable fear of it formerly existed, as in main rods, parallel rods, and crank-pins, and in a large number of parts of machinery subjected to more or less vibration. On the other hand, the steel used for tools is, of course, high-carbon steel.

Tools are almost always made of crucible steel, and they have of course a high percentage of carbon, a high tensile strength, and especially should they be capable of being well hardened and taking a good temper.

The usual steel of commerce may be called carbon steel, because, although it always contains small percentages of other ingredients, nevertheless carbon is the ingredient that principally determines its properties. When iron or steel is alloyed with large percentages of certain substances, the resulting alloys enjoy certain special properties, and these alloys still bear the name of steel. Two of the most prominent of these are manganese steel and nickel steel.

Regarding the first it may be said that although carbon steel becomes practically useless when the manganese reaches about $1\frac{1}{2}$ per cent, nevertheless with manganese exceeding about 7 per cent we obtain manganese steel which is so hard that it is exceedingly difficult to machine it.

The alloy that has come into most prominent notice recently is nickel steel, which consists most commonly of a carbon steel with from 0.2 to 0.4 per cent of carbon and with from 3 to 5 per cent of nickel. With this amount of nickel the tensile strength is very much increased, but more especially is the limit of elasticity increased by a very large amount; and while the contraction of area at fracture and the ultimate elongation per cent are a little less than that of carbon steel with the same percentage of carbon, they are not less than those of carbon steel of the same tensile strength.

It is used for armor-plates, for which it is specially suitable on account of the fact that the nickel renders the steel more sensitive to hardening. It is finding, also, a great many other uses to which it is specially adapted by its peculiar properties.

It has been used for bicycle-spokes, for shafts for ocean steamships, for piston-rods, and for various other purposes. Among the many examples given by Mr. D. H. Browne in a paper before the American Institute of Mining Engineers is a case where the presence of 3.5 per cent nickel increased the ultimate strength of 0.2 per cent carbon steel from 55000 to 85000, and the elastic limit from 28000 to 48000 pounds per square inch, while the contraction of area at fracture was only decreased from 60 per cent to 55 per cent.

The quality of steel to be used for different purposes differs, and while the specifications for any one purpose, made by different engineers, and by different engineering societies, often differ, the work of the American Society for Testing Materials is tending to harmonize them as far as possible. The result of their efforts is shown in the following set of specifications.

AMERICAN SOCIETY FOR TESTING MATERIALS. SPECIFICATIONS FOR STEEL.

STEEL CASTINGS.

Adopted 1901. Modified 1905.

Process of Manufacture.

1. Steel for castings may be made by the open-hearth, crucible, or Bessemer process. Castings to be annealed unless otherwise specified.

Chemical Properties.

Ordinary Castings.

2. Ordinary castings, those in which no physical requirements are specified, shall not contain over 0.40 per cent of carbon, nor over 0.08 per cent of phosphorus.

Tested Castings.

3. Castings which are subjected to physical test shall not contain over 0.05 per cent of phosphorus, nor over 0.05 per cent of sulphur.

Physical Properties.

Tensile Tests.

4. Tested castings shall be of three classes: "hard," "medium," and "soft." The minimum physical qualities required in each class shall be as follows:

	Hard Castings.	Medium Castings.	Soft Castings.
Tensile strength, pounds per square inch	85000	70000	60000
Yield-point, pounds per square inch	38250	31500	27000
Elongation, per cent in 2 inches	15	18	22
Contraction of area, per cent	20	25	30

5. A test to destruction may be substituted for the tensile test in the case of small or unimportant castings by selecting three castings from a lot. This test shall show the material to be ductile and free from injurious defects and suitable for the purposes intended. A lot shall consist of all castings from the same melt or blow, annealed in the same furnace charge. **Drop Test.**

6. Large castings are to be suspended and hammered all over. No cracks, flaws, defects, nor weakness shall appear after such treatment. **Percussive Test.**

7. A specimen one inch by one-half inch ($1'' \times \frac{1}{2}''$) shall bend cold around a diameter of one inch ($1''$) without fracture on outside of bent portion, through an angle of 120° for "soft" castings and of 90° for "medium" castings. **Bending Test.**

Test Pieces and Methods of Testing.

8. The standard turned test specimen one-half inch ($\frac{1}{2}''$) diameter and two inch ($2''$) gauged length shall be used to determine the physical properties specified in paragraph No. 4. It is shown in Fig. 1. (See page 398.) **Test Specimen for Tensile Test.**

9. The number of standard test specimens shall depend upon the character and importance of the castings. A test piece shall be cut cold from a coupon to be moulded and cast on some portion of one or more castings from each melt or blow or from the sink-heads (in case heads of sufficient size are used). The coupon or sink-head must receive the same treatment as the casting or castings before the specimen is cut out, and before the coupon or sink-head is removed from the casting. **Number and Location of Tensile Specimens.**

10. One specimen for bending test one inch by one-half inch ($1'' \times \frac{1}{2}''$) shall be cut cold from the coupon or sink-head of the casting or dastings as specified in paragraph No. 9. The bending test may be made by pressure or by blows. **Test Specimen for Bending.**

Yield-point.

11. The yield-point specified in paragraph No. 4 shall be determined by the careful observation of the drop of the beam or halt in the gauge of the testing-machine.

Sample for Chemical Analysis.

12. Turnings from tensile specimen, drillings from the bending specimen, or drillings from the small test ingot, if preferred by the inspector, shall be used to determine whether or not the steel is within the limits in phosphorus and sulphur specified in paragraphs Nos. 2 and 3.

FINISH.

13. Castings shall be true to pattern, free from blemishes, flaws, or shrinkage cracks. Bearing-surfaces shall be solid, and no porosity shall be allowed in positions where the resistance and value of the casting for the purpose intended will be seriously affected thereby.

INSPECTION.

14. The inspector, representing the purchaser, shall have all reasonable facilities afforded to him by the manufacturer to satisfy him that the finished material is furnished in accordance with these specifications. All tests and inspections shall be made at the place of manufacture, prior to shipment.

STEEL FORGINGS.

Adopted 1901. Modified 1905.

PROCESS OF MANUFACTURE.

1. Steel for forgings may be made by the open-hearth, crucible, or Bessemer process.

CHEMICAL PROPERTIES.

2. There shall be four classes of steel forgings which shall conform to the following limits in chemical composition:

	Forgings of Soft or Low-carbon Steel.	Forgings of Carbon Steel not Annealed.	Forgings of Carbon Steel Oil-tempered or Annealed.	Locomotive Forgings.	Forgings of Nickel Steel, Oil-tempered or Annealed.
	Per Cent.	Per Cent.	Per Cent.	Per Ct.	Per Cent.
Phosphorus shall not exceed	0.10	0.06	0.04	0.05	0.04
Sulphur " " "	0.10	0.06	0.04	0.05	0.04
Manganese " " "	—	—	—	0.60	—
Nickel	—	—	—	—	3.0 to 4.0

PHYSICAL PROPERTIES.

Tensile Tests.

3. The minimum physical qualities required of the different-sized forgings of each class shall be as follows:

Tensile Strength. Lbs. per Sq. In.	Yield-point. Lbs. per Sq. In.	Elongation in 2 Inches. Per Cent.	Contraction of Area. Per Cent.	
				SOFT STEEL OR LOW-CARBON STEEL.
58000	29000	28	35	For solid or hollow forgings, no diameter or thickness of section to exceed 10″.
				CARBON STEEL NOT ANNEALED.
75000	37500	18	30	For solid or hollow forgings, no diameter or thickness of section to exceed 10″.
	Elastic Limit.			CARBON STEEL ANNEALED.
80000	40000	22	35	For solid or hollow forgings, no diameter or thickness of section to exceed 10″.
75000	37500	23	35	For solid forgings, no diameter to exceed 20″ or thickness of section 15″.
70000	35000	24	30	For solid forgings, over 20″ diameter.
				CARBON STEEL OIL-TEMPERED.
90000	55000	20	45	For solid or hollow forgings, no diameter or thickness of section to exceed 3″.
85000	50000	22	45	For solid forgings of rectangular sections not exceeding 6″ in thickness or hollow forgings, the walls of which do not exceed 6″ in thickness.
80000	45000	23	40	For solid forgings of rectangular sections not exceeding 10″ in thickness or hollow forgings, the walls of which do not exceed 10″ in thickness.
80000	40000	20	25	LOCOMOTIVE FORGINGS.
				NICKEL STEEL ANNEALED.
80000	50000	25	45	For solid or hollow forgings, no diameter or thickness of section to exceed 10″.
80000	45000	25	45	For solid forgings, no diameter to exceed 20″ or thickness of section 15″.
80000	45000	24	40	For solid forgings, over 20″ diameter.
				NICKEL STEEL, OIL-TEMPERED.
95000	65000	21	50	For solid or hollow forgings, no diameter or thickness of section to exceed 3″.
90000	60000	22	50	For solid forgings of rectangular sections not exceeding 6″ in thickness or hollow forgings, the walls of which do not exceed 6″ in thickness.
85000	55000	24	45	For solid forgings of rectangular sections not exceeding 10″ in thickness or hollow forgings, the walls of which do not exceed 10″ in thickness.

4. A specimen one inch by one-half inch ($1'' \times \frac{1}{2}''$) shall bend cold 180° without fracture on outside of the bent portion, as follows:

Bending Test.

Around a diameter of $\frac{1}{2}''$, for forgings of soft steel.

Around a diameter of $1\frac{1}{2}''$, for forgings of carbon steel not annealed.

Around a diameter of $1\frac{1}{2}''$, for forgings of carbon steel annealed, if 20″ in diameter or over.

Around a diameter of 1″, for forgings of carbon steel annealed, if under 20″ diameter.

Around a diameter of 1″, for forgings of carbon steel, oil tempered.

Around a diameter of $\frac{1}{2}''$, for forgings of nickel steel annealed.

Around a diameter of 1″, for forgings of nickel steel, oil tempered.

For locomotive forgings no bending tests will be required.

Test Pieces and Methods of Testing.

5. The standard turned test specimen, one-half inch ($\frac{1}{2}''$) diameter and two (2″) gauged length, shall be used to determine the physical properties specified in paragraph No. 3.

Test Specimen for Tensile Test.

It is shown in Fig. 1. (See page 398.)

6. The number and location of test specimens to be taken from a melt, blow, or a forging, shall depend upon its character and importance, and must therefore be regulated by individual cases. The test specimens shall be cut cold from the forging or full-sized prolongation of same parallel to the axis of the forging and half-way between the centre and outside, the specimens to be longitudinal; *i.e.*, the length of the specimen to correspond with the direction in which the metal is most drawn out or worked. When forgings have large ends or collars, the test specimens shall be taken from a prolongation of the same diameter or section as that of the forging back of the large end or collar. In the case of hollow shafting, either forged or bored, the specimen shall be taken within the finished section prolonged, half-way between the inner and outer surface of the wall of the forging.

Number and Location of Tensile Specimens.

7. The specimen for bending test one inch by one-half inch ($1'' \times \frac{1}{2}''$) shall be cut as specified in paragraph No. 6. The bending test may be made by pressure or by blows.

Test Specimen for Bending.

8. The yield-point specified in paragraph No. 3 shall be determined by the careful observation of the drop of the beam, or halt in the gauge of the testing machine. **Yield-point.**

9. The elastic limit specified in paragraph No. 3 shall be determined by means of an extensometer, which is to be attached to the test specimen in such manner as to show the change in rate of extension under uniform rate of loading, and will be taken at that point where the proportionality changes. **Elastic Limit.**

10. Turnings from the tensile specimen or drillings from the bending specimen or drillings from the small test ingot, if preferred by the inspector, shall be used to determine whether or not the steel is within the limits in chemical composition specified in paragraph No. 2. **Sample for Chemical Analysis.**

Finish.

11. Forgings shall be free from cracks, flaws, seams, or other injurious imperfections, and shall conform to the dimensions shown on drawings furnished by the purchaser, and be made and finished in a workmanlike manner.

Inspection.

12. The inspector, representing the purchaser, shall have all reasonable facilities afforded him by the manufacturer to satisfy him that the finished material is furnished in accordance with these specifications. All tests and inspections shall be made at the place of manufacture, prior to shipment.

OPEN-HEARTH BOILER PLATE AND RIVET STEEL.

Adopted 1901.

Process of Manufacture.

1. Steel shall be made by the open-hearth process.

Chemical Properties.

2. There shall be three classes of open-hearth boiler plate and rivet steel; namely, flange, or boiler steel, fire-box steel, and extra-soft steel, which shall conform to the following limits in chemical composition:

	Flange or Boiler Steel. Per Cent.	Fire-box Steel. Per Cent.	Extra Soft Steel. Per Cent.
Phosphorus shall not exceed . .	Acid 0.06 Basic 0.04	Acid 0.04 Basic 0.03	Acid 0.04 Basic 0.04
Sulphur " " " . .	0.05	0.04	0.04
Manganese	0.30 to 0.60	0.30 to 0.50	0.30 to 0.50

Boiler-rivet Steel. 3. Steel for boiler rivets shall be of the extra-soft class as specified in paragraphs Nos. 2 and 4.

Physical Properties.

Tensile Tests. 4. The three classes of open-hearth boiler plate and rivet steel—namely, flange or boiler steel, fire-box steel, and extra-soft steel—shall conform to the following physical qualities:

	Flange or Boiler Steel.	Fire-box Steel.	Extra Soft Steel.
Tensile strength, pounds per square inch . . .	55000 to 65000	52000 to 62000	45000 to 55000
Yield-point, in pounds per square inch, shall not be less than	$\frac{1}{2}$ T. S.	$\frac{1}{2}$ T. S.	$\frac{1}{2}$ T. S.
Elongation, per cent in 8 inches shall not be less than	25	26	28

Modifications in Elongation for Thin and Thick Material. 5. For material less than five-sixteenths inch ($\frac{5}{16}''$) and more than three-fourths inch ($\frac{3}{4}''$) in thickness the following modifications shall be made in the requirements for elongation:

(*a*) For each increase of one-eighth inch ($\frac{1}{8}''$) in thickness above three-fourths inch ($\frac{3}{4}''$) a deduction of one per cent (1%) shall be made from the specified elongation.

(*b*). For each decrease of one-sixteenth inch ($\frac{1}{16}''$) in thickness below five-sixteenths inch ($\frac{5}{16}''$) a deduction of two and one-half per cent ($2\frac{1}{2}$%) shall be made from the specified elongation.

Bending Tests. 6. The three classes of open-hearth boiler plate and rivet steel shall conform to the following bending tests; and for this purpose the test specimen shall be one and one-half inches ($1\frac{1}{2}''$) wide, if possible, and for all material three-fourths inch ($\frac{3}{4}''$) or less in thickness the test specimen shall be of the same thickness as that of the finished material from which it is cut, but for material more than three-fourths inch ($\frac{3}{4}''$) thick the bending-test specimen may be one-half inch ($\frac{1}{2}''$) thick:

Rivet rounds shall be tested of full size as rolled.

(*c*). Test specimens cut from the rolled material, as specified above, shall be subjected to a cold bending test, and also to a quenched bending test. The cold bending test shall be made on the material in the condition in which it is to be used, and prior to the quenched bending test the specimen shall be heated to a light cherry-red, as seen in the dark, and quenched in water the temperature of which is between 80° and 90° Fahrenheit.

(*d*). Flange or boiler steel, fire-box steel, and rivet steel, both before and after quenching, shall bend cold one hundred and eighty degrees (180°) flat on itself without fracture on the outside of the bent portion.

Homogeneity Tests.

7. For fire-box steel a sample taken from a broken tensile-test specimen shall not show any single seam or cavity more than one-fourth inch ($\frac{1}{4}''$) long in either of the three fractures obtained on the test for homogeneity as described below in paragraph 12.

Test Pieces and Methods of Testing.

Test Specimen for Tensile Test.

8. The standard specimen of eight inch (8″) gauged length shall be used to determine the physical properties specified in paragraphs Nos. 4 and 5. The standard shape of the test specimen for sheared plates shall be as shown in Fig. 2. (See page 398.)

For other material the test specimen may be the same as for sheared plates, or it may be planed or turned parallel throughout its entire length; and in all cases, where possible, two opposite sides of the test specimens shall be the rolled surfaces. Rivet rounds and small rolled bars shall be tested of full size as rolled.

Number of Tensile Tests.

9. One tensile-test specimen will be furnished from each plate as it is rolled, and two tensile-test specimens will be furnished from each melt of rivet rounds. In case any one of these develops flaws or breaks outside of the middle third of its gauged length, it may be discarded and another test specimen substituted therefor.

10. For material three-fourths inch ($\frac{3}{4}''$) or less in thickness the bending-test specimen shall have the natural rolled surface on two opposite sides. The bending-test specimens cut from plates shall be one and one-half inches ($1\frac{1}{2}''$) wide, and for material more than three-fourths inch ($\frac{3}{4}''$) thick the bending-test specimens may be one-half inch ($\frac{1}{2}''$) thick. The sheared edges of bending-test specimens may be milled or planed. The bending-test specimens for rivet rounds shall be of full size as rolled. The bending test may be made by pressure or by blows.

Test Specimens for Bending.

11. One cold-bending specimen and one quenched-bending specimen will be furnished from each plate as it is rolled. Two cold-bending specimens and two quenched-bending specimens will be furnished from each melt of rivet rounds. The homogeneity test for fire-box steel shall be made on one of the broken tensile-test specimens.

Number of Bending Tests.

12. The homogeneity test for fire-box steel is made as follows: A portion of the broken tensile-test specimen is either nicked with a chisel or grooved on a machine, transversely about a sixteenth of an inch ($\frac{1}{16}''$) deep, in three places about two inches ($2''$) apart. The first groove should be made on one side, two inches ($2''$) from the square end of the specimen; the second, two inches ($2''$) from it on the opposite side; and the third, two inches ($2''$) from the last, and on the opposite side from it. The test specimen is then put in a vise, with the first groove about a quarter of an inch ($\frac{1}{4}''$) above the jaws, care being taken to hold it firmly. The projecting end of the test specimen is then broken off by means of a hammer, a number of light blows being used, and the bending being away from the groove. The specimen is broken at the other two grooves in the same way. The object of this treatment is to open and render visible to the eye any seams due to failure to weld up, or to foreign interposed matter, or cavities due to gas bubbles in the ingot. After rupture, one side of each fracture is examined, a pocket lens being used, if necessary, and the length of the seams and cavities is determined.

Homogeneity Tests for Fire-box Steel.

13. For the purposes of this specification the yield-point shall be determined by the careful observation of the drop of the beam or halt in the gauge of the testing machine.

Yield-point.

14. In order to determine if the material conforms to the chemical limitations prescribed in paragraph 2 herein, analysis shall be made of drillings taken from a small test ingot. An additional check analysis may be made from a tensile specimen of each melt used on an order, other than in locomotive fire-box steel. In the case of locomotive fire-box steel a check analysis may be made from the tensile specimen from each plate as rolled. **Sample for Chemical Analysis.**

Variation in Weight.

15. The variation in cross section or weight of more than $2\frac{1}{2}$ per cent from that specified will be sufficient cause for rejection, except in the case of sheared plates, which will be covered by the following permissible variations:

(*e*) Plates $12\frac{1}{2}$ pounds per square foot for heavier, up to 100 inches wide when ordered to weight, shall not average more than $2\frac{1}{2}$ per cent variation above or $2\frac{1}{2}$ per cent below the theoretical weight. When 100 inches wide and over, 5 per cent above or 5 per cent below the theoretical weight.

(*f*) Plates under $12\frac{1}{2}$ pounds per square foot, when ordered to weight, shall not average a greater variation than the following:

Up to 75 inches wide, $2\frac{1}{2}$ per cent above or $2\frac{1}{2}$ per cent below the theoretical weight. Seventy-five inches wide up to 100 inches wide, 5 per cent above or 3 per cent below the theoretical weight. When 100 inches wide and over, 10 per cent above or 3 per cent below the theoretical weight.

. . . .

(*g*) For all plates ordered to gauge there will be permitted an average excess of weight over that corresponding to the dimensions on the order equal in amount to that specified in the following table:

Table of Allowances for Overweight for Rectangular Plates when ordered to Gauge.

Plates will be considered up to gauge if measuring not over $\frac{1}{100}$ inch less than the ordered gauge.

The weight of one cubic inch of rolled steel is assumed to be 0.2833 pound.

PLATES $\frac{1}{4}$ INCH AND OVER IN THICKNESS.

Thickness of Plate. Inch.	Width of Plate.		
	Up to 75 Inches. Per Cent.	75 to 100 Inches. Per Cent.	Over 100 Inches. Per Cent.
$\frac{1}{4}$	10	14	18
$\frac{5}{16}$	8	12	16
$\frac{3}{8}$	7	10	13
$\frac{7}{16}$	6	8	10
$\frac{1}{2}$	5	7	9
$\frac{9}{16}$	$4\frac{1}{2}$	$6\frac{1}{2}$	$8\frac{1}{2}$
$\frac{5}{8}$	4	6	8
Over $\frac{5}{8}$	$3\frac{1}{2}$	5	$6\frac{1}{2}$

PLATES UNDER $\frac{1}{4}$ INCH IN THICKNESS.

Thickness of Plate. Inch.	Width of Plate.	
	Up to 50 Inches. Per Cent.	50 Inches and Above. Per Cent.
$\frac{1}{8}$ up to $\frac{5}{32}$	10	15
$\frac{5}{32}$ " " $\frac{3}{16}$	$8\frac{1}{2}$	$12\frac{1}{2}$
$\frac{3}{16}$ " " $\frac{1}{4}$	7	10

FINISH.

16. All finished material shall be free from injurious surface defects and laminations, and must have a workmanlike finish.

BRANDING.

17. Every finished piece of steel shall be stamped with the melt number, and each plate and the coupon or test specimen cut from it shall be stamped with a separate identifying mark or number. Rivet steel may be shipped in bundles securely wired together with the melt number on a metal tag attached.

INSPECTION.

18. The inspector, representing the purchaser, shall have all reasonable facilities afforded to him by the manufacturer to satisfy him that the finished material is furnished in accordance with these specifications. All tests and inspections shall be made at the place of manufacture, prior to shipment.

STRUCTURAL STEEL FOR BUILDINGS.

Adopted 1901.

Process of Manufacture.

1. Steel may be made by either the open-hearth or Bessemer process.

Chemical Properties.

2. Each of the two classes of structural steel for buildings shall not contain more than 0.10 per cent of phosphorus.

Physical Properties.

3. There shall be two classes of structural steel for buildings,—namely, rivet steel and medium steel,—which shall conform to the following physical qualities: **Classes.**

4. **Tensile Tests.**

	Rivet Steel.	Medium Steel.
Tensile strength, pounds per square inch . .	50000 to 60000	60000 to 70000
Yield-point, in pounds per square inch, shall not be less than	½ T. S.	½ T. S.
Elongation, per cent in 8 inches shall not be less than	26	22

5. For material less than five-sixteenths inch ($\frac{5}{16}''$) and more than three-fourths inch ($\frac{3}{4}''$) in thickness the following modifications shall be made in the requirements for elongation: **Modifications in Elongation for Thin and Thick Material.**

(*a*) For each increase of one-eighth inch ($\frac{1}{8}''$) in thickness above three-fourths inch ($\frac{3}{4}''$) a deduction of one per cent (1%) shall be made from the specified elongation.

(*b*) For each decrease of one-sixteenth inch ($\frac{1}{16}''$) in thickness below five-sixteenths inch ($\frac{5}{16}''$) a deduction of two and one-half per cent (2½%) shall be made from the specified elongation.

(*c*) For pins the required elongation shall be five per cent (5%) less than that specified in paragraph No. 4, as determined on a test specimen the centre of which shall be one inch (1″) from the surface.

6. The two classes of structural steel for buildings shall conform to the following bending tests; and for this purpose the test specimen shall be one and one-half inches ($1\frac{1}{2}''$) wide, if possible, and for all material three-fourths ($\frac{3}{4}''$) or less in thickness the test specimen shall be of the same thickness as that of the finished material **Bending. Tests.**

from which it is cut, but for material more than three-fourths inch ($\frac{3}{4}''$) thick the bending-test specimen may be one-half inch ($\frac{1}{2}''$) thick.

Rivet rounds shall be tested of full size as rolled.

(*d*) Rivet steel shall bend cold 180° flat on itself without fracture on the outside of the bent portion.

(*e*) Medium steel shall bend cold 180° around a diameter equal to the thickness of the specimen tested, without fracture on the outside of the bent portion.

TEST PIECES AND METHODS OF TESTING.

Test Specimen for Tensile Test.

7. The standard test specimen of eight-inch ($8''$) gauged length shall be used to determine the physical properties specified in paragraphs Nos. 4 and 5. The standard shape of the test specimen for sheared plates shall be as shown by Fig. 2. (See page 398.) For other material the test specimen may be the same as for sheared plates or it may be planed or turned parallel throughout its entire length and, in all cases where possible, two opposite sides of the test specimen shall be the rolled surfaces. Rivet rounds and small rolled bars shall be tested of full size as rolled.

Number of Tensile Tests.

8. One tensile-test specimen shall be taken from the finished material of each melt or blow; but in case this develops flaws, or breaks outside of the middle third of its gauged length, it may be discarded and another test specimen substituted therefor.

Test Specimen for Bending.

9. One test specimen for bending shall be taken from the finished material of each melt or blow as it comes from the rolls, and for material three-fourths inch ($\frac{3}{4}''$) and less in thickness this specimen shall have the natural rolled surface on two opposite sides. The bending-test specimen shall be one and one-half inches ($1\frac{1}{2}''$) wide, if possible; and for material more than three-fourths inch ($\frac{3}{4}''$) thick the bending-test specimen may be one-half inch ($\frac{1}{2}''$) thick. The sheared edges of bending-test specimens may be milled or planed.

Rivet rounds shall be tested of full size as rolled.

(*f*) The bending test may be made by pressure or by blows.

Annealed Test Specimens.

10. Material which is to be used without annealing or further treatment shall be tested for tensile strength in the condition in which it comes from the rolls. Where it is

impracticable to secure a test specimen from material which has been annealed or otherwise treated, a full-sized section of tensile-test specimen length shall be similarly treated before cutting the tensile-test specimen therefrom.

11. For the purposes of this specification the yield-point shall be determined by the careful observaton of the drop of the beam or halt in the gauge of the testing machine. **Yield-point.**

12. In order to determine if the material conforms to the chemical limitations prescribed in paragraph No. 2 herein, analysis shall be made of drillings taken from a small test ingot. **Sample for Chemical Analysis.**

Variation in Weight.

13. The variation in cross section or weight of more than $2\frac{1}{2}$ per cent from that specified will be sufficient cause for rejection, except in the case of sheared plates, which will be covered by the following permissible variations:

(*g*) Plates $12\frac{1}{2}$ pounds per square foot or heavier, up to 100 inches wide, when ordered to weight, shall not average more than $2\frac{1}{2}$ per cent variation above or $2\frac{1}{2}$ per cent below the theoretical weight. When 100 inches wide, and over 5 per cent above or 5 per cent below the theoretical weight.

(*h*) Plates under $12\frac{1}{2}$ pounds per square foot, when ordered to weight, shall not average a greater variation than the following:

Up to 75 inches wide, $2\frac{1}{2}$ per cent above or $2\frac{1}{2}$ per cent below the theoretical weight. Seventy-five inches wide up to 100 inches wide, 5 per cent above or 3 per cent below the theoretical weight. When 100 inches wide and over, 10 per cent above or 3 per cent below the theoretical weight.

(*i*) For all plates ordered to gauge, there will be permitted an average excess of weight over that corresponding to the dimensions on the order equal in amount to that specified in the following table:

Table of Allowances for Overweight for Rectangular Plates when ordered to Gauge.

Plates will be considered up to gauge if measuring not over $\frac{1}{100}$ inch less than the ordered gauge.

The weight of 1 cubic inch of rolled steel is assumed to be 0.2833 pound.

PLATES $\frac{1}{4}$ INCH AND OVER IN THICKNESS.

Thickness of Plate. Inch.	Width of Plate.		
	Up to 75 Inches. Per Cent.	75 to 100 Inches. Per Cent.	Over 100 Inches. Per Cent.
$\frac{1}{4}$	10	14	18
$\frac{5}{16}$	8	12	16
$\frac{3}{8}$	7	10	13
$\frac{7}{16}$	6	8	10
$\frac{1}{2}$	5	7	9
$\frac{9}{16}$	$4\frac{1}{2}$	$6\frac{1}{2}$	$8\frac{1}{2}$
$\frac{5}{8}$	4	6	8
Over $\frac{5}{8}$	$3\frac{1}{2}$	5	$6\frac{1}{2}$

PLATES UNDER $\frac{1}{4}$ INCH IN THICKNESS.

Thickness of Plate. Inch.	Width of Plate.	
	Up to 50 Inches. Per Cent.	50 Inches and Above. Per Cent.
$\frac{1}{8}$ up to $\frac{5}{32}$	10	15
$\frac{5}{32}$ " " $\frac{3}{16}$	$8\frac{1}{2}$	$12\frac{1}{2}$
$\frac{3}{16}$ " " $\frac{1}{4}$	7	10

FINISH.

14. Finished material must be free from injurious seams, flaws, or cracks, and have a workmanlike finish.

BRANDING.

15. Every finished piece of steel shall be stamped with the melt or blow number, except that small pieces may be shipped in bundles securely wired together with the melt or blow number on a metal tag attached.

INSPECTION.

16. The inspector, representing the purchaser, shall have all reasonable facilities accorded to him by the manufacturer to satisfy him that the finished material is furnished in accordance with these specifications. All tests and inspections shall be made at the place of manufacture, prior to shipment.

STRUCTURAL STEEL FOR BRIDGES.

Adopted 1905.

1. Steel shall be made by the open-hearth process. **Manufacture.**

2. The chemical and physical properties shall conform to the following limits: **Chemical and Physical Properties.**

Elements Considered.	Structural Steel.	Rivet Steel.	Steel Castings.
Phosphorus Max. { Basic	0.04 per cent.	0.04 per cent.	0.05 per cent.
Phosphorus Max. { Acid	0.08 "	0.04 "	0.08 "
Sulphur Max.	0.05 "	0.04 "	0.05 "
Ult. tensile strength. Pounds per sq. in.	Desired 60,000	Desired 50,000	Not less than 65,000
Elong.: Min. per cent. in 8 in. (Fig. 1)	$\frac{1,500,000^*}{\text{Ult. tens. str.}}$	$\frac{1,500,000}{\text{Ult. tens. str.}}$	
Elong.: Min. per cent. in 2 in. (Fig. 2)	22	....	18
Character of fracture	Silky	Silky	Silky or fine granular.
Cold bend without fracture	180° flat †	180° flat ‡	90°. $d=3t$

* See par. 11. † See par. 12, 13 and 14. ‡ See par. 15.

The yield-point, as indicated by the drop of beam, shall be recorded in the test reports.

3. If the ultimate strength varies more than 4,000 lbs. from that desired, a retest may be made, at the discretion of the inspector, on the same gauge, which, to be acceptable, shall be within 5,000 lbs. of the desired ultimate. **Retests.**

4. Chemical determinations of the percentages of carbon, phosphorus, sulphur, and manganese shall be made by the manufacturer from a test ingot taken at the time of the pouring of each melt of steel and a correct copy of such analysis shall be furnished to the engineer or his inspector. Check analyses shall be made from finished material, if called for by the purchaser, in which case an excess of 25 per cent above the required limits will be allowed. **Chemical Determinations.**

5. Specimens for tensile and bending tests for plates, shapes, and bars shall be made by cutting coupons from the finished product, which shall have both faces rolled and both edges milled to the form shown by Fig. 2, page 398; or with both edges **Plates, Shapes and Bars.**

parallel; or they may be turned to a diameter of $\frac{3}{4}$ inch for a length of at least 9 inches, with enlarged ends.

Rivets. 6. Rivet rods shall be tested as rolled.

Pins and Rollers. 7. Specimens shall be cut from the finished rolled or forged bar in such manner that the centre of the specimen shall be 1 inch from the surface of the bar. The specimen for tensile test shall be turned to the form shown by Fig. 1, page 398. The specimen for bending test shall be 1 inch by $\frac{1}{2}$ inch in section.

Steel Castings. 8. The number of tests will depend on the character and importance of the castings. Specimens shall be cut cold from coupons moulded and cast on some portion of one or more castings from each melt or from the sink-heads, if the heads are of sufficient size. The coupon or sink-head, so used, shall be annealed with the casting before it is cut off. Test specimens to be of the form prescribed for pins and rollers.

Conditions for Tests. 9. Material which is to be used without annealing or further treatment shall be tested in the condition in which it comes from the rolls. When material is to be annealed or otherwise treated before use, the specimens for tensile tests, representing such material, shall be cut from properly annealed or similarly treated short lengths of the full section of the bar.

Number of Tests. 10. At least one tensile and one bending test shall be made from each melt of steel as rolled. In case steel differing $\frac{3}{8}$ inch and more in thickness is rolled from one melt, a test shall be made from the thickest and thinest material rolled.

Elongation. 11. For material less than 5-16 inch and more than $\frac{3}{4}$ inch in thickness the following modifications will be allowed in the requirements for elongation:

(*a*) For each 1-16 inch in thickness below 5-16 inch, a deduction of $2\frac{1}{2}$ will be allowed from the specified percentage.

(*b*) For each $\frac{1}{8}$ inch in thickness above $\frac{3}{4}$ inch, a deduction of 1 will be allowed from the specified percentage.

Bending Tests. 12. Bending tests may be made by pressure or by blows. Plates, shapes, and bars less than 1 inch thick shall bend as called for in paragraph 2.

13. Full-sized material for eye-bars and other steel 1 inch thick

and over, tested as rolled, shall bend cold 180° around a pin the diameter of which is equal to twice the thickness of the bar, without a fracture on the outside of bend. **Full-sized Bends**

14. Angles ¾ inch and less in thickness shall open flat, and angles ½ inch and less in thickness shall bend shut, cold, under blows of a hammer, without sign of fracture. This test will be made only when required by the inspector. **Tests on Angles.**

15. Rivet steel, when nicked and bent around a bar of the same diameter as the rivet rod, shall give a gradual break and a fine, silky, uniform fracture. **Tests on Rivet Steel.**

16. Finished material shall be free from injurious seams, flaws, cracks, defective edges, or other defects, and have a smooth uniform, workmanlike finish. Plates 36 inches in width and under shall have rolled edges. **Finish.**

17. Every finished piece of steel shall have the melt number and the name of the manufacturer stamped or rolled upon it. Steel for pins and rollers shall be stamped on the end. Rivet and lattice steel and other small parts may be bundled with the above marks on an attached metal tag. **Marking.**

18. Material which, subsequent to the above tests at the mills and its acceptance there, develops weak spots, brittleness, cracks or other imperfections, or is found to have injurious defects, will be rejected at the shop and shall be replaced by the manufacturer at his own cost. **Rejections.**

19. A variation in cross-section or weight of each piece of steel of more than 2½ per cent from that specified will be sufficient cause for rejection, except in case of sheared plates, which will be covered by the following permissible variations, which are to apply to single plates. **Permissible Variations.**

WHEN ORDERED TO WEIGHT.

Permissible Variations.

20. Plates 12½ pounds per square foot or heavier:

(*a*) Up to 100 inches wide, 2½ per cent above or below the prescribed weight.

(*b*) One hundred inches wide and over, 5 per cent above or below.

21. Plates under 12½ pounds per square foot:

(*a*) Up to 75 inches wide, 2½ per cent above or below.

(*b*) Seventy-five inches and up to 100 inches wide, 5 per cent above or 3 per cent below.

(*c*) One hundred inches wide and over, 10 per cent above or 3 per cent below.

When Ordered to Gauge.

Permissible Variations. 22. Plates will be accepted if they measure not more than 0.01 inch below the ordered thickness.

23. An excess over the nominal weight corresponding to the dimensions on the order, will be allowed for each plate, if not more than that shown in the following tables, one cubic inch of rolled steel being assumed to weigh 0.2833 pound.

24. Plates ¼ inch and over in thickness.

Thickness Ordered.	Nominal Weights.	Width of Plate.			
		Up to 75″.	75″ and up to 100″.	100″ and up to 115′.	Over 115″.
1–4 inch.	10.20 lbs.	10 per cent.	14 per cent.	18 per cent.	
5–16 "	12.75 "	8 " "	12 " "	16 " "	
3–8 "	15.30 "	7 " "	10 " "	13 " "	17 per cent.
7–16 "	17.85 "	6 " "	8 " "	10 " "	13 " "
1–2 "	20.40 "	5 " "	7 " "	9 " "	12 " "
9–16 "	22.95 "	4½ " "	6½ " "	8½ " "	11 " "
5–8 "	25.50 "	4 " "	6 " "	8 " "	10 " "
Over 5–8 "	....	3½ " "	5 " "	6½ " "	9 " "

25. Plates under ¼ inch in thickness.

Thickness Ordered.	Nominal Weights. Pounds per Square Feet.	Width of Plate.		
		Up to 50″.	50″ and up to 70″.	Over 70″.
1–8″ up to 5–32″	5.10 to 6.37	10 per cent.	15 per cent.	20 per cent.
5–32″ " 3–16″	6.37 " 7.65	8½ " "	12½ " "	17 " "
3–16″ " 1–4″	7.65 " 10.20	7 " "	10 " "	15 " "

Inspection and Testing. 26. The purchaser shall be furnished complete copies of mill orders and no material shall be rolled, nor work done, before the purchaser has been notified where the orders have been placed, so that he may arrange for the inspection.

27. The manufacturer shall furnish all facilities for inspecting and testing the weight and quality of all material at the mill where it is manufactured. He shall furnish a suitable testing machine for testing the specimens, as well as prepare the pieces for the machine, free of cost.

28. When an inspector is furnished by the purchaser to inspect material at the mills, he shall have full access, at all times, to all parts of mills where material to be inspected by him is being manufactured.

STRUCTURAL STEEL FOR SHIPS.

Adopted 1901 for bridges and ships. Restricted to ships, 1905.

Process of Manufacture.

1. Steel shall be made by the open-hearth process.

Chemical Properties.

2. Each of the three classes of structural steel for ships shall conform to the following limits in chemical composition:

	Steel Made by the Acid Process. Per Cent.	Steel Made by the Basic Process. Per Cent.
Phosphorus shall not exceed .	0.08	0.06
Sulphur " " " .	0.06	0.06

Physical Properties.

3. There shall be three classes of structural steel for ships,—namely, rivet steel, soft steel, and medium steel,—which shall conform to the following physical qualities: **Classes.**

4. **Tensile Tests.**

	Rivet Steel.	Soft Steel.	Medium Steel.
Tensile strength, pounds per square inch . . .	50000 to 60000	52000 to 62000	60000 to 70000
Yield-point, in pounds per square inch shall not be less than	½ T. S.	½ T. S.	½ T. S.
Elongation, per cent in 8 inches shall not be less than	26	25	22

5. For material less than five-sixteenths inch ($\frac{5}{16}''$) and more than three-fourths inch ($\frac{3}{4}''$) in thickness the following modifications shall be made in the requirements for elongation:

Modifications in Elongation for Thin and Thick Material.

(*a*) For each increase of one-eighth inch ($\frac{1}{8}''$) in thickness above three-fourths inch ($\frac{3}{4}''$) a deduction of one per cent (1%) shall be made from the specified elongation.

(*b*) For each decrease of one-sixteenth inch ($\frac{1}{16}''$) in thickness below five-sixteenths inch ($\frac{5}{16}''$) a deduction of two and one-half per cent ($2\frac{1}{2}\%$) shall be made from the specified elongation.

(*c*) For pins made from any of the three classes of steel the required elongation shall be five per cent (5%) less than that specified in paragraph No. 4, as determined on a test specimen, the center of which shall be one inch (1″) from the surface.

6. Eye-bars shall be of medium steel. Full-sized tests shall show $12\frac{1}{2}$ per cent elongation in fifteen feet of the body of the eye-bar, and the tensile strength shall not be less than 55,000 pounds per square inch. Eye-bars shall be required to break in the body; but, should an eye-bar break in the head, and show twelve and one-half per cent ($12\frac{1}{2}\%$) elongation in fifteen feet and the tensile strength specified, it shall not be cause for rejection, provided that not more than one-third ($\frac{1}{3}$) of the total number of eye-bars tested break in the head.

Tensile Tests of Eye-bars.

7. The three classes of structural steel for ships shall conform to the following bending tests; and for this purpose the test specimen shall be one and one-half inches wide, if possible, and for all material three-fourths inch ($\frac{3}{4}''$) or less in thickness the test specimen shall be of the same thickness as that of the finished material from which it is cut, but for material more than three-fourths inch ($\frac{3}{4}''$) thick the bending-test specimen may be one-half inch ($\frac{1}{2}''$) thick.

Bending Tests.

Rivet rounds shall be tested of full size as rolled.

(*d*) Rivet steel shall bend cold 180° flat on itself without fracture on the outside of the bent portion.

(*e*) Soft steel shall bend cold 180° flat on itself without fracture on the outside of the bent portion.

(*f*) Medium steel shall bend cold 180° around a diameter equal to the thickness of the specimen tested, without fracture on the outside of the bent portion.

Test Pieces and Methods of Testing.

8. The standard test specimen of eight inch (8″) gauged length shall be used to determine the physical properties specified in paragraphs Nos. 4 and 5. The standard shape of the test specimen for sheared plates shall be as shown by Fig. 2, page 398. For other material the test specimen may be the same as for sheared plates, or it may be planed or turned parallel throughout its entire length; and, in all cases where possible, two opposite sides of the test specimens shall be the rolled surfaces. Rivet rounds and small rolled bars shall be tested of full size as rolled.

Test Specimen for Tensile Test.

9. One tensile-test specimen shall be taken from the finished material of each melt; but in case this develops flaws, or breaks outside of the middle third of its gauged length, it may be discarded, and another test specimen substituted therefor.

Number of Tensile Tests.

10. One test specimen for bending shall be taken from the finished material of each melt as it comes from the rolls, and for material three-fourths inch (¾″) and less in thickness this specimen shall have the natural rolled surface on two opposite sides. The bending-test specimen shall be one and one half inches (1½″) wide, if possible, and for material more than three-fourths inch (¾″) thick the bending-test specimen may be one-half inch (½″) thick. The sheared edges of bending-test specimens may be milled or planed.

Test Specimens for Bending.

(*g*) The bending test may be made by pressure or by blows.

11. Material which is to be used without annealing or further treatment shall be tested for tensile strength in the condition in which it comes from the rolls. Where it is impracticable to secure a test specimen from material which has been annealed or otherwise treated, a full-sized section of tensile test, specimen length, shall be similarly treated before cutting the tensile-test specimen therefrom.

Annealed Test Specimens.

12. For the purpose of this specification the yield-point shall be determined by the careful observation of the drop of the beam or halt in the gauge of the testing machine.

Yield-point.

13. In order to determine if the material conforms to the chemical limitations prescribed in paragraph No. 2 herein, analysis shall be made of drillings taken from a small test ingot.

Sample for Chemical Analysis.

Variation in Weight.

14. The variation in cross section or weight of more than $2\frac{1}{2}$ per cent from that specified will be sufficient cause for rejection, except in the case of sheared plates, which will be covered by the following permissible variations:

(*h*) Plates $12\frac{1}{2}$ pounds per square foot or heavier, up to 100 inches wide, when ordered to weight, shall not average more than $2\frac{1}{2}$ per cent variation above or $2\frac{1}{2}$ per cent below the theoretical weight. When 100 inches wide and over, 5 per cent above or 5 per cent below the theoretical weight.

(*i*) Plates under $12\frac{1}{2}$ pounds per square foot, when ordered to weight, shall not average a greater variation than the following:

Up to 75 inches wide, $2\frac{1}{2}$ per cent above or $2\frac{1}{2}$ per cent below the theoretical weight. 75 inches wide up to 100 inches wide, 5 per cent above or 3 per cent below the theoretical weight. When 100 inches wide and over, 10 per cent above or 3 per cent below the theoretical weight.

(*j*) For all plates ordered to gauge there will be permitted an average excess of weight over that corresponding to the dimensions on the order equal in amount to that specified in the following table:

Table of Allowances for Overweight for Rectangular Plates when ordered to Gauge.

Plates will be considered up to gauge if measuring not over $\frac{1}{100}$ inch less than the ordered gauge.

The weight of 1 cubic inch of rolled steel is assumed to be 0.2833 pound.

Plate $\frac{1}{4}$ Inch and over in Thickness.

Thickness of Plate. Inch.	Width of Plate.		
	Up to 75 Inches. Per Cent.	75 to 100 Inches. Per Cent.	Over 100 Inches. Per Cent.
$\frac{1}{4}$	10	14	18
$\frac{5}{16}$	8	12	16
$\frac{3}{8}$	7	10	13
$\frac{7}{16}$	6	8	10
$\frac{1}{2}$	5	7	9
$\frac{9}{16}$	$4\frac{1}{2}$	$6\frac{1}{2}$	$8\frac{1}{2}$
$\frac{5}{8}$	4	6	8
Over $\frac{5}{8}$	$3\frac{1}{2}$	5	$6\frac{1}{2}$

Plates under ¼ Inch in Thickness.

Thickness of Plate. Inch.	Width of Plate.	
	Up to 50 Inches. Per Cent.	50 Inches and Above. Per Cent.
$\frac{1}{8}$ up to $\frac{5}{32}$	10	15
$\frac{5}{32}$ " " $\frac{3}{16}$	8½	12½
$\frac{3}{16}$ " " $\frac{1}{4}$.	7	10

Finish.

15. Finished material must be free from injurious seams, flaws, or cracks, and have a workmanlike finish.

Branding.

16. Every finished piece of steel shall be stamped with the melt number, and steel for pins shall have the melt number stamped on the ends. Rivets and lacing steel, and small pieces for pin plates and stiffeners, may be shipped in bundles, securely wired together, with the melt number on a metal tag attached.

Inspection.

17. The inspector, representing the purchaser, shall have all reasonable facilities afforded to him by the manufacturer to satisfy him that the finished material is furnished in accordance with these specifications. All tests and inspections shall be made at the place of manufacture, prior to shipment.

STEEL AXLES.

Adopted 1901. Modified 1905.

Process of Manufacture.

1. Steel for axles shall be made by the open-hearth process.

Chemical Properties.

2. There shall be three classes of steel axles, which shall conform to the following limits in chemical composition:

	Car and Tender-truck Axles. Per Cent.	Driving and Engine-truck Axles. (Carbon Steel.) Per Cent.	Driving-wheel Axles. (Nickel-steel.) Per Cent.
Phosphorus shall not exceed........	0.06	0.06	0.04
Sulphur " " "	0.06	0.06	0.04
Manganese " " "		0.60	
Nickel..........................			3.0 to 4.0

PHYSICAL PROPERTIES.

Tensile Tests. 3. For car and tender-truck axles, no tensile test shall be required.

4. The minimum physical qualities required in the two classes of driving-wheel axles shall be as follows:

	Driving and Engine-truck Axles. (Carbon Steel.)	Driving and Engine-truck Axles. (Nickel steel.)
Tensile strength, pounds per square inch.........	80,000	80,000
Yield-point, pounds per square inch..............	40,000	50,000
Elongation, per cent in two inches...............	20	25
Contraction of area per cent.....................	25	45

Drop Test. 5. One axle selected from each melt, when tested by the drop test described in paragraph No. 9, shall stand the number of blows at the height specified in the following table without rupture and without exceeding, as the result of the first blow, the deflection given. Any melt failing to meet these requirements will be rejected.

Diameter of Axle at Center. Inches.	Number of Blows.	Height of Drop. Feet.	Deflection. Inches.
$4\frac{1}{4}$	5	24	$8\frac{1}{4}$
$4\frac{3}{8}$	5	26	$8\frac{1}{4}$
$4\frac{7}{16}$	5	$28\frac{1}{2}$	$8\frac{1}{4}$
$4\frac{5}{8}$	5	31	8
$4\frac{3}{4}$	5	34	8
$5\frac{3}{8}$	5	43	7
$5\frac{7}{8}$	7	43	$5\frac{1}{2}$

6. Carbon-steel and nickel-steel driving-wheel axis shall not be subject to the above drop test.

TEST PIECES AND METHODS OF TESTING.

Test Specimen for Tensile Tests.

7. The standard test specimen one-half inch ($\frac{1}{2}''$) diameter and two inch ($2''$) gauged length shall be used to determine the physical properties specified in paragraph No. 4. It is shown in Fig. 1. (See p. 398.)

Number and Location of Tensile Specimens.

8. For driving and engine-truck axles one longitudinal test specimen shall be cut from one axle of each melt. The center of this test specimen shall be half-way between the center and outside of the axle.

Drop Test Described.

9. The points of supports on which the axle rests during tests must be three feet apart from center to center; the tup must weigh 1,640 pounds; the anvil, which is supported on springs, must weigh 17,500 pounds; it must be free to move in a vertical direction; the springs upon which it rests must be twelve in number, of the kind described on drawing; and the radius of supports and of the striking face on the tup in the direction of the axis of the axle must be five (5) inches. When an axle is tested, it must be so placed in the machine that the tup will strike it midway between the ends; and it must be turned over after the first and third blows, and, when required, after the fifth blow. To measure the deflection after the first blow, prepare a straight edge as long as the axle, by reinforcing it on one side, equally at each end, so that, when it is laid on the axle, the reinforced parts will rest on the collars or ends of the axle, and the balance of the straight edge not touch the axle at any place. Next place the axle in position for test, lay the straight edge on it, and measure the distance from the straight edge to the axle at the middle point of the latter. Then, after the first blow, place the straight edge on the now bent axle in the same manner as before, and measure the distance from it to that side of the axle next to the straight edge at the point farthest away from the latter. The difference between the two measurements is the deflection. The report of the drop test shall state the atmospheric temperature at the time the tests were made.

Yield-point.

10. The yield-point specified in paragraph No. 4 shall be determined by the careful observation of the drop of the beam or halt in the gauge of the testing machine.

Sample for Chemical Analysis.

11. Turnings from the tensile-test specimen of driving and engine-truck axles, or drillings taken midway between the center and outside of car, engine, and tender-truck axles, or drillings from the small test ingot, if preferred by the inspector, shall be used to determine whether the melt is within the limits of chemical composition specified in paragraph No. 2.

Finish.

12. Axles shall conform in sizes, shapes, and limiting weights to the requirements given on the order or print sent with it. They shall be made and finished in a workmanlike manner, and shall be free from all injurious cracks, seams, or flaws. In centering, sixty- (60) degree centers must be used, with clearance given at the point to avoid dulling the shop lathe centers.

Branding.

13. Each axle shall be legibly stamped with the melt number and initials of the maker at the places marked on the print or indicated by the inspector.

Inspection.

14. The inspector, representing the purchaser, shall have all reasonable facilities afforded to him by the manufacturer to satisfy him that the finished material is furnished in accordance with these specifications. All tests and inspections shall be made at the place of manufacture, prior to shipment.

STEEL TIRES.

Adopted 1901.

Process of Manufacture.

1. Steel for tires may be made by either the open-hearth or crucible process.

Chemical Properties.

2. There will be three classes of steel tires which shall conform to the following limits in chemical composition:

	Passenger Engines. Per Cent.	Freight-engine and Car-wheels. Per Cent.	Switching-engines. Per Cent.
Manganese shall not exceed	0.80	0.80	0.80
Silicon shall not be less than	0.20	0.20	0.20
Phosphorus shall not exceed	0.05	0.05	0.05
Sulphur shall not exceed	0.05	0.05	0.05

PHYSICAL PROPERTIES.

3. The minimum physical qualities required in each of the three classes of steel tires shall be as follows: Tensile Tests.

	Passenger-engines.	Freight-engine and Car-wheels.	Switching-engines.
Tensile strength, pounds per square inch.	100,000	110,000	120,000
Elongation, per cent in two inches.	12	10	8

4. In the event of the contract calling for a drop test, a test tire from each melt will be furnished at the purchaser's expense, provided it meets the requirements. This test tire shall stand the drop test described in paragraph No. 7, without breaking or cracking, and shall show a minimum deflection equal to $D^2 \div (40T^2+2D)$, the letter "D" being internal diameter and the letter "T" thickness of tire at center of tread. Drop Test.

TEST PIECES AND METHODS OF TESTING.

5. The standard turned test specimen, one-half inch ($\frac{1}{2}''$) diameter and two inch ($2''$) gauged length, shall be used to determine the physical properties specified in paragraph No. 3. It is shown in Fig. 1. (See p. 398.) Test Specimen for Tensile Tests.

6. When the drop test is specified, this test specimen shall be cut cold from the tested tire at the point least affected by the drop test. If the diameter of the tire is such that the whole circumference of the tire is seriously affected by the drop test, or if no drop test is required, the test specimen shall be forged from a test ingot cast when pouring the melt, the test ingot receiving, as nearly as possible, the same proportion of reduction as the ingots from which the tires are made. Location of Tensile Specimens.

7. The test tire shall be placed vertically under the drop in a running position on solid foundation of at least ten tons in weight and subjected to successive blows from a tup weighing 2,240 pounds, falling from increasing heights until the required deflection is obtained. Drop Test Described.

8. Turnings from the tensile specimen, or drillings from the small test ingot, or turnings from the tire, if preferred by the inspector, shall be used to determine whether the melt is within the limits of chemical composition specified in paragraph No. 2. Sample for Chemical Analysis.

Finish.

9. All tires shall be free from cracks, flaws, or other injurious imperfections, and shall conform to dimensions shown on drawings furnished by the purchaser.

Branding.

10. Tires shall be stamped with the maker's brand and number in such a manner that each individual tire may be identified.

Inspection.

11. The inspector representing the purchaser shall have all reasonable facilities afforded to him by the manufacturer to satisfy him that the finished material is furnished in accordance with these specifications. All tests and inspections shall be made at the place of manufacture, prior to shipment.

STEEL RAILS.

Adopted 1901.

Process of Manufacture.

1. (*a*) Steel may be made by the Bessemer or open-hearth process.

(*b*) The entire process of manufacture and testing shall be in accordance with the best standard current practice, and special care shall be taken to conform to the following instructions:

(*c*) Ingots shall be kept in a vertical position in pit heating furnaces.

(*d*) No bled ingots shall be used.

(*e*) Sufficient material shall be discarded from the top of the ingots to insure sound rails.

Chemical Properties.

2. Rails of the various weights per yard specified below shall conform to the following limits in chemical composition:

	50 to 59+ Pounds. Per Cent.	60 to 69+ Pounds. Per Cent.	70 to 79+ Pounds. Per Cent.	80 to 89+ Pounds. Per Cent.	90 to 100 Pounds. Per Cent.
Carbon.	0.35–0.45	0.38–0.48	0.40–0.50	0.43–0.53	0.45–0.55
Phosphorus shall not exceed.	0.10	0.10	0.10	0.10	0.10
Silicon shall not exceed.	0.20	0.20	0.20	0.20	0.20
Manganese.	0.70–1.00	0.70–1.00	0.75–1.05	0.80–1.10	0.80–1.10

PHYSICAL PROPERTIES.

3. One drop test shall be made on a piece of rail not more than six feet long, selected from every fifth blow of steel. The rail shall be placed head upwards on the supports, and the various sections shall be subjected to the following impact tests:

Drop Test.

Weight of Rail. Pounds per Yard.	Height of Drop. Feet.
45 to and including 55....	15
More than 55 " " 65....	16
" " 55 " " 75....	17
" " 75 " " 85....	18
" " 85 " " 100....	19

If any rail break when subjected to the drop test, two additional tests will be made of other rails from the same blow of steel, and, if either of these latter tests fail, all the rails of the blow which they represent will be rejected; but, if both of these additional test pieces meet the requirements, all the rails of the blow which they represent will be accepted. If the rails from the tested blow shall be rejected for failure to meet the requirements of the drop test, as above specified, two other rails will be subjected to the same tests, one from the blow next preceding, and one from the blow next succeeding the rejected blow. In case the first test taken from the preceding or succeeding blow shall fail, two additional tests shall be taken from the same blow of steel, the acceptance or rejection of which shall also be determined as specified above; and, if the rails of the preceding or succeeding blow shall be rejected, similar tests may be taken from the previous or following blows, as the case may be, until the entire group of five blows is tested, if necessary.

The acceptance or rejection of all the rails from any blow will depend upon the result of the tests thereof.

TEST PIECES AND METHODS OF TESTING.

4. The drop-test machine shall have a tup of two thousand (2,000) pounds weight, the striking face of which shall have a radius of not more than five inches (5″), and the test rail shall be placed head upwards on solid supports three test (3′) apart. The anvil-block shall weigh at least twenty thousand (20,000) pounds, and the supports shall be a part of, or firmly secured to, the anvil.

Drop-testing Machine.

The report of the drop test shall state the atmospheric temperature at the time the tests were made.

Sample for Chemical Analysis.

5. The manufacturer shall furnish the inspector daily with carbon determinations of each blow, and a complete chemical analysis every twenty-four hours, representing the average of the other elements contained in the steel. These analyses shall be made on drillings taken from a small test ingot.

FINISH.

Section.

6. Unless otherwise specified, the section of rail shall be the American Standard, recommended by the American Society of Civil Engineers, and shall conform, as accurately as possible, to the templet furnished by the railroad company, consistent with paragraph No. 7, relative to specified weight. A variation in height of one-sixty-fourth of an inch ($\frac{1}{64}''$) less and one-thirty-second of an inch ($\frac{1}{32}''$) greater than the specified height will be permitted. A perfect fit of the splice-bars, however, shall be maintained at all times.

Weight.

7. The weight of the rails shall be maintained as nearly as possible, after complying with paragraph No. 6, to that specified in contract. A variation of one-half of one per cent ($\frac{1}{2}\%$) for an entire order will be allowed. Rails shall be accepted and paid for according to actual weights.

Length.

8. The standard length of rails shall be thirty feet (30′). Ten pe cent (10%) of the entire order will be accepted in shorter lengths, varying by even feet down to twenty-four feet (24′). A variation of one-fourth of an inch ($\frac{1}{4}''$) in length from that specified will be allowed.

Drilling.

9. Circular holes for splice-bars shall be drilled in accordance with the specifications of the purchaser. The holes shall accurately conform to the drawing and dimensions furnished in every respect, and must be free from burrs.

Finish.

10. Rails shall be straightened while cold, smooth on head, sawed square at ends, and prior to shipment shall have the burr occasioned by the saw-cutting removed, and the ends made clean. No. 1 rails shall be free from injurious defects and flaws of all kinds.

BRANDING.

11. The name of the maker, the month and year of manufacture,

shall be rolled in raised letters on the side of the web, and the number of the blow shall be stamped on each rail.

Inspection.

12. The inspector, representing the purchaser, shall have all reasonable facilities afforded to him by the manufacturer to satisfy him that the finished material is furnished in accordance with these specifications. All tests and inspections shall be made at the place of manufacture, prior to shipment.

No. 2 Rails.

13. Rails that possess any injurious physical defects, or which for any other cause are not suitable for first quality, or No. 1 rails, shall be considered as No. 2 rails, provided, however, that rails which contain any physical defects which seriously impair their strength shall be rejected. The ends of all No. 2 rails shall be painted in order to distinguish them.

STEEL SPLICE-BARS.

Adopted 1901.

Process of Manufacture.

1. Steel for splice-bars may be made by the Bessemer or open-hearth process.

Chemical Properties.

2. Steel for splice-bars shall conform to the following limits in chemical composition:

	Per Cent.
Carbon shall not exceed	0.15
Phosphorus shall not exceed	0.10
Manganese	0.30–0.60

Physical Properties.

3. Splice-bar steel shall conform to the following physical qualities: **Tensile Tests.**

Tensile strength, pounds per square inch	54,000 to 64,000
Yield-point, pounds per square inch	32,000
Elongation, per cent in eight inches shall not be less than	25

Bending Tests.

4. (*a*) A test specimen cut from the head of the splice-bar shall bend 180° flat on itself without fracture on the outside of the bent portion.

(*b*) If preferred, the bending tests may be made on an unpunched splice-bar, which, if necessary, shall be first flattened, and shall then be bent 180° flat on itself without fracture on the outside of the bent portion.

Test Pieces and Methods of Testing.

Test Specimen for Tensile Tests.

5. A test specimen of eight inch (8″) gauged length, cut from the head of the splice-bar, shall be used to determine the physical properties specified in paragraph No. 3.

Number of Tensile Tests.

6. One tensile-test specimen shall be taken from the rolled splice-bars of each blow or melt; but in case this develops flaws, or breaks outside of the middle third of its gauged length, it may be discarded, and another test specimen substituted therefor.

Test Specimen for Bending.

7. One test specimen cut from the head of the splice-bar shall be taken from a rolled bar of each blow or melt, or, if preferred, the bending test may be made on an unpunched splice-bar which, if necessary, shall be flattened before testing. The bending test may be made by pressure or by blows.

Yield-point.

8. For the purposes of this specification the yield-point shall be determined by the careful observation of the drop of the beam or halt in the gauge of the testing machine.

Sample for Chemical Analysis.

9. In order to determine if the material conforms to the chemical limitations prescribed in paragraph No. 2 herein, analysis shall be made of drillings taken from a small test ingot.

Finish.

10. All splice-bars shall be smoothly rolled and true to templet. The bars shall be sheared accurately to length and free from fins and cracks, and shall perfectly fit the rails for which they are intended. The punching and notching shall accurately conform in every respect to the drawing and dimensions furnished. A variation in weight of more than $2\frac{1}{2}$ per cent from that specified will be sufficient cause for rejection.

BRANDING.

11. The name of the maker and the year of manufacture shall be rolled in raised letters on the side of the splice-bar.

INSPECTION.

12. The inspector, representing the purchaser, shall have all reasonable facilities afforded to him by the manufacturer, to satisfy him that the finished material is furnished in accordance with these specifications. All tests and inspections shall be made at the place of manufacture, prior to shipment.

§ 226. **Strength of Steel.**—The literature upon steel is exceedingly voluminous, and many books and articles written upon the metallurgy of steel, such as "Metallurgy of Steel," by Henry M. Howe, and "The Manufacture and Properties of Iron and Steel," by H. H. Campbell, contain a great many tests, which have, as a rule, to do with its properties and the effects of different compositions and treatments. They do not often contain, however, tests upon full-size pieces, such as columns for bridges or buildings, beams, large riveted joints, full-size parts of machinery, etc. The greater part of this latter class of tests are to be found in the reports of the various testing laboratories, such as those of the laboratories at Munich, at Berlin, and at Zurich in Europe, and the Watertown Arsenal reports and the Technology Quarterly in America; and also in various articles in the Proceedings of the various Engineering Societies in Europe and America. A number of these have already been mentioned among the references to tests of wrought-iron, and the greater part of them contain also experiments on steel.

References to such full-size tests of steel as are quoted here will be given in connection with the tests themselves.

A detailed study of the effect of the different ingredients and combinations of ingredients, upon the strength, elasticity, and ductility of steel, is a very complicated matter; it belongs to the study of Metallurgy and is beyond the scope of this work. Nevertheless, the engineer needs, of course, some general knowledge of these matters, and especially of the effect, within certain limits, of different percentages of carbon.

This subject has been dealt with by Mr. Wm. R. Webster in the Trans. Am. Inst. Mining Engineers, of October, 1892, August, 1893, and October, 1898, and in the Journal of the Iron and Steel Institute, No. 1, 1894; also by Mr. A. C. Cunningham in the Trans. Am. Soc. Civil Engineers of December, 1897; and by Mr. H. H. Campbell, in his book, "Metallurgy of Iron and Steel." Of course none of them claims anything more than approximation for their various rules and formulæ, and then only in the case of what they call normal steel, i.e., such steel as is most frequently manufactured by the mills.

Mr. Webster made an investigation of the effects of carbon, phosphorus, manganese, and sulphur upon the tensile strength of the steel. He gives a set of tables from which to determine, approximately, the tensile strength of normal steel, of a given chemical composition. His investigations were principally made upon basic Bessemer, and basic open-hearth steel.

Mr. Campbell gives a formula for the tensile strength of acid, and another for the tensile strength of basic steel, and states that they represent the facts with a good degree of accuracy. His formulæ are as follows:

For acid steel,

$$38600+121C+89P+R=\text{ultimate strength};$$

For basic steel,

$$37430+95C+8.5Mn+105P+R=\text{ultimate strength};$$

where C indicates carbon, P phosphorus, and Mn manganese, in units of 0.001 per cent, and R depends upon the finishing temperature, and may be plus or minus.

Mr Cunningham gives the following rule: To find the approximate tensile strength of structural steel; to a base of 40000 add 1000 pounds for every 0.01 per cent of carbon, and 1000 pounds for every 0.01 per cent of phosphorus, neglecting all other elements in normal steel.

In this connection a set of tests will be quoted which were made on the government testing-machine at Watertown Arsenal, upon specimens of steel containing different percentages of carbon, the tests themselves forming a portion of a series denominated in the government report as the "Temperature Series." The account of the tests to be quoted is to be found in their report for 1887.

Ten grades of open-hearth steel are here represented, in which the carbon ranges from 0.09 to 0.97 per cent, varying by tenths of a per cent as nearly as was practicable to obtain the steel.

The other elements do not follow any regular succession.

TENSILE TESTS OF STEEL BARS—TEMPERATURE SERIES.

Tests at Atmospheric Temperature.

No. of Test.	Carbon, Per Cent.	Manganese, Per Cent.	Silicon, Per Cent.	Diameter, Inches.	Sectional Area, Sq. In.	Load in 1886, Lbs. per Sq. In.	Length of Rest, Months.	Elastic Limit, Lbs. per Sq. In.	Tensile Strength per Sq. In. of Original Area. Lbs.	Elongation in 30 In. Per Cent.	Contraction of Area at Fracture, Per Cent.	Mechanical Work at Elastic Limit, in Inch-Lbs.	Mechanical Work at Tensile Strength, in Inch-Lbs.	Pounds per Sq. In. on Ruptured Section.
753	0.09	0.11		1.009	0.80	21000	3	30000	52475	23.6	63.5	15.85	9808.36	106434
754	0.20	0.45		1.009	0.80	25000	3	39500	68375	21.2	49.1	26.40	10651.90	113704
755	0.31	0.57		0.798	0.50	25000	6	46500	80600	18.0	43.5	37.27	10660.77	126640
756	0.37	0.70		0.798	0.50	25000	6	50000	85160	17.5	45.3	42.50	10935.48	134600
757	0.51	0.58	0.02	0.798	0.50	30000	6	58000	98760	14.9	41.6	58.00	11380.62	152380
758	0.57	0.93	0.07	0.798	0.50	30000	6	55000	117440	10.1	14.0	52.43	11169.34	134880
759	0.71	0.58	0.08	0.757	0.45	35000	12	57000	116000	8.8	26.2	56.53	9231.21	151510
760	0.81	0.56	0.17	0.798	0.50	40000	12	70000	149600	5.0	5.4	84.35	7872.20	158140
761	0.89	0.57	0.19	0.757	0.45	45000	12	75000	141290	4.3	4.4	95.00	6418.53	147860
762	0.97	0.80	0.28	0.757	0.45	50000	12	79000	152550	4.3	5.8	108.62	7550.23	161910

The following tables include sets of miscellaneous tests of various kinds of steel.

Bessemer Steel.

Section, Diameter.	Maximum Load. (Lbs. per sq. in.)	Elastic Limit. (Lbs. per sq. in.)	Reduction of Area. (Per cent.)	Modulus of Elasticity.
.7426	70983	40397	54.7	29139000
.7481	57760	33000	53.5	28885000
.7463	58408	28575	58.9	32799000
.7285	62761	34787	65.2	32135000
.7476	50505	19364	72.5	29479000
.7442	51230	19550	69.7	30653000
.7500	51110	21503	71.5	28457000
.7500	51518	21503	50.1	27665000
.7300	73865	46584	56.8	29600000
.7500	50294	26029	27.0	28055000
.7400	97655	54641	44.8	30539000
.7400	87086	47666	46.8	30090000
.7350	87508	50673	48.0	30057000
.7600	65235	49598	62.5	30310000
.7350	87014	50673	45.0	30058000
.7400	87356	49991	38.6	30090000
.7500	84317	48665	46.2	28868000
.7420	86720	49650	47.2	29887000
.7691	66465	35526	61.5	29149000
.7730	66077	35159	61.5	30244000
.7690	66745	35526	62.9	30075000
.7690	66445	39832	62.5	30560000
.7690	66142	35526	60.7	27864000
.7680	66530	35618	60.9	29225000
.7690	67068	39832	61.5	30075000

Open-hearth Steel.

Section, Diameter.	Maximum Load. (Lbs. per sq. in.)	Elastic Limit. (Lbs. per sq. in.)	Reduction of Area. (Per cent.)	Modulus of Elasticity.
.7600	64169	47395	56.7	29392600
.7500	63083	44137	64.0	30179000
.7600	64477	47394	60.1	30780000
.7700	62449	46171	63.5	30481000
.7700	62556	46171	64.3	29073000
.7700	62857	46171	59.5	29073000
.7600	64315	45189	64.1	29843000
.7650	63527	44600	64.6	29008000
.7600	64830	42984	61.8	28527000
.7550	65020	45790	57.9	29338000
.7600	65140	45190	64.9	31288000
.7575	65240	43270	62.3	30040000
.7575	65125	45487	59.9	29912000
.7600	64500	40780	61.9	30060000
.7550	65089	41320	61.3	29134000
.87	43300	21900	75.6	28500000
.73	44900	21500	75.7	29900000
.72	46300	22100	73.6	29900000
.60	46800	24800	73.3	30800000
.60	46700	24800	75.0	30000000

Machine-steel.

Section, Diameter.	Maximum Load. (Lbs. per sq. in.)	Elastic Limit. (Lbs. per sq. in.)	Reduction of Area. (Per cent.)	Modulus of Elasticity.
.7608	91795	62693	53.3	29316000
.7629	96256	66723	44.2	29391000
.7633	96767	65561	44.1	29586000
.7520	92087	59665	40.4	30482000
.7593	92091	62940	54.0	30848000
.7598	92191	58445	50.4	28968000
.7625	86941	62413	48.7	28340000
.7623	95750	62445	46.2	30604000
.7620	96045	62495	44.6	28802000
.7620	91220	62495	51.6	32706000
.7560	96684	63490	42.8	29400000
.7634	96567	66638	43.3	30518000
.7609	92804	60755	51.3	28884000
.7597	86678	58460	50.2	29867000
.7580	96119	54292	41.8	29818000
.7600	86741	58415	45.7	28738000
.7513	99635	62032	27.4	21813000
.7613	106980	60413	48.4	29291000
.7590	96142	54149	45.6	26643000
.7699	94513	59071	46.6	28148000
.7622	91435	64654	54.2	27164000
.7613	86775	60413	47.8	29291000
.7567	96249	61150	45.2	28776000
.7579	95303	67606	44.9	30005000
.7540	95518	54870	40.8	29416000
.7554	84990	45752	55.5	29751000

Boiler-plate.

Section.	Load. (Lbs. per sq. in.)	Elastic Limit. (Lbs. per sq. in.)	Reduction of Area. (Per cent.)	Modulus of Elasticity.
.379 x 1.458	59450	31670	47.3	29459000
.384 x 1.48	58770	39590	56.3	30270000
.365 x 1.65	58115	32380	45.6	29305000
.369 x 1.49	61657	37284	51.9	30135000
.398 x 1.511	54370	30760	58.0	28608000
.376 x 1.496	53156	32889	57.1	28511000
.4095 x 1.3647	54930	33110	....	29826000
.375 x 1.494	55173	29451	58.5	28849000
.3737 x 1.4974	54220	31270	52.5	27490000
.475 x 1.0295	47954	....	67.3	......
.4702 x 1.0064	51035	....	67.5	....
.4292 x 1.0235	55560	....	64.9	....
.4258 x 1.0123	54984	....	67.7	....
.4225 x 1.0025	60680	......	56.3	....
.4125 x 1.0025	61500	....	55.3	....
.40 x 1.02	57190	26552	58.5	25800000
.50 x 1.02	60352	28431	58.8	36199000
.49 x 1.02	63825	31010	52.0	25273000
.49 x 1.02	60024	29012	58.8	26677000
.50 x 1.02	59803	29012	50.5	30012000
.49 x 1.02	61024	29012	58.8	30012000
.50 x 1.02	60393	29412	60.2	29412000
.49 x 1.02	63625	30012	46.1	31866000
.39 x 1.28	50480	26041	62.5	32051000
.38 x 1.27	53543	29009	59.8	26168000
.41 x 1.27	58144	27846	50.1	35455000

Diameter of Cross-section. (Inches.)	Elastic Limit. (Lbs. per sq. in.)	Maximum Load. (Pounds.)	Maximum Load. (Lbs. per sq. in.)	Reduction of Area. (Per cent.)	Modulus of Elasticity. (Lbs. per sq. in.
Bessemer Steel Wire.					
.1290	....	1013	77500	64.4	30000000
.1280	66100	1021	79400	55.9	30400000
.1288	68300	1010	77500	61.4	30900000
.1291	67200	970	74100	63.5	30000000
.1283	66500	996	77000	57.1	28500000
.1283	69600	1021	79000	57.1	30000000
.1289	69700	1005	77000	60.5	31200000
.1281	71400	1043	80900	63.9	29200000
.1283	65800	1004	77700	62.6	30700000
.1286	68500	1000	77000	63.2	31000000
Bessemer Spring-steel Wire.					
.0911	75000	950	146000	34.6	24500000
.0910	69900	974	149000	51.6	25900000
.0905	79600	969	150000	42.0	23000000
.0911	72900	950	146000	37.5	24200000
.0905	....	931	143000	39.6	25400000

Tests of Steel Eye-bars.

Tests of Steel Eye-bars made on the Government Machine.—In the Tests of Metals at Watertown Arsenal for 1883 is the record of the tests of six eye-bars of steel, presented by the president of the Keystone Bridge Company.

The following is an extract from the report in regard to these eye-bars:

"The eye-bars were made of Pernot open-hearth steel, furnished by the Cambria Iron Company of Johnstown, Penn.

"The furnace charges, about 15 tons each of cast-iron, magnetic ore, spiegeleisen, and rail-ends, preheated in an auxiliary furnace, required six and one-half hours for conversion.

"All these bars were rolled from the same ingot.

"Samples were tested at the steel-works taken from a test ingot about one inch square, from which were rolled $\frac{3}{4}$-inch round specimens.

"The annealed specimen was buried in hot ashes while still red-hot, and allowed to cool with them.

"The following results were obtained by tensile tests: —

	Elastic Limit, in lbs., per Sq. In.	Ultimate Strength, in lbs., per Sq. In.	Contraction of Area.	Modulus of Elasticity.	Carbon.
			%.		%.
$\frac{3}{4}$-inch round rolled bar .	48040	73150	45.7	28210000	0.27
$\frac{3}{4}$-inch round rolled and annealed bar	42210	69470	54.2	29210000	0.27

"The billets measured 7 inches by 8 inches, and were bloomed down from 14-inch square ingot.

"They were rolled down to bar-section in grooved rolls at the Union Iron Mills, Pittsburgh.

"The reduction in the roughing-rolls was from 7 inches by 8 inches to $6\frac{1}{4}$ inches by 4 inches; and in the finishing-rolls, to $6\frac{1}{2}$ inches by 1 inch.

"The eye-bar heads were made by the Keystone Bridge Company, Pittsburgh, by upsetting and hammering, proceeding as follows: —

"The bar is heated bright red for a length of (approximately) 27 inches, and upset in a hydraulic machine; after which the bar is reheated, and drawn down to the required thickness, and given its proper form in a hammer-die.

"The bars are next annealed, which is done in a gas-furnace longer than the bars. They are placed on edge on a car in the annealing-furnace, separated one from another to allow free circulation of the heated gases. They are heated to a red heat, when the fires are drawn, and the furnace allowed to cool. Three or four days, according to conditions, are required before the bars are withdrawn.

"The pin-holes are then bored.

"The analyses of the heads before annealing were:—

"Carbon, by color	0.270	per cent.
Silicon	0.036	"
Sulphur	0.075	"
Phosphorus	0.090	"
Manganese	0.380	"
Copper	Trace.	

"The bars were tested in a horizontal position, secured at the ends, which were vertical.

"To prevent sagging of the stem, a counterweight was used at the middle of the bar.

"Before placing in the testing-machine, the stem from neck to neck was laid off into 10-inch sections, to determine the uniformity of the stretch after the bar had been fractured.

"A number of intermediate 10-inch sections were used as the gauged length, obtaining micrometer measurements of elongation, and the elastic limit for that part of the stem which was not acted upon during the formation of the heads. Elongations were also measured from centre to centre of pins, taken with an ordinary graduated steel scale.

"The moduli of elasticity were computed from elongations taken between loads of 10000 and 30000 lbs. per square inch, deducting the permanent sets.

"The behavior of bars Nos. 4582 and 4583, after having been strained beyond the elastic limit, is shown by elongations of the gauged length measured after loads of 40000 and 50000 lbs. per square inch had been applied; and with bar No. 4583, after its first fracture under 64000 lbs. per square inch, a rest of five days intervening between the time of fracture and the time of measuring the elongations.

"Considering the behavior between loads of 10000 and 30000 lbs. per square inch, we observe the elongations for the primitive readings are nearly in exact proportion to the increments of load.

"Loads were increased to 40000 lbs. per square inch, passing the elastic limit at about 37000 lbs. per square inch; the respective permanent stretch of the bars being 1.31 and 1.26 per cent.

"Elongations were then immediately redetermined, which show a reduction in the modulus of elasticity, as we advanced with each increment, of 5000 lbs. per square inch.

"Corresponding measurements after the bars had been loaded with 50000 lbs. per square inch reach the same kind of results.

"The first fracture of bar No. 4583, under 64000 lbs. per square inch, occurred at the neck, leaving sufficient length to grasp in the hydraulic jaws of the testing-machine, and continue observations on the original gauged length. This was done after the fractured bar had rested five days.

"The elongations now show the modulus of elasticity constant or nearly so, the only difference in measurements being in the last figures, up to 50000 lbs. The readings were then immediately repeated, and the same uniformity of elongations obtained.

"An illustration of the serious influence of defective metal in the heads is found in the first fracture of bar No. 4583.

"There was about 27 per cent excess of metal along the line of fracture over the section of the stem."

TABULATION OF STEEL EYE-BAR TESTS.

Number of Test.	Length, in inches.	Gauged Length, in inches.	Width, in inches.	Thickness, in inches.	Elastic Limit, in lbs., per Square Inch.	Ultimate Strength, in lbs., per Square Inch.	Elongation, per cent. In Gauged Length.	Elongation, per cent. Centre to Centre of Pins.	Contraction of Area, per cent.	Modulus of Elasticity.	Maximum Compression on Pin-Holes, in lbs., per Square Inch.	Position of Fracture.	Appearance of Fracture.
4582	221.88	160	6.48	0.98	37480	67800	15.80	15.9	33.7	29880000	86100	Broke in stem.	Silky and granular.
4583	221.60	160	6.46	0.98	36650	64000	-	-	-	30270000	79430	Broke across neck.	Granular.
4583*a*	-	160	-	-	-	71560	13.77	-	36.5	-	-	Broke in stem.	Fine, silky, and granular.
4583*b*	-	-	-	-	-	72050	-	-	38.4	-	-	" " "	Fine, silky trace of granulation.
4584	262.70	200	6.46	0.98	37600	68720	12.30	12.6	34.1	29630000	83650	" " "	Silky and granular.
4585	261.85	200	6.46	0.96	35810	65850	12.00	11.5	39.2	29960000	80000	" " "	Fine, silky, and granular.
4586	262.12	200	6.45	0.97	33230	64410	16.40	15.1	49.5	29670000	79060	" " "	Fine and silky.
4587	261.72	200	6.46	0.97	37640	68290	13.90	13.5	42.4	29960000	83960	" " "	Fine, silky, and granular.

ELONGATIONS OF No. 4582 FOR EACH INCREMENT OF 5000 LBS. PER SQUARE INCH.

Loads, in lbs., per Square Inch.	Elongations.		
	Primitive Loading.	After Load of 40000 lbs. per Square Inch.	After Load of 50000 lbs. per Square Inch.
10000	–	–	–
15000	0.0274	0.0300	0.0311
20000	0.0269	0.0305	0.0322
25000	0.0269	0.0320	0.0337
30000	0.0269	0.0330	0.0341

ELONGATIONS OF No. 4583 FOR EACH INCREMENT OF 5000 LBS. PER SQUARE INCH.

Loads, in lbs., per Square Inch.	Elongations.			Elongations after 64000 lbs. per Square Inch.	
	Primitive Loading.	After 40000 lbs. per Square Inch.	After 50000 lbs. per Square Inch.	First Reading.	Second Reading.
10000	–	–	–	–	–
15000	0.0272	0.0291	0.0302	0.0311	0.0310
20000	0.0272	0.0305	0.0315	0.0308	0.0310
25000	0.0268	0.0314	0.0325	0.0311	0.0310
30000	0.0267	0.0326	0.0340	0.0312	0.0310
35000	–	–	–	0.0311	–
40000	–	–	–	0.0312	–
45000	–	–	–	0.0310	–
50000	–	–	–	0.0315	–

In the Tests of Metals for 1886 is given the following table of tensile tests of steel eye-bars, furnished by the Chief Engineer of the Statue of Liberty.

Dimensions.			Elastic Limit per Square Inch.	Tensile Strength per Square Inch.	Elongation.		Contraction of Area.	Modulus of Elasticity per Square Inch.	Maximum Compression on Pin-holes per Square Inch.	Fracture.	
Length, Center to Center of Pin-holes.	Width.	Thickness.			In Gauged Length.	Center to Center of Pin-holes.				Location.	Appearance.
Ins.	Ins.	Ins.	Lbs.	Lbs.	%	%	%	Lbs.	Lbs.		
308.00	5.16	1.02	34610	64870	7.4	7.3	....	31400000	74173		
308.00	5.14	1.02	34730	69330	10.4	10.3	....	29279000	84093		
308.00	5.15	1.02	37330	70286	11.7	11.5	....	29017000	80043		
308.10	5.14	1.02	35000	70229	11.6	11.4	13.4	....	79826	Stem	Granular, radiating from a button of hard metal.
308.00	5.13	1.02	35950	71680	11.8	11.5	....	....	81323		
307.95	5.15	1.02	35000	70895	12.1	11.8	....	30162000	80737		

The gauged length of the bars was 260 inches. The moduli of elasticity computed between 25000 and 30000 pounds per square inch.

In connection with the work upon the bridge over the Mississippi at Memphis, Mr. Geo. S. Morison, the Chief Engineer, had 56 full-size steel eye-bars tested. The results are given in his Report, dated March, 1894, and furnish valuable information regarding the behavior of the steel, and the design, and construction of the bars. Only the following table (see page 494) will be given here, containing a portion of the results of the tests upon 31 of the bars, all made of basic open-hearth steel, and all of which broke in the body.

This table will aid the reader in comparing the tensile strength and the limit of elasticity of full-size steel eye-bars, with those obtained from the tests of small samples of the steel.

In Engineering News of Feb. 2, 1905, is an article containing a comparison of full-size and specimen tests of eleven steel eye-bars, made at the Phœnix Iron Co. Each of these bars was 15 inches wide; two of them were $1\frac{1}{4}$ inches thick; one was $1\frac{11}{16}$ inches thick, six were 2 inches thick, and two were $2\frac{1}{16}$ inches thick. The specimen tests gave tensile strengths varying from 60310 to 67000 pounds per square inch, and limits of elasticity varying from 31550 to 41760 pounds per square inch.

	Full-size Eye-bars.					Sample Bars from same Melt.				
	Width. Ins.	Thickness. Ins.	Length Center to Center. Ins.	Elastic Limit per Sq. In. Lbs.	Maximum Load per Sq. In. Lbs.	Original Area. Sq. In.	Elongation, Per Cent.	Elastic Limit per Sq. In. Lbs.	Maximum Load per Sq. In. Lbs.	Phosphorus, Per Cent.
1	10.07	1.50	160.63	35100	67490	.9500	27.5	41580	73050	.027
3	9.95	1.73	358.93	37680	70160	.9918	24.4	42650	75620	.015
4	9.98	1.75	361.23	39700	65500	.9520	28.8	40280	70280	.062
5	10.05	1.50	162.38	33140	65060	.9500	27.5	41580	73050	.027
6	6.08	1.13	291.26	29690	56700	.9756	28.1	40490	69700	.026
8	10.07	1.67	287.37	32860	65600	1.1590	20.0	43750	75000	.021
9	9.92	1.95	284.28	31110	61060	1.0140	28.8	42210	69730	.046
10	9.94	0.99	287.88	33990	63220	.9868	28.1	40230	69720	.025
11	10.05	2.20	222.88	29330	63100	.9635	28.8	38090	71300	.017
13	10.12	1.86	464.03	31970	53860	1.0201	27.0	40200	71860	.017
15	7.12	1.17	314.04	30270	51500	1.0180	28.8	33400	57170	.014
16	10.07	2.20	338.73	28080	55160	1.1220	24.2	38320	70220	.023
18	10.03	1.81	251.58	29670	62140	1.0200	26.3	40200	71080	.028
21	9.97	1.37	250.28	32700	65400	1.0670	25.0	39360	69360	.041
23	7.02	1.31	385.73	28980	52010	1.1700	31.3	34190	58460	.039
24	7.01	1.26	385.78	28410	54740	1.0170	28.1	41400	67840	.010
25	9.99	1.62	249.98	30500	58870	.9338	25.0	40910	70360	.014
27	9.96	2.05	341.28	33360	73550	.9700	25.5	40410	69900	.063
28	10.13	1.30	249.48	32520	60710	.9504	27.0	40400	70490	.023
30	9.98	1.81	284.82	28000	58720	.5557	29.5	40000	66800	.008
31	10.15	1.83	221.98	32290	62270	.9746	21.3	40530	72240	.056
34	10.04	0.99	361.68	29970	58680	1.1720	27.0	40610	70480	.060
35	7.01	1.27	258.68	28640	56830	1.0200	28.1	40790	68730	.030
42	7.98	1.20	254.63	31930	63870	1.0100	21.9	40900	69800	.024
43	8.03	2.32	338.58	32840	62400	1.0620	23.1	41710	71000	.066
44	7.00	1.18	258.68	27870	53520	1.0560	31.9	32480	58050	.027
46	9.09	1.25	206.58	32590	57410	.9734	28.7	38110	60920	.014
53	8.11	1.79	279.98	28940	58010	1.114	23.0	40480	66880	.030
55	7.00	1.00	289.23	31380	59850	1.020	28.1	40790	68730	.030

The decrease of tensile strength in the full-size eye-bars varied from 6.3 per cent to 11.9 per cent, while the decrease in elastic limit varied from 8.3 per cent to 17 per cent.

STEEL COLUMNS.

In the Trans. Am. Soc. C. E., of June, 1889, will be found a paper by Mr. J. G. Dagron, giving an account of a set of tests of eight full-size Bessemer-steel bridge columns, made for the Susquehanna River Bridge of the Baltimore and Ohio R.R. The steel varied in tensile strength from 83680 to 84440 pounds per square inch, in elastic limit from 51190 to 53890 pounds per square inch, in elongation in 8 inches from 18.75 per cent to 20.75 per cent, and in contraction of area from 30.55 per cent to 39.7 per cent. The columns were made by the Keystone Bridge Company and tested in their hydraulic press, with the columns in a horizontal position, and with the pins horizontal.

The results obtained are given by the accompanying table:

No. of Column.	Depth. Inches.	Sectional Area. Sq. Ins.	Length Center to Center Pin-holes.	Ratio of Length to Radius of Gyration.	Square of Radius of Gyration.	Ultimate Strength, in Lbs. per Sq. In.	Modulus of Elasticity. Lbs. per Sq. In.
1	8	8.24	16′ 0″	42.05	20.86	41020	27705000
2	8	8.24	16′ 0″	42.05	20.86	41650	27705000
3	8	8.24	20′ 0″	52.564	20.86	39440	26113000
4	8	8.24	20′	52.564	20.86	41050	25816000
5	8	8.24	24′ 0″	63.075	20.86	40230	29504000
6	8	8.24	24′ 0″	63.075	20.86	40070	28398000
7	9	13.23	25′ 7-½″	58.795	27.34	35570	26557000
8	9	13.23	25-7-½″	58.795	27.34	38810	29478000

The columns failed as follows:

No. 1. Failed by bending downwards at rivet in latticing, 1 foot 10½ inches from the center, buckling flange angles and web-plate.

No. 2. Failed by bending upwards at rivet in latticing at center, buckling flange angles and web-plate. One angle was fractured at point of buckling, and also at the two adjacent rivets in latticing

No. 3. Failed by bending upwards between latticing, 3 feet from center, buckling flange angles and web-plate.

No. 4. Failed by bending upwards between latticing, 4 inches from center, buckling flange angles and web-plate.

No. 5. Failed by bending upwards between latticing, 9½ inches from center, buckling flange angles and web-plate.

No. 6. Failed by bending upwards between latticing, 1 foot $5\frac{3}{4}$ inches from center, buckling flange angles and web plate.

No. 7. Failed by bending upwards at rivet in latticing, 3 inches from center, buckling flange angles and web-plate.

No. 8. Failed by bending upwards at rivet in latticing, 1 foot from center, buckling flange angles and web-plate.

In every case, after test, the rivets of each column were found by hammer test to be perfectly right.

The following table gives the results of a set of tests by direct compression, of eight connecting-rods specially made for these tests, by the Baldwin Locomotive Works, and tested in the Laboratory of Applied Mechanics of the Mass. Institute of Technology.

Rod.	Length, Center to Center of Pin-holes.	$\frac{l}{p}$	Area.		Tensile Properties of the Steel.					Breaking-strength per Sq. In. of the Rod.	
			Least Section.	Mean Section.	Elastic Limit per Sq. In.	Tensile Strength per Sq. In.	Elongation in 8 Inches.	Contraction of Area.	Modulus of Elasticity per Sq. In.	Least Section.	Mean Section.
	Ins.		Sq. In.	Sq. In.	Lbs.	Lbs.	Pr. Ct.	Pr. Ct.	Lbs.	Lbs.	Lbs.
A	89.38	100.5	7.19	7.60	37730	80280	25.8	30.9	28000000	38700	36700
B	98.38	109.4	7.19	7.78	45650	78830	20.8	34.1	28300000	40600	37500
C	107.38	118.5	6.73	7.21	43900	77840	20.4	42.5	30000000	39300	36700
D	111.75	125.0	7.27	7.78	47560	79270	22.3	43.2	28500000	36100	33700
E	116.25	130.0	7.38	7.96	45820	79250	.....	.....	30500000	39300	36400
F	120.63	134.8	7.21	7.55	49440	81660	24.1	39.9	28800000	39300	37500
G	125.13	139.7	7.06	7.67	39590	79690	24.4	45.5	30300000	38000	35000
H	134.13	149.4	7.28	7.78	39470	78650	21.0	28.3	30800000	37400	35000

TRANSVERSE STRENGTH OF STEEL.

The following table gives the results of tests of a number of steel I beams, made in the Laboratory of Applied Mechanics of the Mass. Institute of Technology.

No. of Test.	Depth. Inches.	Moment of Inertia. Ins.	Span. Feet and Inches.	Breaking Load. Lbs.	Modulus of Rupture per Sq. In. Lbs.	Modulus of Elasticity per Sq. In. Lbs.	Remarks.
290	7	38.00	14′ 6″	10500	42874	29030000	From Phœnix Co.
293	8	57.11	14′ 6″	14200	44270	29410000	" " "
295	9	81.34	14′ 6″	16700	40200	29890000	" " "
337	6	24.86	14′ 7″	8200	44900	28170000	N. J. Iron & Steel Co.
340	7	39.63	12′ 11″	12000	42100	27480000	" " " "
343	8	51.67	14′ 7″	14900	46400	29040000	" " " "
631*a*	10	134.00	14′ 0″	24200	38500	28400000	Carnegie Steel Co.
638	10	134.00	14′ 0″	25100	39500	29300000	" " "
674	10	129.00	14′ 0″	24900	41300	27450000	" " "
675	10	131.20	14′ 0″	25600	41700	27850000	" " "

In Heft IV of the Mitth. der Materialprüfungsanstalt in Zurich are given the following results of tests of the transverse strength of ten steel plate girders :

Depth of Web. (Inches.)	Span. (Inches.)	Modulus of Rupture. (Lbs. per sq. in.)	Modulus of Elasticity. (Lbs. per sq.in.)
19.76	177.17	53325	29193660
19.76	177.17	55316	27430380
15.75	141.73	55174	26662500
15.75	141.73	55316	28738620
19.69	177.17	53325	29193560
19.69	177.17	55316	27430380
23.62	212.60	57591	28795500
23.62	212.60	52472	28155600
27.56	248.03	54320	27529920
27.56	248.03	53041	28752840

COLD CRYSTALLIZATION OF IRON AND STEEL.

The question of cold crystallization of wrought-iron and steel is one that has been agitated from the earliest times, and, although Kirkaldy tried to dispose of it finally by offering evidence showing that it does not exist, nevertheless we find the same old question cropping out every little while, and although the bulk of the evidence is admitted to be against it, and, as it seems to the writer, there is no evidence in its favor, we find every now and then some one who thinks that certain observed phenomena can be explained in no other way.

The most usual phenomenon which cold crystallization is called upon to explain is the crystalline appearance of the fracture of some piece of wrought-iron or steel that has been in service for a long time, and which has, as a rule, been subjected to more or less jars or shocks. The cases most frequently cited are those of axles of some sort which have been broken, and, in the case of which, the fracture has had a crystalline appearance, and where samples cut from the other parts of the axle and tested have shown a fibrous fracture. The assumption has therefore been made that the iron was originally fibrous, and that crystallization has been caused by the shocks or the jarring to which it has been subjected in the natural service for which it was intended.

Kirkaldy showed (see his sixty-six conclusions) that when fibrous iron was broken suddenly, or when the form of the piece was such as not to offer any opportunity for the fibres to stretch, the fibres always broke off short, and the fracture was at right angles to their length, and hence followed the crystalline appearance; whereas if the breaking was gradual, and the fibres had a chance to stretch, they produced a fibrous appearance: in short, he claimed that the difference between the crystalline or the fibrous appearance of the fracture was only a difference of appearance, and not a change of internal structure from fibrous to crystalline.

The facts that Kirkaldy showed in this regard are generally acknowledged to-day, and doubtless answer by far the greater part of those who claimed cold crystallization at the time that he wrote, and also a great many of those who claim its existence to-day.

But it is easy, if suitable means be taken, to distinguish cases of crystalline appearance of fracture from cases where there are actual crystals in the piece ; and it is rather about those cases where the iron near the fracture actually contains distinct crystals that what discussion there is to-day that is worth considering takes place.

The number of such cases is, of course, small, but every once in a while some one is cited, and the claim is put forward that the iron was originally fibrous, and that these crystals must therefore have been produced without heating the iron to a temperature where chemical change is known to occur.

Inasmuch as the one who claims the existence of cold crystallization is announcing a theory which is manifestly opposed to the well-known chemical law that crystallization requires freedom of molecular motion, and hence can only take place from solution, fusion, or sublimation, it follows that the burden of proof rests with him, and before he can substantiate his theory in any single case he must prove beyond the possibility of doubt, 1°, that the iron or steel was originally fibrous, i.e., not only that fibrous iron was used in manufacturing the pieces, but also that it had not been overheated during its manufacture, and, 2°, that it has never been overheated during its period of service.

It is because the writer is not aware of any case where these two circumstances have been proved to hold that he says that he knows of no evidence for cold crystallization. In this connection it is not worth while to quote very much of the extensive literature on the subject; hence only a little of the most modern evidence will be given here.

On page 1007 *et seq.* of the report of tests on the government testing-machine at Watertown Arsenal for 1885 is given an account of a portion of a series of tests upon wrought-iron railway axles, and the following is quoted from that report: —

"This series of axle tests, begun September, 1883, is carried on for the purpose of determining whether a change in structure takes place in a metal originally ductile and fibrous to a brittle, granular, or crystalline state, resulting from exposure to such conditions as are met with in the ordinary service of a railway axle.

"Twelve axles were forged from one lot of double-rolled muck-bars, and in their manufacture were practically treated alike. Each axle was made from a pile composed of nine bars, each 6 in. wide, $\frac{3}{4}$ in. thick, and 3 ft. 3 in. long, and was finished in four heats, two heats for each end.

"The forging was done by the Boston Forge Company in their improved hammer dies, which finish the axle very nearly to its final dimensions.

"Two axles were taken for immediate test, to show the quality of the finished metal before it had performed any railway service, and serve as standards to compare with the remaining ten axles, to be tested after they had been in use.

"The axles are in use in the tender-trucks of express locomotives of the Boston and Albany Railroad. Mr. A. B. Underhill, superintendent of motive-power, contributes the axles and furnishes the record of their mileage."

The results of some measurements of deflection are given concerning one of the axles in tender 134, after it had run 95000 miles; and then follows: —

"Regarding the axle for the time being as cylindrical, 3.96

inches diameter, the modulus of elasticity by computation will be 28541000 pounds.

"Applying this modulus to the deflections observed in rear axle of the rear trucks of tender No. 150, the maximum fibre strain is found to be 9935 pounds per square inch when the tender was partially loaded, and 14900 pounds per square inch when fully loaded.

"Taken together, the tensile and compressive stresses, which are equal, amount to 19870 and 29800 pounds per square inch respectively, as the range of stresses over which the metal works.

"This definition of the limits of stresses must be regarded as approximate. There are influences which tend to increase the maximum fibre strain, such as unevenness of the track, the side thrust of the wheel-flanges against the rails. On the other hand, the inertia of the axle, particularly under high rates of speed, would exert a restraining influence on the total deflection.

"Nine tensile specimens were taken from each axle; three from each end, including the section of axle between the box and wheel bearings, and three from the middle of its length. They are marked M.B., with the number of the axle; also a sub-number and letter to indicate from what part of the axle each was taken.

"The tensile test-pieces showed fibrous metal, and generally free from granulation.

"The muck-bar had a higher elastic limit and lower tensile strength, and less elongation than the axles. The moduli of elasticity of the two are almost identical.

"Between loads of 15000 and 25000 pounds per square inch the muck-bar had a modulus of elasticity of 29400000 pounds, the axles (average of all specimens) between 5000 and 20000 pounds per square inch was 29367000 pounds. Individually the axles showed the modulus of elasticity to be substantially the same in each."

Two specimens were subjected to their maximum load and removed from the testing-machine before breaking in order to see whether the straining followed by rest will cause any change.

"It does not appear from these tests that 95000 miles run has produced any effect on the quality of the metal."

On page 1619 *et seq.* of the Report for 1886 is given an account of the tests made on some more of these axles which had run 163138 miles, and the following is quoted from that account: —

"Specimens from muck-bar axle No. 4 after the axle had run 163138 miles.

"Comparing these results with earlier tests of this series, the tensile strength of the metal in this axle is lower, and the modulus of elasticity less than shown by the preceding axles.

"The variations in strength, elasticity, and ductility are no greater, however, than those met in different specimens of new iron of nominally the same grade, and while apparently there is a deterioration in quality, it needs confirmation of a more decisive nature from the remaining axles before attributing this result to the influence of the work done in service."

Another set of tests made at Watertown Arsenal is to be found on page 1044 *et seq.* of the Report for 1885. There were tested —

1°. Two side-rods of a passenger locomotive which had been in service about twelve years.

2°. One side-rod of a passenger engine which had been run twenty-eight years and eight months.

3°. One main-rod which had been run thirty-two years and eight months in freight and five years in passenger service.

In none of these tests were there any evidences of crystallization, as the metal was in all cases fibrous when fractured.

In the report is said: —

"There are no data at command telling what the original qualities of the metal of these bars were: it is sufficient, however, to find toughness and a fibrous appearance in the iron to prove that brittleness or crystallization has not resulted from long exposure to the stresses and vibrations these bars have sustained."

The only other evidence that will be referred to is the paper of Mr. A. F. Hill upon the "Crystallization of Iron and Steel," contained in the Proceedings of the Society of Arts of the Massachusetts Institute of Technology for 1882–83. In this article Mr. Hill covers the ground very fully, and distinctly asserts that —

"The fact is that there is at present not a single well-authenticated instance of iron or steel ever having become crystallized from use under temperatures below 900° F."

He claims to have investigated a great many cases where cold crystallization has been claimed, and to have found, in every case where crystals existed, that at some period of its manufacture or working the metal was overheated. He says: —

"That the crystalline appearance of a fracture is not necessarily an indication of the presence of genuine crystals is proven by the well-known fact that a skilful blacksmith can fracture fine fibrous iron or steel in such a manner as to let it appear either fibrous and silky, or coarse and crystalline, according to his method of breaking the bar. On the other hand, where there is genuine crystallization, no skill of manipulation will avail to hide that fact in the fracture. The most striking illustrations of this that have come under my notice are the fractures of the beam-strap of the Kaaterskill, and of the connecting-rod of the chain-cable testing-machine at the Washington Navy Yard. The photographs of both fractures are submitted to you, and the similarity of their appearance is most singular. Yet what a difference in the development of the longitudinal sections by acid treatment, which are also presented to you.

"In the Kaaterskill accident the fractures of both the upper and lower arms of the strap were found to be short and square. The appearance of the fractured faces showed no trace of fibre, and was altogether granular. Yet the longitudinal section, taken immediately through the break, and developed by acid treatment, shows the presence of but few and small crystals, and the generally fibrous character of the iron used in the strap.

"In the connecting-rod of the chain-cable testing-machine we find the crystalline appearance of the fracture less, if anything, than that of the beam-strap, while the development of the longitudinal section by acid treatment reveals most beautifully, in this case, the thoroughly crystalline character of the metal. As is well known, this rod, after many years of service, finally broke under a comparatively light strain, and having all along been supposed to have been carefully made, and from well-selected scrap, its intensely crystalline structure, as revealed by the fractures, has done service for quite a number of years as *pièce de résistance* in all the 'cold-crystallization' arguments which have been served up in that time."

He then goes on to say that he cut the rod in a longitudinal direction, and treated the section with acid; that some of the crystals shown are so large as to be discernible with the naked eye; that the treated section furnished incontrovertible evidence that the rod, aside from the fact of being badly dimensioned anyhow, was made of poor material, badly heated, and insufficiently hammered, all records, suppositions, and assertions to the contrary notwithstanding; that there are a large number of crystals composed of a substance, presumably a ferro-carbide, which is not soluble in nitric acid, and is found in steel only; that the deduction from the large amount of this substance is that the pile was formed of rather poorly selected scrap, with steel scrap mixed in; that evidences of bad heating are abundant throughout; and that the strongest evidence against the presumption that these crystals were formed during the service of the rod, or while the metal was cold, is found in the groupings of the crystals during their formation, as shown in the tracing developed by the acid; that they are not of the same chemical composition, the lighter parts containing much more carbon than the darker ones; it is therefore pretty evident that with the grouping of the crystals a segregation of like chemical compounds took place, and this of course would have been impossible in the solid state. He then cites an experiment he made, in which he took a slab of best selected scrap weighing about 200 pounds and forged it down to a 3-inch by 3-inch square bar, one-half being properly forged, and the other half being exposed to a sharp flame bringing it quickly to a running heat, keeping it at this heat some time, and then hammering lightly and then treating it a second time in a similar manner; the result being, that while no difference was discernible in the appearance of the two portions, when cut and treated with acid the portion that was properly made showed itself to be a fair representative of the best quality of iron, while in the other portion the crystallization was strongly marked, the majority of the crystals being large and well developed.

He also says: —

"The fact is, all hammered iron or steel is more or less crystalline, the lesser or greater degree of crystallization depending altogether upon the greater or lesser skill employed in working the metal, and also largely upon the size of the forging. Crystallization tends to lower very sensibly the elastic limit of iron and steel, and therefore hastens the deterioration of the metal under strain. It is for this reason that large and heavy forgings ought to be, and measurably are, excluded as much as possible from permanent structures. In machine construction we cannot do without them, and must therefore accept the necessity of replacing more or less frequently the parts doing the heaviest work."

The evidence given above seems to the writer to be sufficient, and to warrant the conclusions stated on pages 475, 476.

EFFECT OF TEMPERATURE UPON THE RESISTING PROPERTIES OF IRON AND STEEL.

Much the best and most systematic work upon this subject has been done at the Watertown Arsenal, and an account of it is to be found in "Notes on the Construction of Ordnance, No. 50," published by the Ordnance Department at Washington, D. C., U.S.A.

Other references are the following: —

Sir William Fairbairn: Useful Information for Engineers.
Committee of Franklin Institute: Franklin Institute Journal.
Knutt Styffe and Christer P. Sandberg: Iron and Steel.
Kollman: Engineering, July 30, 1880.
Massachusetts R. R. Commissioners' Report of 1874.
Bauschinger: Mittheilungen, Heft 13, year 1886.

A summary of the Watertown tests, largely quoted from the above-mentioned report, will be given here, and then a few remarks will suffice for the others.

The subjects upon which experiments were made at Watertown were the effect of temperatures upon —

1°. The coefficient of expansion.
2°. The modulus of elasticity.
3°. The tensile strength.
4°. The elastic limit.
5°. The stress per square inch of ruptured section.
6°. The percentage contraction of area.
7°. The rate of flow under stress.
8°. The specific gravity.

9°. The strength when strained hot and subsequently ruptured cold.

10°. The color after cooling.

11°. Riveted joints.

1°. THE COEFFICIENTS OF EXPANSION.

These were determined from direct measurements upon the experimental bars, first measuring their lengths on sections 35 inches long, while the bars were immersed in a cold bath of ice-water, and again measuring the same sections after a period of immersion in a bath of hot oil.

The range of temperature employed was about 210 degrees Fahr., as shown by mercurial thermometers.

Observations were repeated, and again after the steel bars had been heated and quenched in water and in oil.

The average values are exhibited in the following: —

TABLE I.

First Series of Bars.

Metal.	Chemical Composition.			Coefficients of Expansion per Degree Fahr., per Unit of Length.
	C.	Mn.	Si.	
Wrought-iron.				.0000067302
Steel.	.09	.11		.0000067561
"	.20	.45		.0000066259
"	.31	.57		.0000065149
"	.37	.70		.0000066597
"	.51	.58	.02	.0000066202
"	.57	.93	.07	.0000063891
"	.71	.58	.08	.0000064716
"	.81	.56	.17	.0000062167
"	.89	.57	.19	.0000062335
"	.97	.80	.28	.0000061700
Cast- (gun) iron.				.0000059261
Drawn copper.				.0000091286

Subsequent determinations of the coefficient of expansion of a second series of steel bars gave —

TABLE II.

Chemical Composition.						Coefficients of Expansion per Degree Fahr., per Unit of Length.
C.	Mn.	Si.	S.	P.	Cu.	
.17	1.13	.023	.122	.079	.04	.0000067886
.20	.69	.037	.13	.078	.26	.0000068567
.21	1.26	.08	.14	.059	.00	.0000067623
.26	1.07	.11	.096	.08	.047	.0000067476
.26	1.26	.07	.112	.06	.038	.0000067102
.26	1.28	.07	.115	.062	.035	.0000067175
.28	1.23	.09	.168	.09	.178	.0000067794
.43	.97	.05	.08	.096	.024	.0000066124
.43	1.08	.037	.08	.114	.233	.0000066377
.53	.75	.10	.078	.087	.174	.0000064181
.55	1.02	.05	.078	.12	.15	.0000066122
.72	.70	.18	.07	.13	.23	.0000064330
.72	.76	.20	.056	.086	.186	.0000063080
.79	.86	.21	.084	.093	.096	.0000063562
1.07	.07	.13	.01	.018	.006	.0000061528
1.08	.12	.19	.011	.02	trace	.0000061702
1.12	.10	.09	.013	.018	trace	.0000060716
1.14	.10	.15	trace	.018	trace	.0000062589
1.17	.10	.10	trace	.018	o	.0000061332
1.31	.13	.19	.011	.026	trace	.0000061478

Ten bars of the first series were now heated a bright cherry-red and quenched in oil at 80° Fahr., the hot bars successively raising the temperature of the oil to about 240° Fahr., the bath being cooled between each immersion.

The behavior of the bars under rising temperature, when examined for coefficients of expansion, seemed somewhat erratic, the highest temperature reached being 235°; but this behavior was subsequently explained by the permanent changes in length found when the bars were returned to the cold bath.

Generally the bars were found permanently shortened at the close of these observations.

The bars were again heated bright cherry-red and quenched in water at 50° to 55° Fahr., the water being raised by the quenching to 110° to 125° Fahr.

After resting 72 hours, measurements were taken in the cold bath, followed by a rest of 18 hours, when they were heated and measured in the hot bath, after which they were measured in the cold bath; the maximum temperature reached with the hot bath being 233°.7 Fahr., erratic behavior occurring still.

They were next heated in an oil bath at 300° Fahr., and kept at this temperature 6 hours, then cooled in the bath; 15 hours later they were heated to 243° Fahr., and again measured hot, and then cold. These downward readings showed the quenched in water bars to have their coefficients elevated above the normal, as shown in the following table, these being the same steel bars as in Table I, and in the same order:—

TABLE III.

Coefficients of Expansion per Degree per Unit of Length.	Apparent Shortening of Bars Due to Six Hours at 300° Fahr., and the following Immersion in the Hot Bath.
.0000067641	— .0006
.0000066622	.0002
.0000066985	.0016
.0000067377	.0023
.0000069776	— .0004
.0000067041	.0082
.0000066939	.0064
.0000068790	.0054
.0000072906	.0055
.0000071578	.0048

Finally the bars were annealed by heating bright red and cooling in pine shavings, the effect of which was to approximately restore the rate of expansion to the normal, as shown by Table I for these ranges of temperature.

2°. MODULUS OF ELASTICITY.

These were obtained with the first series of bars at atmospheric temperatures, and at higher temperatures, up to 495° Fahr.

There occurred invariably a decrease in the modulus of elasticity with an increase in temperature, and, in the case of the specimens tested, the low carbon steels showed a greater reduction in the modulus than the high carbon steels, the first specimen having a modulus of elasticity at the minimum temperature 30612000, and at the maximum 27419000, while the last specimen had at the minimum temperature 29126000, and at the maximum 27778000.

3°. TENSILE STRENGTH.

The tests were made upon the first series of steel bars, wrought-irons marked *A* and *B*, a muck-bar railway axle, and cast-iron specimens from a slab of gun-iron.

The specimens were 0″.798 diameter, and 5″ length of stem, having threaded ends 1″.25 diameter.

Wrought-iron *A* was selected because it was found very hot short at a welding temperature. It had been strained with a tensile stress of 42320 pounds per square inch seven years previous to being cut up into specimens for the hot tests.

The specimens while under test were confined within a sheet-iron muffle, through the ends of which passed auxiliary bars screwed to the specimens, the auxiliary bars being secured to the testing-machine.

The heating was done by means of gas-burners arranged below the specimen and within the muffle.

The temperature of the test-bar was estimated from the expansion of the metal, observed on a specimen length of six inches, using the coefficients which were determined at lower temperatures, as hereinbefore stated, assuming there was a uniform rate of expansion.

Access to the specimen for the purpose of measuring the expansion was had through holes in the top of the muffle. The temperature was regulated by varying the number of gas-burners in use, the pressure of the gas, and also by means of diaphragms placed within the muffle for diffusing the heat.

The approximate elongations under different stresses were determined during the continuance of a test from measurements made on the hydraulic holders of the testing-machine, at a convenient distance from the hot muffle, correcting these measurements from data obtained by simultaneous micrometer readings made on the specimen and the hydraulic holders at atmospheric temperatures.

While it does not seem expedient in one series of tests to obtain complete results upon the tensile properties at high temperatures, yet, incidentally, much additional valuable information may be obtained while giving prominence to one or more features.

From these elongations the elastic limits were established where the elongations increased rapidly under equal increments of load. Proceeding with the test until the maximum stress was reached, recorded as the tensile strength, observing the elongation at the time, then, when practicable, noting the stress at the time of rupture."

For the detailed tables of tests the student is referred to the "Notes on the Construction of Ordnance."

The elastic limits and tensile strengths are computed in pounds per square inch, both on original sectional areas of the specimens and on the minimum or reduced sections, as measured at the close of the hot tests.

From the results it appears that the tensile strength of the steel bars diminishes as the temperature increases from zero Fahr., until a minimum is reached between 200° and 300° Fahr., the milder steels appearing to reach the place of minimum strength at lower temperatures than the higher carbon bars.

From the temperature of this first minimum strength the bars display greater tenacity with increase of temperature, until the maximum is reached between the temperatures of about 400° to 650° Fahr.

The higher carbon steels reach the temperature of maximum strength abruptly, and retain the highest strength over a limited range of temperature. The mild steels retain the increased tenacity over a wider range of temperature.

From the temperature of maximum strength the tenacity diminishes rapidly with the high carbon bars, somewhat less so with mild steels, until the highest temperatures are reached, covered by these experiments.

The greatest loss observed in passing from 70° Fahr. to the temperature of first minimum strength was 6.5 per cent at 295° Fahr.

The greatest gain over the strength of the metal at 70° was 25.8 per cent at 460° Fahr.

The several grades of metal approached each other in tenacity as the higher temperatures were reached. Thus steels differing in tensile strength nearly 90000 pounds per square inch at 70°, when heated to 1600° Fahr. appear to differ only about 10000 pounds per square inch.

The rate of speed of testing which may modify somewhat the results with ductile material at atmospheric temperatures has a very decided influence on the apparent tenacity at high temperatures.

A grade of metal which, at low temperatures, had little ductility, displayed the same strength whether rapidly or slowly fractured from the temperature of the testing-room up to 600° Fahr.; above this temperature the apparent strength of the rapidly fractured specimens largely exceeded the others.

At 1410° Fahr. the slowly fractured bar showed 33240 pounds per square inch tensile strength, while a bar tested in two seconds showed 63000 pounds per square inch.

Cast-iron appeared to maintain its strength with a tendency to increase until 900° Fahr. is reached, beyond which the strength diminishes. Under the higher temperatures it developed numerous cracks on the surface of the specimens preceding complete rupture.

4°. ELASTIC LIMIT.

The report says of this that it appears to diminish with increase of temperature. Owing to a period of rapid yielding without increase of stress, or even under reduced stress, the elastic limit is well defined at moderate temperatures with most of the steels.

Mild steel shows this yielding point up to the vicinity of 500°; in hard steels, if present, it appears at lower temperatures.

The gradual change in the rate of elongation at other times often leaves the definition of the elastic limit vague and doubtful, especially so at high temperatures. The exclusion of determinable sets would in most cases place the elastic limit below the values herein given.

In approaching temperatures at which the tensile properties are almost eliminated exact determinations are correspondingly difficult, the tendency being to appear to reach too high values.

5°. STRESS ON THE RUPTURED SECTION.

This, generally, follows with and resembles the curve of tensile strength.

Specimens of large contraction of area, tested at high temperature, have given evidence on the fractured ends of having separated at the centre of the bar before the outside metal parted.

Elongation under Stress.

Although the metal is capable of being worked under the hammer at high temperatures, it does not then possess sufficient strength within itself to develop much elongation, general elongation being greatest at lower temperatures.

Greater rigidity exists under certain stresses at intermediate temperatures than at either higher or lower temperatures.

Thus one of the specimens tested at 569° Fahr. showed less elongation under stresses above 50000 pounds per square inch than the bars strained at higher or lower temperatures.

Two other specimens showed a similar behavior at 315° and 387° respectively, and likewise other specimens.

In bars tested at about 200° to 400° Fahr. there are displayed alternate periods of rigidity and relaxation under increasing stresses, resembling the yielding described as occurring with some bars immediately after passing the elastic limit.

The repetition of these intervals of rigidity and relaxation is suggestive of some remarkable change taking place within the metal in this zone of temperature.

6°. PERCENTAGE CONTRACTION OF AREA.

This varies with the temperature of the bar; it is somewhat less in mild and medium hard steels at 400° to 600° than at atmospheric temperatures.

Above 500° or 600° the contraction increases with the temperature of the metal; with three exceptions, which showed diminished contraction at 1100° Fahr., until at the highest temperatures some of them were drawn down almost to points.

7°. RATE OF FLOW UNDER STRESS.

The full effect of a load superior to the elastic limit is not immediately felt in the elongation of a ductile metal, and the same is true at higher temperatures.

The flow caused by a stress not largely in excess of the elastic limit has a retarding rate of speed, and eventually ceases altogether ; whereas under a high stress the rate of flow may accelerate, and end in rupture of the metal.

Hence the apparent tensile strength may be modified within limits by the time employed in producing fracture.

8°. SPECIFIC GRAVITY.

In general, the specific gravity is materially diminished in the vicinity of the fractured ends of tensile specimens, and this diminution takes place in the different grades of steel, in bars ruptured under different conditions of temperature, stress, and contraction of area.

9° BARS STRAINED HOT, AND SUBSEQUENTLY RUPTURED COLD.

The effect of straining hot on the subsequent strength cold appears to depend upon the magnitude of the straining force and the temperature in the first instance.

There is a zone of temperature in which the effect of hot straining elevates the elastic limit above the applied stress, and above the primitive value, and if the straining force approaches the tensile strength, there is also a material elevation of that value when ruptured cold. These effects have been observed within the limits of about 335° and 740° Fahr.

After exposure to higher temperatures there occurs a gradual loss in both the elastic limit and tensile strength, and generally a noticeable increase in the contraction of area.

10°. COLOR AFTER COOLING.

This was not sensibly changed by temperatures below 200°. After 300° the metal was light straw-colored; after 400°, deep straw; from 500° to 600°, purple, bronze-colored, and blue; after 700°, dark blue and blue black.

After 800° some specimens still remained dark blue. After heating above about 800° the final color affords less satisfactory means of approximately judging of the temperature, the color remaining a blue black, and darker when a thick magnetic oxide is formed.

At about 1100° the surface oxide reaches a tangible thickness, a heavy scale of 0″.001 to 0″.002 thickness forming as higher temperatures are reached. The red oxide appears at about 1500°.

11°. IN THE TESTS OF RIVETED JOINTS

of steel boiler-plates at temperatures ranging from 70° to about 700° Fahr. the indications of the tensile tests of plain bars were corroborated.

Joints at 200° Fahr. showed less strength than when cold; at 250° and higher temperatures the strength exceeded the cold joints; and when overstrained at 400° and 500° there was found, upon completing the test cold, an increase in strength.

Rivets which sheared cold at 40000 to 41000 pounds per square inch, at 300° Fahr. sheared at 46000 pounds per square inch; and at 600° Fahr., the highest temperature at which the joints failed in this manner, the shearing-strength was 42130 pounds per square inch.

In addition to the work at Watertown which has just been detailed two other matters will be referred to here.

1°. It is well known that wrought-iron and steel are very brittle at a straw heat and a pale blue, as shown by the fact that when the attempt is made to bend a specimen at these temperatures it results in cracking it some time before a complete bending can be effected, even in the case of metal which is so ductile that it can be bent double cold, red hot, or at a flanging heat, without showing any signs of cracking.

2°. Bauschinger defines the elastic limit as the load at which the stress is no longer proportional to the strain; whereas he calls stretch-limit (Streckgrenze) the load at which the strain diagram makes a sudden change in its direction; i.e., where instead of showing a gradually increasing ratio of strain to stress it shows a sudden and rapid increase.

From his experiments (see Heft 13 of the Mittheilungen, year 1886) he draws the following conclusions: —

(*a*) That the effect of heating and subsequent cooling in lowering both the elastic and the stretch limits in mild steel begins at about 660° Fahr. when the cooling is sudden, and at about 840° Fahr. when it is slow, and for wrought-iron at about 750° with either rapid or slow cooling.

(*b*) That the operation of heating above those temperatures, and of subsequent slow or quick cooling, is that both the elastic and the stretch limit are lowered, and the more so the greater the heating; also, that this effect is greater on the elastic than on the stretch limit.

(*c*) Quick cooling after heating higher than the above-stated temperatures lowers the elastic and the stretch limit, especially the first, much more than slow cooling, dropping the elastic limit almost immediately at a heat of about 930° and certainly at a red heat to nothing or nearly nothing in wrought-iron, and in both mild and hard steel, while slow cooling cannot bring about such a great drop of the elastic limit, even from more than a red heat.

Effect of Cold-Rolling on Iron and Steel.—It has already been stated, p. 410, that it was discovered independently by Commander Beardslee and Professor Thurston, that if a load were gradually applied to a piece of iron or steel which exceeded its elastic limit, and the piece then allowed to rest, the elastic limit and the ultimate strength would thus be increased. This may be accomplished with soft iron and steel by cold-rolling or cold-drawing, but cannot be taken advantage of in hard iron or steel.

Professor Thurston, who has investigated this matter at great length, and made a large number of tests on the subject, gives the following as the results of cold-rolling: —

Increase in	Per Cent.
Tenacity	25 to 40
Transverse stress	50 to 80
Elastic limit (tension, torsion, and transverse),	80 to 125
Elastic resilience	300 to 400
Elastic resilience (transverse)	150 to 425

He also says, in regard to the modulus of elasticity, —

"Collating the results of several hundred tests, the author [Professor Thurston] found that the modulus of elasticity rose, in cold-rolling, from about 25000000 lbs. per square inch to 26000000, the tenacity from 52000 lbs. to nearly 70000, the elastic limit from 30000 lbs. to nearly 60000 lbs.; and the extension was reduced from 25 to $10\frac{1}{2}$ per cent.

"Transverse loads gave a reduction of the modulus of elasticity to the extent of about 1000000 lbs. per square inch, an increase in the modulus of rupture from 73600 to 133600, and reduction of deflection at maximum load of about 25 per cent. The resistance of the elastic limit was doubled, and occurred at a much greater deflection than with untreated iron."

On the other hand, the two steel eye-bars referred to on

p. 472 show a decrease of modulus of elasticity with increasing overstrain.

Whitworth's Compressed Steel.—Sir Joseph Whitworth produces steel of great strength by applying to the molten metal, directly after it leaves the furnace, a pressure of about 14000 lbs. per square inch; this being sufficient to reduce the length of an eight-foot column by one foot. He claims, according to D. K. Clark, to be able to obtain with certainty a strength of 40 English tons with 30 per cent ductility, and mild steel of a strength of 30 English tons with 33 or 34 per cent ductility.

The following tests were made on the Watertown machine, upon some specimens of Whitworth steel taken from a section of a jacket which was shrunk upon a wrought-iron tube, and removed from shrinkage by the application of high furnace heat:

TENSILE TESTS.

Diameter, Inches.	Tensile Strength, lbs. per Sq. In.	Elastic Limit, lbs. per Sq. In.	Contraction of Area, per cent.
0 564	103960	55000	41.9
0.564	90040	48000	47.2
0.564	104200	57000	24.6
0.564	100120	57000	44.6
0.564	93040	53000	39.2
0.564	104160	60000	24.6
0.564	93160	47000	39.2

COMPRESSIVE TESTS.

Length, Inches.	Diameter, Inches.	Compressive Strength, lbs. per Sq. In.	Elastic Limit, lbs. per Sq. In.
5	0.798	102100	61000
5	0.798	89000	57000
3.94	0.798	101600	53000
3.94	0.798	101600	54000

§ 227. **Factor of Safety.** — In order to determine the proper dimensions of any loaded piece, it becomes necessary to fix, in some way, upon the greatest allowable stress per square inch to which the piece shall be subjected.

The most common practice has been to make this some fraction of the breaking-strength of the material per square inch.

As to how great this factor should be, depends upon —

1°. The use to which the piece is to be subjected;

2°. The liability to variation in the quality of the material;

3°. The question whether we are considering, as the load upon the piece, the average load, or the greatest load that can by any possibility come upon it;

4°. The question as to whether the structure is a temporary or a permanent one;

5°. The amount of injury that would be done by breakage of the piece;

and other considerations.

The factors most commonly recommended are, 3 for a dead or quiescent load, and 6 for a live or moving load.

A common American and English practice for iron bridges is to use a factor of safety of 4 for both dead and moving load. In machinery a factor as large as 6 is desirable when there is no liability to shocks; and when there is, a larger factor should be used.

A method sometimes followed for tension and compression pieces is, to prescribe that the stretch under the given load should not exceed a certain fixed fraction of the length. This requires a knowledge of the modulus of elasticity of the material.

In the case of a piece subjected to a transverse load, it is the most common custom to determine its dimensions in accordance with the principle of providing sufficient strength; and for this purpose a certain fraction (as one-fourth) of the modulus of rupture is prescribed as the greatest allowable safe stress per square inch at the outside fibre. Thus, for wrought-iron from 10000 to 12000 lbs. per square inch is often adopted as the greatest allowable stress at the outside fibre, this being about one-fourth of the modulus of rupture.

The other method for dimensioning a beam is, to prescribe its stiffness; i.e., that it shall not deflect under its load more than a certain fraction of the span. This fraction is taken as $\frac{1}{400}$ to $\frac{1}{600}$.

This latter method depends upon the modulus of elasticity of the beam; and while it is the most advisable method to follow, and as a rule would be safer than the other method, nevertheless, in the case of very stiff and brittle material it might be dangerous; hence we ought to know also the breaking-weight and the limit of elasticity of the beam we are to use, and not allow it to approach either of these. This precaution will be especially important to observe in the case of steel beams, which are only now being introduced.

On the other hand, in moving machinery a factor of safety of six is usually required when there is no unusual exposure to shocks, as in smooth-running shafting, etc.; and when there are irregular shocks liable to come upon the piece, a greater factor is used.

WOHLER'S RESULTS.

§ 228. **Repeated Stresses.**—The extensive experiments of Wöhler for the Prussian government, which were subsequently carried on by his successor, Spangenberg, were made to determine the effect of oft-repeated stresses, and of changes of stress, upon wrought-iron and steel.

In the ordinary American and English practice, it is customary, in determining the dimensions of a piece, as of a bridge member, to ascertain the greatest load which the piece can ever be called upon to bear, and to fix the size of the piece in accordance with this greatest load.

Wöhler called attention to the fact that the load that would break a piece depends upon both the greatest and least load that it would ever be called upon to bear. Thus, a tension-rod which is subjected to alternate changes of load extending from 20000 to 80000 lbs. would require a greater area for safety than one which was subjected to loads varying only between the limits of 60000 and 80000 lbs.; and this would require more area than one which was subjected to a steady load of 80000 lbs.

Wöhler expresses this law as follows, in his "Festigkeits versuche mit Eisen und Stahl."

"The law discovered by me, whose universal application for iron and steel has been proved by these experiments, is as follows: The fracture of the material can be effected by variations of stress repeated a great number of times, of which none reaches the breaking-limit. The differences of the stresses which limit the variations of stress determine the breaking-strength. The absolute magnitude of the limiting stresses is only so far of influence as, with an increasing stress, the differences which bring about fracture grow less.

"For cases where the fibre passes from tension to compression and *vice versa*, we consider tensile strength as positive and compressive strength as negative; so that in this case the difference of the extreme fibre stresses is equal to the greatest tension plus the greatest compression."

Besides the ordinary tests of tensile, compressive, shearing, and torsional strength, he made his experiments mainly on the following two cases: —

1°. Repeated tensile strength; the load being applied and wholly removed successively, and the number of repetitions required for fracture counted.

2°. Alternate tension and compression of equal amounts successively applied, the number of repetitions required for fracture being counted.

In making these two sets of tests, he made the first set in two ways: —

(*a*) By applying direct tension.

(*b*) By applying a transverse load, and determining the greatest fibre stress.

The second set of tests was made by loading at one end a piece of shaft fixed in direction at the other, and then causing it to revolve rapidly, each fibre passing alternately from tension to an equal compression, and *vice versa*.

He also tried a few experiments where the lower limit of stress was neither zero nor equal to the upper limit, with a minus sign, also some experiments on torsion, on shearing, and on repeated torsion.

When Wöhler had made his experiments, and published his results, there were a number of attempts made by different persons to deduce formulæ which should depend upon these experiments for their constants, and which should serve to determine the breaking-strength for any given variation of stresses.

Only two of these formulæ will be given here, viz.:

1° That of Launhardt for one kind of stress,

2° That of Weyrauch for alternate tension and compression.

LAUNHARDT'S FORMULA.

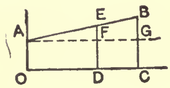
The constants used in this formula are:

1°. t, the carrying-strength (Tragfestigkeit) of the material per unit of area, which is the same as the tensile strength as determined by the ordinary tensile testing-machine.

2°. u, the primitive breaking-strength (Ursprungsfestigkeit), i.e., the greatest stress per unit of area of which the piece can bear, without breaking, an unlimited number of repetitions, the load being entirely removed between times. These two quantities have been determined experimentally by Wöhler; and it is the object of Launhardt's formula to deduce, in terms of t, u, and the ratio between the greatest and least loads to which the piece is ever subjected, the value a of the breaking-strength per unit of area when these loads are applied.

Let the greatest stress per unit area be a.
the least stress per unit area be c.

Plot the values of $\frac{c}{a}$ as abscissæ, and those of a as ordinates, making $OA=u$ (since when $\frac{c}{a}=0$, $a=u$), $OC=1$, and $CB=t$ (since when $\frac{c}{a}=1$, $a=t$). Then will any curve which passes through the points A and B have for its ordinates values of a that will satisfy the conditions that when $c=0$, $a=u$, and when $c=t$, $a=t$. By assuming for this curve, the straight line AB we obtain $DE=AO+FE=AO+(BG)\frac{OD}{OC}$, and hence



$$a=u+(t-u)\frac{c}{a}, \tag{1}$$

which is Launhardt's formula.

Moreover, if we denote by max L the greatest load on the entire piece, and by min L the least, we shall have

$$\frac{c}{a}=\frac{\min L}{\max L}.$$

Hence

$$a=u+(t-u)\frac{\min L}{\max L}, \tag{2}$$

this being in such a form as can be used. Or we may write it thus:

$$a=u\left\{1+\frac{t-u}{u}\frac{\min L}{\max L}\right\}, \tag{3}$$

this being the more common form.

The values of the constants as determined by Wöhler's experiments, and the resulting form of the formula for Phœnix axle-iron and for Krupp cast-steel, have already been given in § 172.

In the same paragraph are given the corresponding values of b, the safe working-strength, the factor of safety being three.

WEYRAUCH'S FORMULA FOR ALTERNATE TENSION AND COMPRESSION.

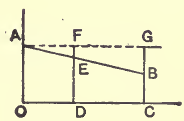
The constants used in this formula are:

1°. u, the primitive breaking-strength, which has been already defined.

2°. s, the vibration breaking-strength (Schwingungsfestigkeit) i.e., the greatest stress per unit of area, of which the piece can bear, without breaking, an unlimited number of applications, when subjected alternately to a tensile, and to a compressive stress of the same magnitude.

He lets a=greatest stress per unit of area, c=greatest stress of the opposite kind per unit of area. If a is tension, c is compression, and *vice versa*.

Plot the values of $\frac{c}{a}$ as abscissæ, and those of a as ordinates, making $OA=u$ (since when $\frac{c}{a}=0$, $a=u$), $OC=1$, and $CB=s$ (since when $\frac{c}{a}=1$, $a=s$). Then will any curve which passes through the points A and B have for its ordinates values of a that will satisfy the conditions that when $c=0$, $a=u$, and when $c=s$, $a=s$.



By assuming for this curve the straight line AB we obtain $DE=DF-FE=DF-(BG)\frac{OD}{OC}$, and hence

$$a=u-(u-s)\frac{c}{a}, \qquad (4)$$

which is the Weyrauch formula.

Moreover, if we write

$$\frac{c}{a}=\frac{\max L'}{\max L},$$

where max L=greatest load on the piece, and max L'=greatest load of opposite kind, so that, if L is tension, L' shall be compression, and *vice versa*, we shall have

$$a=u-(u-s)\frac{\max L'}{\max L}, \qquad (5)$$

this being in a form suitable to use, the more common form being

$$a=u\left\{1-\frac{u-s}{u}\frac{\max L'}{\max L}\right\}. \qquad (6)$$

The values of the constants as determined from Wöhler's experiments, and the resulting form of the formulæ for Phœnix axle-iron and for Krupp cast-steel, are given in § 176.

GENERAL REMARKS.

In each case the value of a given by the formula (3) or (6) is the breaking-strength per unit of area.

If either of these values of a be divided by 3, we have, according to Weyrauch, the safe working-strength.

WÖHLER'S EXPERIMENTAL RESULTS.

Wöhler himself made his tests upon the extremes of fibre stresses of which a piece could bear, without breaking, an unlimited number of applications. He gives, as a summary of these results, the following: —

In iron, —

Between	+16000	lbs.	per sq.	in. and	−16000	lbs.	per sq.	in.
"	+30000	"	"	"	0	"	"	
"	+44000	"	"	"	+24000	"	"	

In axle-steel, —

Between	+28000	lbs.	per sq.	in. and	−28000	lbs.	per sq.	in.
"	+48000	"	"	"	0	"	"	
"	+80000	"	"	"	+35000	"	"	

In untempered spring steel, —

Between	+50000	lbs.	per sq.	in. and	0	lbs.	per sq.	in.
"	+70000	"	"	"	+25000	"	"	
"	+80000	"	"	"	+40000	"	"	
"	+90000	"	"	"	+60000	"	"	

For shearing in axle-steel, —

Between	+22000	lbs.	per sq.	in. and	−22000	lbs.	per sq.	in.
"	+38000	"	"	"	0	"	"	

This table would justify the use, in Launhardt's and Weyrauch's formulæ, of the following values of u and s; viz., —

In iron, —

$$u = 30000 \text{ lbs. per sq. in.,}$$
$$s = 16000 \text{ lbs. per sq. in.}$$

In axle steel, —

$$u = 48000 \text{ lbs. per sq. in.,}$$
$$s = 28000 \text{ lbs. per sq. in.}$$

In untempered spring steel, —

$$u = 50000 \text{ lbs. per sq. in.}$$

And it would require, that if, with these values of u, and the values of t given in §§ 172 and 176, we put

$$c = 24000$$

in Launhardt's formula for iron, we ought to obtain approximately

$$a = 44000;$$

and if we put $c = 35000$ in that for steel, we should obtain approximately

$$a = 80000.$$

FACTOR OF SAFETY.

We have seen that Weyrauch recommends, to use with Wöhler's results, a factor of safety of three for ordinary bridge work and similar constructions.

Wöhler himself, however, in his "Festigkeits versuche mit Eisen und Stahl," says, —

1°. That we must guard against any danger of putting on the piece a load greater than it is calculated to resist, by assuming as its greatest stress the actually greatest load that can ever come upon the piece; and

2°. This being done, that the only thing to be provided for is the lack of homogeneity in the material.

3°. That any material which requires a factor of safety greater than two is unfit for use. This advice would hardly be accepted by engineers, however.

He also claims that the reason why it is safe to load car-springs so much above their limit of elasticity, and so near their breaking-load, is, that the variation of stress to which they are subjected is very inconsiderable compared with the greatest stress to which they are subjected.

GENERAL REMARKS.

It is to be observed, —

1°. The tests were all made on a good quality of iron and of steel, consequently on materials that have a good degree of homogeneity.

2°. The specimens were all small, and the repetitions of load succeeded each other very rapidly, no time being given for the material to rest between them.

3°. No observations were made on the behavior of the piece during the experiment before fracture.

4°. As long as we are dealing only with tension, we can say without error that

$$\frac{c}{a} = \frac{\min L}{\max L};$$

but as soon as both stresses or either become compression, if the piece is long compared with its diameter, we cannot assert with accuracy the above relation, nor that

$$\frac{c}{a} = \frac{\max L'}{\max L};$$

and hence results based on these assumptions must be to a certain extent erroneous.

5°. When a piece is subjected to alternate tension and compression, it must be calculated so as to bear either: thus, if sufficient area is given it to enable it to bear the tension, it may not be able to bear the compression unless the metal is so distributed as to enable it to withstand the bending that results from its action as a column.

While Wöhler's tests were mostly confined to ascertaining breaking-strengths, the later experimenters upon this subject, especially Prof. Bauschinger at Munich, Mr. Howard at the Watertown Arsenal, and Prof. Sondericker at the Mass. Institute of Technology, have all undertaken to study the elastic changes developed in the material by repeated stresses, and also, to some extent, the effect upon resistance to repeated stress, of flaws, of indentations, and of sudden changes of section, including sharp corners.

They all agree in the conclusion that flaws and indentations (even though very slight) and sharp corners, including keyways, reduce the resistance to repeated stress very considerably.

A brief account will be given of some of their principal conclusions.

BAUSCHINGER'S TESTS ON REPEATED STRESSES.

Bauschinger's tests upon repeated stress include work upon the properties of metals at or near the elastic limit. Of the properties which he enumerates, the following will be quoted here:

(*a*) The sets within the elastic limit are very small, and increase proportionally to the load, while above that point they increase much more rapidly.

(*b*) With repeated loading, inside of the elastic limit, dropping to zero between times, we find each time the same total elongations.

(*c*) While within the elastic limit the elongations remain constant as long as the load is constant; with a load above the elastic limit the final elongations under that load are only reached after a considerable length of time.

(*d*) If by subjecting a rod to changing stresses between an upper and lower limit, of which at least the upper is above the original elastic limit, the latter were either unchanged or lowered, or if, in the case of its being raised, it were to remain below the upper limit, then the repetition of such stresses must finally end in rupture, for each new application of the stress increases the strain; but if both limits of the changing stress are and remain below the elastic limit, the repetition will not cause breakage.

(*e*) Bauschinger says that by overstraining, the stretch limit is always raised up to the load with which the stretching was done; but in the time of rest following the unloading the stretch limit rises farther, so that it becomes greater than the maximum load with which the piece was stretched, and this rising continues for days, months, and years; but, on the other hand, that the elastic limit is lowered by the overstraining, often to zero; and that a subsequent rest gradually raises it until it reaches, after several days, the load applied, and in time rises above this; that, as a rule, the modulus of elasticity is also lowered under the same circumstances, and is also restored by rest, and rises after several years above its original magnitude.

(*f*) By a tensile load above the elastic limit the elastic limit for compression is lowered, and *vice versa* for a compressive load; and a comparatively small excess over the elastic limit for one kind of load may lower that for the opposite kind down to zero at once. Moreover, an elastic limit which has been lowered in this way is not materially restored by a period of rest—at any rate, of three or four days.

(*g*) With gradually increasing stresses, changing from tension to compression, and *vice versa*, the first lowering of the elastic limit occurs when the stresses exceed the original elastic limit.

(*h*) If the elastic limit for tension or compression has been lowered by an excessive load of the opposite kind, i.e., one exceeding the original elastic limit, then, by gradually increasing stresses, changing between tension and compression, it can again be raised, but only up to a limit which lies considerably below the original elastic limit.

EXPERIMENTS WITH A REPEATED TENSION MACHINE.

Bauschinger states that in 1881 he acquired a machine similar to that used by Wöhler for repeated application of a tensile stress.

The plan of the experiments which he made with it, and which are detailed in the 13th Heft of the Mittheilungen, is as follows:

From a large piece of the material there were cut at least four, and sometimes more, test-pieces for the Wöhler machine. One of them was tested in the Werder machine to determine its limit of elasticity and its tensile strength; the others were tested in the Wöhler machine, so arranged that the upper limit of the repeated stress should be, for the first specimen, near the elastic limit ; for the second, somewhat higher, etc., the lower limit being in all cases zero.

From time to time the test-pieces, after they had been subjected to some hundred thousands, or some millions, of repetitions, were taken from the Wöhler machine and had their limits of elasticity determined in the Werder machine.

The tables of the tests are to be found in the Mittheilungen, and from them Bauschinger draws the following conclusions:

1°. With repeated tensile stresses, whose lower limit was zero, and whose upper limit was near the original elastic limit, breakage did not occur with from 5 to 16 millions of repetitions.

Bauschinger says that in applying this law to practical cases we must bear in mind two things: (*a*) that it does not apply when there are flaws, as several specimens which contained flaws, many of them so small as to be hardly discoverable, broke with a much smaller number of repetitions; (*b*) another caution is that we should make sure that we know what is really the original elastic limit, as this varies very much with the previous treatment of the piece, especially the treatment it received during its manufacture, and it may be very small, or it may be very near the breaking-strength.

2°. With oft-repeated stresses, varying between zero and an upper stress, which is in the neighborhood of or above the original elastic limit, the latter is raised even above, often far above, the upper limit of stresses, and the higher the greater the number of repetitions, without, however, its being able to exceed a known limiting value.

3°. Repeated stresses between zero and an upper limit, which is below the limiting value of stress which it is possible for the elastic limit to reach, do not cause rupture; but if the upper limit lies above this limiting value, breakage must occur after a limited number of repetitions.

4°. The tensile strength is not diminished with a million repetitions, but rather increased, when the test-piece after having been subjected to repeated stresses is broken with a steady load.

5°. He discusses here the probability of the time of formation of what he considers to be a change in the structure of the metal at the place of the fracture.

Besides the above will be given the numerical values which Bauschinger obtained for carrying strength and for primitive safe strength as average values.

1°. For wrought-iron plates:

$t = 49500$ lbs. per sq. in.
$u = 28450$ " " " "

2°. For mild-steel plates (Bessemer):

$t = 62010$ lbs. per sq. in.
$u = 34140$ " " " "

3°. For bar wrought-iron, 80 mm. by 10 mm.:

$t = 57600$ lbs. per sq. in.
$u = 31290$ " " " "

4°. For bar wrought-iron, 40 mm. by 10 mm.:

$t = 57180$ lbs. per sq. in.
$u = 34140$ " " " "

5°. For Thomas-steel axle:

$t = 87050$ lbs. per sq. in.
$u = 42670$ " " " "

6°. For Thomas-steel rails:

$t = 84490$ lbs. per sq. in.
$u = 39820$ " " " "

7°. For Thomas-steel boiler-plate:

$$t = 57600 \text{ lbs. per sq. in.}$$
$$u = 34140 \text{ " " " "}$$

For Thomas-steel axle, and Thomas-steel rails, Bauschinger's obtained for the vibration breaking-strength the same values as those for primitive breaking-strength. His experiments on the other five materials, however, give lower values for s than for u. These values will not be quoted here, however, because they were obtained from experiments upon rotating bars of rectangular section transversely loaded.

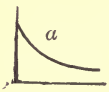
EXPERIMENTS UPON ROTATING SHAFTING SUBJECTED TO TRANSVERSE LOADS, BY PROF. SONDERICKER.

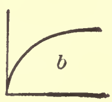
Accounts of these tests are to be found in the Technology Quarterly of April, 1892, and of March, 1899. In every case the (transverse) loads were so applied, that a certain portion, greater than ten inches in length, was subjected to a uniform bending-moment. At various times, the shaft was stopped, the load was removed, then replaced, and again removed, and measurements made of the strains and sets. The diameter of the shaft was, in every case, approximately one inch. Some extracts from the paper of March, 1899, will be given. The investigations were conducted along two lines.

1°. The determination of elastic changes, resulting from the repeated stresses, and the influence of such changes in producing fracture.

2°. The influence of form, flaws, and local conditions generally in causing fracture.

Accurate measurements of the elastic strains, and sets were made at intervals during each test. Characteristic curves of set





indicate the general character of the changes which occurred in the set, the abscissæ being the number of revolutions, and the ordinates the amount of the set, a is the characteristic curve for wrought-iron, and also occurred in one kind of soft steel. No change is produced until the elastic limit is reached, and then the change consists in a decrease of set. *b* is the characteristic curve for all the steels tested with the single exception mentioned. It is the reverse of the preceding, beginning commonly below the elastic limit, and consisting of an increase of set; rapid at first, but finally ceasing. Under heavy loads, the increase of set is very rapid, and ceases comparatively quickly. Accompanying the change of set there is a change in the elastic strain in the same direction but much smaller in amount. From the fact that these changes finally cease, we conclude that, if of sufficiently small magnitude, they do not necessarily result in fracture.

The table on page 536 gives a number of his results.

Regarding these results he says:

1°. In several cases, changes would have been detected under smaller stresses had observations been taken.

2°. Changes of set may be expected to begin at stresses varying from $\frac{1}{3}$ to $\frac{1}{2}$ of the tensile strength.

3°. The set does not appear to have a notable influence in causing fracture until it reaches 0''.001 or 0''.002 in a length of ten inches.

4°. The effect of rest is to decrease the amount of set. In most cases, however, the set lost is soon regained, when the bar is again subjected to repeated stress, especially in the case of the harder steels.

Prof. Sondericker also cites a few experiments to determine the loss of strength due to indentations, grooves, and keyways. In one case, the result of cutting a groove around the steel shaft about 0''.003 deep was a loss of strength of about 40 per cent, while similar results were obtained with indentations, and with square shoulders. He also cites the case of two pieces of steel shafting united by a coupling, where the result of cutting the necessary keyways in the shafts caused, apparently, a loss of about 50 per cent.

Mark.	Material.	Tensile Properties of the Metal. Elastic Limit per Sq. In. Lbs.	Tensile Properties of the Metal. Tensile St'gth per Sq. In. Lbs.	Stress per Sq. In. Lbs.	Revolutions at which Change was First Observed.	Revolutions.	Maximum Observed Sets. Inches.	Remarks.
D	Wt.-Iron	15700	45080	30000	42300	86400	< .01200	Broke at one end at shoulder, and at other where arm was attached.
40	"	24000	50700	24000	1500000	1500000		
				26000		2427000		Broke.
1	"	25900	51390	26000	2214000	2285000	.00026	Broke near center.
2	"	25900	51390	32000	486000	486000	.00136	Broke at mark burned by electric current.
3	"	23400	50510	24000	6593000	6593000	.00037	
				24000		4059000	.00011	
				25000		8962000	.00016	
				26000		3932000	.00022	
				27000		8155000	.00037	
				28000		589000	.00038	Broke at shoulder.
4	"	23400	50510	28000	2506000	2506000	.00042	Broke at center.
33	Steel	24800	47400	32000	85900	89750	.00771	Broke outside of arm near bearing; color blue black.
34	"	24800	47400	32000	103500	116600	.00832	Do.
21	"	30400	62590	32000	4395000	4395000	.00029	
				34000		8339000	.00032	
				36000		4627000	.00041	
				36000		1428000	.00008	After resting unloaded 18 days.
				38000		3769000	.00023	
				40000		4523000	.00054	
				42000		505000	.00072	Broke near shoulder.
54	"	42000	63130	45000	163000	163000	.00312	Broke at shoulder.
50	"	23200	73760	30000	339000	339000	.00100	
				35000		16400	.00282	Not broken.
25	"	38300	78010	40000	5031000	5031000	.00028	
				42000		2483000	.00046	Broke at shoulder.
26	"	38300	78010	40000	20838000	20838000	.00037	
				42000		3311000	.00044	Not broken.
18	"	50000	81010	36000	6463000	6982000	.00052	Broke where arm was attached.
19	"	50000	81010	36000	7252000	7686000	.00069	Broke at shoulder.
20	"	50000	81010	34000	21221000	21221000	.00028	
				36000		13577000	.00067	
				38000		2263000	.00113	
				38000		9237000	.00116	After resting 6 mos. unloaded.
				40000		932000	.00177	Broke at shoulder.
53	"	58000	96580	50000	24000	146500	.00249	Broke at shoulder.
58	"	58000	96580	45000	50100	50100	.00020	
				50000		156900	.00289	Broke near middle.
29	"	54000	104480	40000	5257000	5257000	.00046	
				42000		7125000	.00067	
				44000		4626000	.00100	
				46000		6760000	.00145	
				48000		4965000	.00196	
				50000		170000	.00197	
				50000		1000	.00203	After 24 days rest unloaded; not broken.
55	"	50000	104830	35000	276900	276900	.00060	
				40000		237900	.00274	
				50000		22530	.00615	Broke near shoulder; color dark straw.
57	"	50000	104830	60000	14300	14900	.00768	Broke near shoulder; color dark blue.

TESTS OF ROTATING SHAFTING UNDER TRANSVERSE LOAD, BY MR. HOWARD AT THE WATERTOWN ARSENAL.

A large number of tests of this character have been made at the Watertown Arsenal. A few extracts will be given from the remarks of Mr. Howard upon the subject, which may be found in the Technology Quarterly of March, 1899, as follows:

"In the Watertown tests, two principal objects have been in view, namely, to ascertain the total number of repetitions of stresses necessary to cause rupture, and to observe through what phases the physical properties of the metal pass prior to the limit of ultimate endurance. The Watertown tests have included cast-iron, wrought-iron, hot and cold rolled metal, and steels ranging in carbon from 0.1 per cent to 1.1 per cent, also milled steels. The fibre-stresses have ranged from 10000 pounds per square inch on the cast-iron bars up to 60000 pounds per square inch on the higher tensile-strength steel bars.

The speed of rotation was from 400 per minute up to 2200 per minute, in different experiments. Observations were made on the deflection of the shafts, and on the sets developed. It was early observed that intervals of rest were followed by temporary reduction in the magnitude of the sets. In the Report of Tests of Metals of 1888, he says the deflections tend to diminish under high speeds of rotation, when the loads exceed the elastic limit of the metal, and tend to cause permanent sets; but, on the other hand, when the elastic limit is not passed, the deflections are the same within the range of speeds yet experimented upon.

Efforts were inaugurated at this time to ascertain the effect of repeated alternate stresses on the tensile properties of the metal, and it appeared that such treatment tended to raise the tensile strength of the metal before rupture ensued.

Concerning the limit of indefinite endurance to repeated stress we know but very little. In most experiments rupture occurs after a few thousand repetitions, so high have been the applied stresses. Examples are not uncommon in railway practice of axles having made 200000000 rotations. In order to establish a practical limit of endurance, indefinite endurance, if we choose to call it so, our experimental stresses will need to be somewhat lowered, or new grades of metal found.

The following table which accompanied the Watertown Arsenal Exhibit at the Louisiana Purchase Exposition gives a summary of some of the repeated stress tests upon three different grades of steel:

STEEL BARS.

Tensile Tests and Repeated Stress Tests on Different Carbon Steels.

Description.	Tensile Tests.					Repeated Stress Tests.		
	Elastic Limit per Sq. In. Lbs.	Tensile St'gth per Sq. In. Lbs.	Elongation in 4 Ins. Per ct.	Contraction of Area Per ct.	Mechanical Work at Rupture per Cu.In. Ft.-lbs.	Maximum Fiber Stress per Sq. In. Lbs.	Number of Rotations at Rupture.	Mechanical Work at Rupture per Cu. In. Ft.-lbs.
0.17 Carbon steel.	51000	68000	33.5	51.9	982	60000	6470	32835
						50000	17790	62635
						45000	70400	201960
						40000	293500	665290
						35000	5757920	9992390
						30000	*23600000	*29500000
0.55 Carbon steel.	57000	106100	16.2	18.7	1.047	60000	12490	63387
						50000	93160	328000
						45000	166240	476900
						40000	455350	1032130
						35000	900720	1563125
						30000	*19870000	*24838000
0.82 Carbon steel.	63000	142250	8.5	6.5	888	60000	37250	189044
						55000	93790	399780
						50000	213150	750465
						45000	605460	1736910
						40000	*17560000	*40973000
						35000	*19220000	*33635000

* Not ruptured.

GENERAL REMARKS.

That the amount of detailed information regarding repeated stresses is small compared with what is needed will be evident when we consider the number of cases in which metal is subjected to such stresses in practice, among which are shafting, connecting-rods, parallel rods, propeller-shafts, crank-shafts, railway axles, rails, riveted and other bridge members, etc. In the case of some of them, notably, railway axles, attempts have been made to base specifications for the material upon such tests as have become available upon repeated stresses.

§ 229. **Shearing-strength of Iron and Steel.**—Some of the most common cases where the shearing resistance of iron and steel is brought into play are:

1°. In the case of a torsional stress, as in shafting.

2°. In the case of pins, as in bridge-pins, crank-pins, etc.

3°. In the case of riveted joints.

The so-called apparent outside fibre-stress at fracture, as determined from experiments on torsional strength, is found to be not far from the tensile strength of the metal, and is, of course, greater than the shearing-strength, for the same reasons as render the modulus of rupture greater than the actual outside fibre-stress at fracture in transverse tests.

Moreover, the shearing strength of wrought-iron rivets is shown by experiment to be about $\frac{4}{5}$ the tensile strength of the rivet metal.

In regard to cast-iron, Bindon Stoney found the shearing and tensile strength about equal.

The cases where shearing comes in play in wrought-iron and steel will therefore be treated separately.

§ 230. **Torsional Strength of Wrought-iron and Steel.**—The method formerly followed, and in use by some at the present day, was to compute the strength of a shaft from the twisting-moment only, neglecting the bending, but varying the working-strength per square inch to be used according to the character of the service. It is generally the fact, however, that when shafting is running the pulls of the belts create a bending backwards and forwards, bringing the same fibre alternately into tension and compression; and this is combined with the shearing-stresses developed due to the twisting-moment alone. At the two extremes of these general cases are:

1°. The case when the portion of a shaft between two hangers has no pulleys upon it, and when the pulls on the neighboring spans are not so great as to deflect this span appreciably. That is a case of pure torsion: and if the shaft is running smoothly, with no jars or shocks, and no liability to have a greater load thrown upon it temporarily, we may compute it by the usual torsion formula, given in § 212; using for breaking-strength of wrought-iron and steel the so-called apparent outside fibre-stress at fracture as determined from torsional tests, and a factor of safety six, and such a proceeding will probably give us a reasonable degree of safety.

2°. The case when, pulleys being placed otherwise than near the hangers, the belt-pulls are so great that the torsion becomes insignificant compared with the bending, and then it would be proper to compute our shaft so as not to deflect more than $\frac{1}{1200}$ of its span under the load, or better, not more than $\frac{1}{1600}$: of course we should compute also the breaking transverse load, and see that we have a good margin of safety.

In other cases, the methods pursued, the first two of which are incorrect, have been

1°. By using the ordinary torsion formula combined with a large factor of safety.

2°. By computing the shaft also for deflection, and providing that its deflection shall not exceed $\frac{1}{1200}$ or $\frac{1}{1600}$ of its span.

This, however, neglects the torsion, and also the rapid change of stress upon each fibre from tension to compression.

3°. By using the formula of Grashof or of Rankine for combined bending and twisting, with the constants that have been derived from experiments on simple tension or simple torsion.

The results given on pages 544 and 545 are from pieces of shafting of considerable length. As has been stated, the so-called "apparent outside fibre-stress at fracture" appears to be not very far from the tensile strength of the material, and the torsional modulus of elasticity appears to be from three-eighths to two-fifths of the tensile modulus of elasticity.

Under certain circumstances the bending may have the greatest influence, while the twisting may be predominant in others, or their influence may be equally divided. Which of these is the case will depend upon the location of the hangers and of the pulleys, the width of the belts, etc., etc.

As to the formulæ which take into account both twisting and bending, there are two, both of which are based upon the theory of elasticity. The first, which is the most correct from a theoretical point of view, is that given by Grashof and other writers on the theory of elasticity, and is

$$f=\frac{r}{I}\left\{\frac{m-1}{2m}M_1+\frac{m+1}{2m}\sqrt{M_1^2+M_2^2}\right\},$$

where M_1 = greatest bending-moment;
M_2 = greatest twisting-moment;
r = external radius of shaft;
I = moment of inertia of section about a diameter;
f = greatest allowable stress at outside fibre;
m = a constant depending on the nature of the material.

In the case of iron or steel the value of m is often taken as 4, though it is, in most cases, nearer 3. When $m=4$ we have

$$f=\frac{r}{I}\left\{\tfrac{3}{8}M_1+\tfrac{5}{8}\sqrt{M_1^2+M_2^2}\right\}.$$

The other formula, which is also based upon the theory of elasticity, but which is not as correct, is that given by Rankine, and is

$$f=\frac{r}{2I}\left\{M_1+\sqrt{M_1^2+M_2^2}\right\}.$$

With a view to determine the behavior of shafting under a combination of twisting and bending, suitable machinery was erected in the engineering laboratories of the Mass. Institute of Technology, and a number of tests were made.

The principal points of the method of procedure are the following, viz.:

1st. The shaft under test is in motion, and is actually driving an amount of power which is weighed on a Prony brake.

2d. A tr nsverse load is applied which may be varied at the option of the experimenter, and which is weighed on a platform scale.

3d. The proportion between the torsional and transverse loads may be adjusted to correspond with the proportion between the power transmitted and the belt-pull sustained by a shaft in actual use.

4th. Tests are made not only of breaking-strength, but also angle of twist and deflection under moderate loads are measured.

The following table will give the results of the tests on iron shafts, and they will then be discussed :

No. of Test.	Time of running, minutes.	Total revolutions.	H. P. transmitted.	M_1, max. bending moment. In.-lbs.	M_2, max. twisting moment. In.-lbs.	f_1, max. bend. fibre stress.	f_2, max. twist. fibre stress.	f, Grashof.	f, Rankine.	Diam. ins.
8	37.5	7040	11.717	11514.1	3926.4	60024	10234	62162	61755	1″.25
9	200	38839	8.181	10507.8	2656.8	54777	6925	55876	55671	1″.25
10	162	31641	5.291	9891.0	1714.6	51562	4469	52062	51976	1″.25
11	553	108002	4.331	9241.7	1399.2	48179	3647	48539	48769	1″.25
12	408	80694	6.276	9241.7	2027.6	48179	5287	48911	48769	1″.25
13	98	19333	6.342	8917.1	2028.2	46485	5287	47245	47105	1″.25
14	423	82741	6.283	8917.1	2029.7	46485	5290	47246	47106	1″.25
15	565	108739	6.192	8592.5	2031.6	44793	5295	45582	45436	1″.25
16	443	88208	6.338	8267.8	2026.8	43100	5283	43914	43713	1″.25
17	951	185233	6.283	3781.5	2029.7	38503	10333	41768	41117	1″
19	......	....	14.834	8218	4744	84185	24152	91928	90368	1″
20	......	....	7.562	7976	2394	82112	12188	84819	83031	1″
21	......	....	9.972	8917	3232	90793	16454	94434	93716	1″
22	......	....	15.159	8917	4848	90793	24681	98612	97103	1″
23	......	....	2.955	7652	945	77913	4811	78314	78239	1″

In 19 to 23 inclusive the number of revolutions was small and the outside fibre stress at fracture was correspondingly large.

Two specimens of the 1″.25 shafting and two of the 1″ were tested for tension, the results being as follows:

		Breaking-strength, per sq. in.
1″.25 diameter	No. 1	46800
	No. 2	49865
	Average	48333
1″ diameter	No. 1	58687
	No. 2	61812
	Average	60250

As to conclusions:

1st. It is plain from these results that a shaft whose size is determined by means of the results of a quick test would be too weak, and that our constants should be obtained from tests which last for a considerable length of time.

2d. A perusal of the tables will show that the results obtained apply more to the bending than to the twisting of a shaft, as the transverse load used in these tests was so large compared with the twist as to exert the controlling influence. This will be plain by a comparison of the values of f_1, f_2, and f.

3d. Nevertheless, the bending-moments actually used were generally less than such as might easily be realized in practice with the twisting-moments used.

4th. It seems fair to conclude that, in the greater part of cases where shafting is used to transmit power, as in line-shafting or in most cases of head shafting, the breaking is even more liable to occur from bending back and forth than from twisting, and hence that in no such case ought we to omit to make a computation for the bending of the shaft as well as the twist.

5th. As to the precise value of the greatest allowable outside fibre stress to be used in the Grashof formula, it is plain that it is not correct to use a value as great as the tensile strength of the iron, and while the tests show that this figure should not for common iron exceed 40000 lbs. per square inch, it is probable that tests where a longer time is allowed for fracture will show a smaller result yet.

TORSIONAL TESTS OF WROUGHT-IRON.

Norway Iron.					
Diameter. (Inches.)	Distance between Grips. (Inches.)	Maximum Twisting Moment. (Inch Lbs.)	Number of Turns between Grips at Fracture.	Apparent Outside Fibre Stress. (Lbs. per sq. in.)	Shearing Modulus of Elasticity. (Lbs. per sq. in.)
2.00	70.40	72360	16.50	46065	11406000
2.02	72.00	74970	16.00	46600	13215000
2.03	71.30	72000	14.00	43757	12902000
2.02	70.40	74520	17.00	46321	12247000
2.02	69.80	72000	14.25	45837	12738000
2.03	70.30	74880	15.50	45692	11361000
2.03	71.13	74880	20.00	44658	11957000
2.03	70.20	79560	16.00	48437	11554000
2.03	84	74880	15.5	45590	11900000
1.54	54	35100	12.0	48950	9840000
1.52	49	34200	11.25	49600	11410000
1.53	53	33840	8.50	48120	11600000
1.53	49	33840	11.10	48110	... .
1.53	49	34920	14.56	49650	11840000
1.52	53	34200	8.98	49600	12480000
2.25	70	111960	5.80	50060	11830000
2.27	75	106920	12.30	46600	10900000
2.25	70	108360	10.90	48500	11800000
2.25	76	109800	9.90	49100	11700000
2.23	70	113670	11.00	52200	12000000
2.26	70	107640	10.00	47500	11400000
Refined Iron.					
2.03	72.10	84240	4.00	51285	12471000
2.03	71.20	66960	1.25	40646	12576000
2.03	70.50	70200	3.50	41743	11372000
2.03	72.10	72000	2.30	43834	10960000
2.03	71.80	61920	2.80	36820	11393000
2.03	71.30	68760	2.50	40887	11360000
2.03	69.30	78120	2.80	46453	12871000
1.30	71.75	22320	14.60	52127	11436000
1.50	71.25	36360	10.30	54867	11482000
1.33	71.75	45360	6.70	50852	12359000
1.50	79.50	32760	14.60	49435	11071000
2.27	62	123800	4.71	53920	12720000
2.26	63	122950	5.60	54250	12230000
2.29	64	124600	5.24	52840	12510000
2.25	60	116640	3.70	52150	12190000
2.25	61	121590	4.40	54370	12510000
2.27	66	123750	5.00	53890	12840000
1.77	71	38970	2.10	35790	11200000
1.75	61.1	56150	7.1	53400	11200000
1.75	64.0	55350	8.7	52600	11200000
1.72	64.5	45090	5.4	44900	11800000
1.75	61.0	53360	6.1	50700	11500000
[illegible]	63.0	55710	1.49	52900	11300000

Burden's Best.					
Diameter of Cross-section. (Inches.)	Distance between Grips. (Inches.)	Maximum Twisting Moment. (Inch Lbs.)	Number of Turns between Grips at Fracture.	Apparent Outside Fibre Stress. (Lbs. per sq. in.)	Shearing Modulus of Elasticity. (Lbs. per sq. in.)
2.01	63.8	85050	9.50	53300	11300000
2.01	59.0	86400	8.62	54200	11500000
2.01	53.0	84510	6.87	53000	11200000
2.00	58.8	87480	8.40	55700	11600000
2.00	65.5	85410	8.52	54400	11600000
2.01	60.2	85590	8.82	53700	11200000
2.02	58.5	85140	8.05	52600	11300000
2.00	57.0	82650	7.31	52600	11500000
2.02	57.8	86580	8.54	53500	11200000
2.02	59.5	86040	8.61	53200	11200000
2.02	60.0	87840	8.93	54300	11300000
2.02	60.0	88200	8.48	54400	11200000
2.01	53.3	87480	7.85	54900	11300000
2.01	59.5	83970	8.01	52700	11400000
2.01	59.5	84780	8.32	53200	11200000
2.03	61.0	83520	8.98	50900	11100000
2.00	63.0	84050	9.24	53500	11700000
2.02	60.3	85950	7.94	53100	11200000
2.01	60.0	84600	8.62	53100	11400000
2.02	61.0	83520	8.50	51600	11000000
2.00	60.5	86040	8.80	54000	11600000
2.01	51.0	85680	7.35	53700	11200000
2.01	36.8	87480	8.66	54900	11500000
2.01	47.8	85860	7.24	53900	11300000
2.01	59.4	85050	8.92	53300	11500000
2.01	60.9	86400	8.75	54200	11300000
2.00	59.5	86400	9.08	55000	11700000
2.01	58.5	85650	8.30	53700	11500000
2.00	58.5	84870	7.87	54200	11800000
2.01	58.3	87300	9.45	54800	11500000
2.01	58.4	86490	8.73	54200	11900000
2.01	58.4	87120	8.09	54600	11600000
2.00	59.8	84870	8.80	54000	11700000

The above tables show the results of tests made in the engineering laboratories of the Massachusetts Institute of Technology upon the torsional strength of various kinds of wrought-iron. The figures in the column headed "Apparent outside fibre-stress" are obtained from the formula $f = \frac{Mr}{I}$, where M = maximum twisting-moment, r = outside radius of shaft, and I = polar moment of inertia of section. Of course it is not the outside fibre-stress.

TORSIONAL TESTS OF BESSEMER STEEL.

Diameter, Inches.	Distance between Grips, Inches.	Gauged Lengths, Inches.	Elastic Limit, In.-lbs.	Outside Fibre-stress at Elastic Limit, Lbs. per Sq. In.	Maximum Twisting Moment, In.-lbs.	Apparent Outside Fibre-stress Calculated from Maximum Twisting-Moment, Lbs. per Sq. In.	Shearing Modulus of Elasticity, Lbs. per Sq. In.	Number of Turns between Grips of Fracture.
1.75	60.00	......	......	......	66960	63632	12418000	11.88
1.39	59.75	......	......	......	39600	75031	11243000	15.40
1.39	59.50	......	......	......	37620	71250	12594000	15.00
2.00	56.00	30	......	......	101520	64630	11820000	7.85
2.00	55.00	30	......	......	100260	63830	10320000	7.87
2.02	56.00	36	......	......	111960	70630	11990000	10.94
1.85	44.00	24	......	......	81240	65350	10250000	3.87
2.01	55.00	36	......	......	112590	70610	13410000	5.76
2.02	190.00	144	......	......	134100	82860	11830000	6.77
1.50	96.00	75	20160	30400	56000	84500	12200000	16.30
1.50	94.00	75	19800	29900	53280	80400	12200000	1.570
1.52	93.00	75	21600	31300	53280	77300	10700000	16.10
1.53	94.00	75	21600	30800	52560	74700	10900000	15.80
1.49	60.00	40	......	......	43920	66280	11800000	8.60
1.50	58.00	40	14400	21700	44640	67400	11700000	13.30
1.50	57.50	40	17000	25600	44820	67600	11900000	11.50
1.50	57.60	40	17000	25500	45810	69100	11900000	9.90
1.50	59.00	40	16000	24100	45450	68600	11600000	10.80
1.50	58.80	40	18000	27200	44460	67100	11700000	10.50
1.50	59.10	40	16200	24400	45000	67900	11500000	13.20
1.50	58.20	40	18000	27200	44920	67800	11700000	10.80
1.50	58.00	40	18000	27200	45540	68700	11700000	11.40

§ 232. **Riveted Joints.** — The most common way of uniting plates of wrought-iron or steel is by means of rivets. It is, therefore, a matter of importance to know the strength of such joints, and also the proportions which will render their efficiencies greatest; i.e., that will bring their strength as near as possible to the strength of the solid plate.

In § 177 was explained the mode of proportioning riveted joints usually taught, based upon the principle of making all the resistances to giving way equal, and assuming, as the modes of giving way, those there enumerated. This theory does not, however, represent the facts of the case, as —

1°. The stresses which resist the giving-way are of a more complex nature than those there assumed, so that the efficiency of a joint constructed in the way described above may not be as great as that of one differently constructed;

2°. The effects of punching, drilling, and riveting, come in to modify further the action; and

3°. The purposes for which the joint is to be used, often fix some of the dimensions within narrow limits beforehand.

In order to know, therefore, the efficiency of any one kind of joint, we must have recourse to experiment. And here again we must not expect to draw correct conclusions from experiments made upon narrow strips of plate riveted together with one or two rivets; but we need experiments upon joints in wide plates containing a sufficiently long line of rivets to bring into play all the forces that we have in the actual joint. The greater part of the experiments thus far made have been made upon narrow strips, with but few rivets. The number of tests of the other class is not large, and of those that have been made, the greater part merely furnish us information as to the behavior of the particular form of joint tested, and do not teach us how to proportion the best or strongest joint in any given plates, as no complete and systematic series of tests has thus far been carried out, though such a series has been begun on the government testing-machine at the Watertown Arsenal.

The only tests to which it seems to the writer worth while to make reference here are:

1°. A portion of those made by a committee of the British Institution of Mechanical Engineers, inasmuch as, although a very large part were made upon narrow strips with but few rivets, nevertheless a portion were made upon wide strips.

2°. The tests on riveted joints that have been made on the government testing-machine at Watertown Arsenal.

1°. The account of this series is to be found at intervals from 1880 to 1885 inclusive, with one supplementary set in 1888, in the proceedings of the British Institution of Mechanical Engineers; but as all except the supplementary set has also been published in London *Engineering*, these latter references will be given here as follows:

Engineering for 1880, vol. 29, pages 110, 128, 148, 254, 300, 350.
" " 1881, vol. 31, " 427, 436, 458, 508, 588.
" " 1885, vol. 39, " 524.
" " 1885, vol. 40, " 19, 43.
Also, Proc. Brit. Inst. Mechl. Engrs., Oct. 1888.

2°. The second series, referred to above, or those made on the government testing-machine at Watertown Arsenal, are to be found in their reports of the following years, viz., 1882, 1883, 1885, 1886, 1887, and 1895.

3°. Report of tests of structural material made at the Watertown Arsenal, Mass., June, 1891.

While it is from tests upon long joints that we can derive correct and reliable information to use in practice, and hence while the experiments already made give us a considerable amount of information, nevertheless as the tests have not yet been carried far enough to furnish all the information we need, and to settle cases that we are liable to be called upon to decide, therefore, before quoting the above experiments, a few of the rules and proportions more or less used at the present time, and the modes of determining them, will be first explained.

In this regard we must observe that practical considerations render it necessary to make the proportions different when the joint is in the shell of a steam-boiler, from the case when it is in a girder or other part of a structure.

In the case of boiler-work, the joint must be steam-tight, and hence the pitch of the rivets must be small enough to render it so: whereas in girder-work this requirement does not exist; and hence the pitch can, as far as this requirement goes, be made greater.

It is probable, that, with good workmanship, we might be able to secure a steam-tight joint with considerably greater pitches than those commonly used in boiler-work; and now and then some boiler-maker is bold enough to attempt it.

Some years ago punching was the most common practice; but now drilling has displaced punching to such an extent that all the better class of boiler-work is now drilled, and drilling is also used to a very considerable extent in girder-work. When drilling is used, the plates, etc., to be united should be clamped together and the holes drilled through them all together. In this regard it should be said:

1°. When the holes are drilled, and hence no injury is done to the metal between the rivet-holes, this portion of the plate comes to have the properties of a grooved specimen, and hence has a greater tensile strength per square inch than a straight specimen of the same plate, as the metal around the holes has not a chance to stretch. This excess tenacity may amount to as much as 25 per cent in some cases, though it is usually nearer 10 or 12 per cent, depending not only on the nature of the material, but also on the proportions.

2°. When the holes are punched, we have, again, a grooved specimen, but the punching injures the metal around the hole, and this injury is greater the less the ductility of the metal: thus, much less injury is done by the punch to soft-steel plates than to wrought-iron ones, and less to thin than to thick plates. This injury may reach as much as 35 per cent, or it may be very small. Besides this, in punching there is liability of cracking the plate, and of not having the holes in the two plates that are to be united come exactly opposite each other. A number of tests on the tenacity of punched and drilled plates of wrought-iron, and of mild steel, made on the government testing-machine at Watertown Arsenal, are given on page 564 *et seq.*

The hardening of the metal by punching also decreases the ductility of the piece.

The injury done by punching may be almost entirely removed in either of the following ways:—

1°. By annealing the plate.

2°. By reaming out the injured portion of the metal around the hole; i.e., by punching the hole a little smaller than is desired, and then reaming it out to the required size.

There is a certain friction developed by the contraction of the rivets in cooling, tending to resist the giving way of the joint; and some have advocated the determination of the safe load upon a riveted joint on the basis of the friction developed, instead of on the basis of strength—notably M. Dupuy in the Annales des Ponts et Chausées for January, 1895; but this seems to the author an erroneous and unsafe method of proceeding: 1°, because tests show that slipping occurs at all loads, beginning at loads much smaller than the safe loads on the joint; 2°, because all friction disappears before the breaking load is reached.

Hence it is safer to disregard friction in designing a tensile riveted joint.

The shearing-strength of the rivets would appear to be about two thirds the tensile strength of the rivet metal.

Before proceeding to give an account of Kennedy's tests, and of those made at the Watertown Arsenal, which form the principal basis for determining the constants, i.e., the tearing-strength of the plate, the shearing-strength of the rivet iron, and the ultimate compression on the bearing surface, it will be best to outline the proper method of designing a riveted joint, and for this purpose a discussion of a few cases of tensile riveted joints, as given by Prof. Peter Schwamb, will be given by way of illustration.

The letters used will be as follows, viz.:

d = diameter of driven rivet in inches;
t = thickness of plate in inches;
t_1 = thickness of one cover-plate in inches;
f_s = shearing-strength of rivet per square inch;
f_t = tearing-strength of plate per square inch;
f_c = crushing-strength of rivet or plate per square inch;
p = pitch of rivets in inches;
p_d = diagonal pitch in inches;
l = lap in inches.

In every case of a tension-joint we begin by selecting a repeating section and noting all the ways in which it may fail. It would seem natural, then, to determine the diameter of the rivet to be used by equating the resistance to shearing and the resistance to crushing, and in some cases it is desirable to adopt the resulting diameter of rivet; but there are also many cases where there is good reason for adopting either a larger or a smaller rivet, and others where there is good reason for determining the trial diameter in some other way.

Thus we may find that the rivet which presents equal resistance to shearing and crushing may be too large to be successfully worked, or it may require a pitch too large for the purposes for which the joint is to be used; or, on the other hand, it may be so small that it would lead to a pitch too small to be practicable; or it might, in a complicated joint, where there are a good many ways of possible failing, lead to a low efficiency. In all cases, a commercial diameter must be selected.

Single-riveted Lap-joint.—Repeating section containing one rivet may fail by—

1°, shearing one rivet. Resistance $= f_s \dfrac{\pi d^2}{4}$.

2°, tearing the plate. Resistance $= f_t(p-d)t$.

3°, compression. Resistance $= f_c td$.

Equating 1° and 3° gives $d = \dfrac{4t}{\pi} \dfrac{f_c}{f_s}$ (1)

A larger rivet will crush, a smaller one will shear.

The diameter given by (1) will frequently be found to be larger than can be successfully worked.

Equating 2° and 3° gives $p = d\left(1 + \dfrac{f_c}{f_t}\right)$. (2)

Equating 1° and 2° gives $p = d\left(1 + \dfrac{\pi d}{4t} \dfrac{f_s}{f_t}\right)$. (3)

If the value of d given in (1) is used, then (2) and (3) give the same result. If, however, a different value of d is used, then the pitch should be determined by (2) for a larger and by (3) for a smaller rivet.

It may be well to note that whenever compression fixes the pitch, the *computed* efficiency

$$\frac{p-d}{p} = \frac{f_c}{f_t + f_c}$$

is independent of the diameter of the rivet, and that this is the maximum efficiency obtainable with this style of joint.

SINGLE-RIVETED DOUBLE-SHEAR BUTT-JOINT.

The combined thickness of the two cover-plates should always be greater than t, and, this being the case, we proceed as follows :

Repeating section containing one rivet may fail by—

1°, shearing one rivet in two places. Resistance $= f_s \dfrac{\pi d^2}{2}$.

2°, tearing the plate. Resistance $= f_t(p - d)t$.

3°, compression Resistance $= f_c td$.

Equating 1° and 3° gives $d = \dfrac{2t}{\pi} \dfrac{f_c}{f_s}$ (4)

A larger rivet will crush, a smaller one will shear.

The diameter given by (4) is just one half that given by (1), and will frequently be found to lead to a pitch too small to use in practice. In such cases we should use a larger rivet.

Equating 2° and 3° gives $p = d\left(1 + \dfrac{f_c}{f_t}\right)$. (5)

Equating 1° and 2° gives $p = d\left(1 + \dfrac{\pi d}{2t} \dfrac{f_s}{f_t}\right)$. (6)

If the value of d given by (4) be used, then (5) and (6) give the same result. If, however, a different value of d be used, then the pitch should be determined by (5) for a larger and by (6) for a smaller rivet.

For the diagonal pitch, in the case of staggered riveting, we should have, at least, according to Kennedy's sixth conclusion (see page 566) $2(p_d - d) = \frac{4}{3}(p - d)$ and hence $p_d = \frac{2}{3}p + \frac{1}{3}d$.

DOUBLE-RIVETED LAP-JOINTS.

Repeating section containing two rivets may fail by—

1°, shearing two rivet sections. Resistance $= f_s \dfrac{\pi d^2}{2}$.

2°, tearing plate straight across. Resistance $= f_t(p - d)t$.

3°, compression on two rivets. Resistance $= f_c(2td)$.

Equating 1° and 3° gives $d = \dfrac{4t}{\pi} \dfrac{f_c}{f_s}$. (7)

A larger rivet will crush, a smaller one will shear.

The diameter given by (7) would usually be found too large.

Equating 2° and 3° gives $p=d\left(1+\frac{2f_c}{f_t}\right)$. (8)

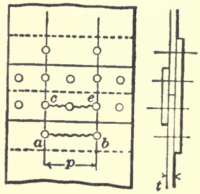
Equating 1° and 2° gives $p=d\left(1+\frac{\pi d}{2t}\frac{f_s}{f_t}\right)$. (9)

The pitch should be determined by (8) for a larger and by (9) for a smaller rivet than that given by (7).

For p_d we should have, as in the last case, according to Kennedy, $p_d=\frac{2}{3}p+\frac{1}{3}d$.

EXAMPLE OF A SPECIAL JOINT.

The joint shown in the cut is one where a part of the rivets are in single and a part in double shear.



Repeating section containing five rivet sections may fail by—

1°, tearing on ab.

$$\text{Resistance}=f_t(p-d)t.$$

2°, shearing five rivet sections.

$$\text{Resistance}=f_s\frac{5\pi d^2}{4}.$$

3°, tearing on ce, and shearing one rivet on ab.

$$\text{Resistance}=f_t(p-2d)t+f_s\frac{\pi d^2}{4}.$$

4°, tearing on ce, and crushing one rivet.

$$\text{Resistance}=f_t(p-2d)+f_ct_1d.$$

5°, crushing two rivets and shearing one.

$$\text{Resistance}=f_c(2td)+f_s\frac{\pi d^2}{4}.$$

6°, crushing on three rivets. $\text{Resistance}=f_c(2td+t_1d)$.

7°, crushing three rivets, where $t_1 \geqq t$.

$$\text{Resistance}=3f_ctd.$$

In this case, we should so proportion the joint that its efficiency may be determined from its resistance to tearing along *ab*. Hence all its other resistances should be equal to or greater than this.

Hence equate 1° and 3°, and calculate the resulting diameter of rivet, which will generally be too small, and hence we select a larger rivet, so that 3° may be greater than 1°.

Having fixed the diameter of rivet, determine the pitch in each of three ways, viz., by equating 1° and 2°, by equating 1° and 6°, and by equating 1° and 5°, and adopt the least value of p.

In this joint as used $f_c t_1 d > f_s \frac{\pi d^2}{4}$, and hence 6° is greater than 5°.

LAP.

To compute the lap, the following method is a good one. Consider the plate in front of the rivet as a rectangular beam fixed at the ends and loaded at the middle, whose span $= d$, breadth $= t$ (for cover-plate t_1), depth $= h = l - d/2$. Assume for modulus of rupture f_t and for center load W, where

1°. When rivet fails by single shear $W = f_s \frac{\pi d^2}{4}$.

2°. When rivet fails by double shear $W = f_s \frac{\pi d^2}{2}$.

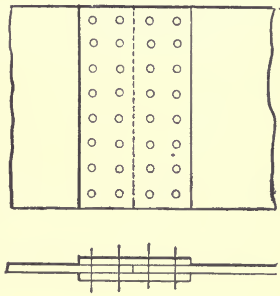
3°. When rivet fails by crushing and lap in plate is sought $W = f_c t d$.

4°. When rivet fails by crushing and lap in cover-plate is sought $W = f_c t_1 d$.

JOINTS IN THE WEB OF A PLATE GIRDER.

While no experiments on the strength of such joints have been published, the constants necessary for use in the ordinary method of calculating them are: 1°, the allowable outside fibre-stress; 2°, the allowable shearing-stress on the outer rivet; and, 3°, the allowable compression on the bearing-surface.

As an example of the usual method of calculation of such a joint, let us consider a chain-riveted butt-joint with two covering strips (as shown in the cut) as being a joint in the web of a plate girder which has equal flanges, and let us determine the allowable amount of bending-moment which the web alone (without the flanges) can resist. The modifications necessary when the flanges are unequal, and hence when the neutral axis is not at the middle of the depth, will readily suggest themselves.



The stress on any one rivet is proportional to its distance from the neutral axis of the girder, and hence, in this case, from the middle of the depth.

Use the following letters, viz.:

$f_t =$ allowable stress per sq. in. at outer edge of web-plate; $f_s =$ allowable shearing-stress per sq. in. on outer rivet; $f_c =$ allowable bearing-pressure per sq. in. on outer rivet; $t =$ thickness of plate; $h =$ total depth of web-plate; $h_1 =$ total depth of girder; $d =$ diameter of driven rivet; $a = \dfrac{\pi d^2}{4} =$ area of driven rivet section; $r =$ number of vertical rows on each side; $2n =$ number of rivets in each vertical row; $y_1 =$ distance from neutral axis to centre of nearest rivet; $y_2 =$ distance from neutral axis to centre of second rivet, etc., etc.; $y_n =$ distance from neutral axis to centre of outer rivet.

Then, for allowable bending-moment, we must take the least of the three following, viz :

1°, that determined from the shearing f_s;

2°, that determined from the compression f_c;

3°, that determined from maximum fibre-stress f_t, observing that if $f =$ greatest allowable fibre-stress in girder, then

$$f_t = f\frac{h}{h_1}.$$

To determine these proceed as follows:

1°. Greatest allowable shear on each outer rivet is

$$2f_s a = f_s \frac{\pi d^2}{2};$$

hence allowable stress on rivet at distance y_m from neutral axis

$$\frac{2f_s a}{y_n} y_m = \frac{f_s \pi d^2}{2y_n} y_m,$$

and the moment of this stress is

$$\frac{2f_s a}{y_n} y_m^2 = \frac{f_s \pi d^2}{2y_n} y_m^2.$$

Hence greatest allowable moment on joint for shearing is

$$\frac{f_s \pi d^2 r}{y_n} \{y_1^2 + y_2^2 + y_3^2 + \cdots + y_n^2\}. \qquad (1)$$

2°. Greatest allowable compression on outer rivet is $f_c td$; hence allowable stress on rivet at distance y_m from neutral axis is

$$\frac{f_c td}{y_n} y_m;$$

and the moment of this stress is

$$\frac{f_c td}{y_n} y_m^2.$$

Hence greatest allowable moment on joint for compression is

$$\frac{2f_c tdr}{y_n}\{y_1^2 + y_2^2 + y_3^2 + \ldots + y_n^2\}. \qquad (2)$$

3°. The section of the plate is a rectangle, width t and height h, with the spaces where the rivet-holes are cut left out. It will be near enough to take for the stress to be deducted on account of the rivet-hole at distance y_m from neutral axis

$$f_t\left(\frac{y_m}{\frac{h}{2}}\right)td,$$

and for its moment

$$\frac{2f_t td}{h} y_m^2.$$

Hence greatest allowable moment on joint for tearing is

$$f_t\frac{th^2}{6} - \frac{4f_t td}{h}\{y_1^2 + y_2^2 + y_3^2 + \ldots + y_n^2\}. \qquad (3)$$

This mode of calculation for (3) would seem to be warranted from the fact that the rivets do not fill the holes, although many deduct only the effect of the holes on the tension side, and consider that those on the compression side do not weaken the metal. The greatest allowable bending-moment on the joint is the smallest of (1), (2), and (3), and it is plain that, in order to make the calculation, we need to know what to use for f_t, f_s, and f_c, or, since $f_t = f\frac{h}{h_1}$, what to use for f, f_s, and f_c; and while f should be determined from the tests on the transverse strength of the metal, whether wrought-iron or steel, the best evidence we have as to the proper values of f_s and f_c is furnished by the tests on tension-joints, which have already been discussed.

Moreover, we might determine the diameter of rivet by equating (1) and (2), but we should generally find it desirable to use a larger rivet, and then we should determine the pitch by equating (2) and (3) if a larger, or (1) and (2) if a smaller, rivet is used.

Moreover, the rivets in common use in such cases are either $\frac{3}{4}''$ or $\frac{7}{8}''$ in diameter.

TESTS OF THE COMMITTEE OF THE BRITISH INSTITUTION OF MECHANICAL ENGINEERS.

The Committee on Riveted Joints of the British Institution of Mechanical Engineers consisted of Messrs. W. Boyd, W. O. Hall, A. B. W. Kennedy, R. N. J. Knight, W. Parker, R. H. Twedell, and W. C. Unwin.

Before beginning operations Prof. Unwin was asked to prepare a preliminary report, giving a summary of what had already been done by way of experiment, and also to make recommendations as to the course to be pursued in the tests.

This preliminary report is contained in vol. xxix. of *Engineering*, on the pages already cited. In regard to its recommendations it is unnecessary to speak here, as the records of the tests show what was done; but in regard to the summary of what had been done, it may be well to say that he gives a list of forty references to tests that had been made before 1880, beginning with those of Fairbairn in 1850, and ending with some made by Greig and Eyth in 1879, together with a brief account of a number of them.

Almost all of this work was done, however, with small strips with but few rivets, and will not be mentioned here. Inasmuch, however, as Fairbairn's proportional numbers have been very extensively published, and are constantly referred to by the books and by engineers, it may be well to quote a portion of what Unwin says in that regard, as follows:

"The earliest published experiments on riveted joints, and probably the first experiments on the strength of riveting ever made, are contained in the memoir by Sir Wm. Fairbairn in the Transactions of the Royal Society.

"The author first determined the tenacity of the iron, and found, for the kinds of iron experimented upon, a mean tenacity of 22.5 tons per square inch with the stress applied in the direction of the fibre, and 23 with the stress across it. That the plates were found stronger in a direction at right angles to that in which they were rolled is probably due to some error in marking the plates.

"Making certain empirical allowances, Sir Wm. Fairbairn adopted the following ratios as expressing the relative strength of riveted joints:

Solid plate	100
Double-riveted joint	70
Single-riveted joint	50

These well-known ratios are quoted in most treatises on riveting, and are still sometimes referred to as having a considerable authority.

"It is singular, however, that Sir Wm. Fairbairn does not appear to have been aware that the proportion of metal punched out in the line of fracture ought to be different in properly designed double and single riveted joints. These celebrated ratios would therefore appear to rest on a very unsatisfactory analysis of the experiments on which they are based. Sir Wm. Fairbairn also gives a well-known table of standard dimensions for riveted joints. It is not very clear how this table has been computed, and it gives proportions which make the ratio of tearing to shearing area different for different thicknesses of plate. There is no good reason for this."

As to the tests which constitute the experimental work of the committee, these were made by or under the direction of Prof. A. B. W. Kennedy, of London. Steel plates and steel rivets were used throughout, the steel containing about 0.18 per cent of carbon, and having a tensile strength varying from about 62000 to about 70000 pounds per square inch, and hence being a little harder than would correspond to our American ideas of what is suitable for use in steam-boilers. The greater portion of the work was performed by the use of a testing-machine of 100000 pounds capacity, and hence one which did not admit of testing wide strips with a sufficient number of rivets to correspond to the cases which occur in practice; indeed, only eighteen of the tests were made on such strips. Nevertheless, a brief summary of what was done will be given here, though some of the conclusions which he drew are al ready, and others are liable to be, proved untrue by tests of wide strips. The tests made by Prof. Kennedy up to 1885 consisted of fourteen series numbered I to V, VA and VI to XIII, and covering 290 experiments, 64 on punched or drilled plates, 97 on joints, 44 on the tenacity of the plates used in the joints, 33 on the tenacity and shearing-resistance of the rivet-steel used in the joints, and the remaining 52 on various other matters.

The first three series were upon the tenacity of the steel used, and showed it to be, as stated, from 62000 to 70000 pounds per square inch, with an ultimate elongation of 23 to 25 per cent in a gauged length of ten inches; the tenacity of the rivet-steel being practically the same as that of the plates. The fourth series showed the shearing-strength of the rivet-steel to be about 55000 pounds per square inch when tested in one way, and 59000 pounds per square inch when tested in another way which corresponded, as Kennedy claims, better to the conditions of a rivet, though neither was by using a riveted joint.

The tests of series V and VA were made upon pieces of plate which had been punched or drilled, in other words, on grooved specimens; and, as might be expected, these specimens showed invariably an increase in tensile strength over the straight specimens. In the $\frac{1}{4}''$ and $\frac{3}{8}''$ plates drilled with holes 1 inch in diameter and 2 inches pitch, the net metal between the holes had a tenacity 11 to 12 per cent greater than that of the untouched plate. Even with punched holes the metal had a similar excess of tenacity of over 6 per cent. The remaining eight series, VI to XIII inclusive, were made on riveted joints, the first five on single-riveted lap-joints, and the last three, or XI, XII, and XIII, on double-riveted lap and butt joints.

Series VI was made on twelve joints in $\frac{3}{8}$-inch plates which contained only two rivets each, the proportions not being intended to be those of practice, but such as should give, to some extent, limiting values for the resistances of the plate to tearing, and of the rivets to shearing and pressure. The results were rather irregular; and the main conclusion which he drew, was, that if the joint is not to break by shearing, the ratio of the tearing to the shearing area must be computed on a much lower value of shearing-strength per square inch than the experiments of series IV had shown; indeed, some of the joints of series VI gave way by shearing the rivets at loads no greater than 36000 pounds per square inch of shearing-area.

Series VII was made upon six (single-riveted lap) joints in $\frac{3}{8}$-inch plate, with only three $\frac{3}{4}$-inch rivets in each joint, and with varying pitch and lap; all these joints breaking by shearing the rivets. His conclusion from these tests was, that the lap need not be more than 1.5 times the diameter of the rivet.

Series VIII was made on eighteen (single-riveted lap) joints in six sets of three each, and these are the only single-riveted lap-joints which he tested, having as many as seven rivets each. The results are given in the accompanying table.

Before giving the table, it may be said that No. 652 was intended to have such proportions as to be equally likely to give way by tearing or by shearing, the intensity of the shearing-strength being assumed as two-thirds that of the tensile strength of the steel, while the bearing-pressure per square inch was intended to be about 7.5 per cent greater than the tension. No. 653 was proportioned with excess of shearing or rivet-area, No. 654 with defect of shearing-area, No. 655 with excess of tearing or plate area, No. 656 with defect of tearing-area, and No. 657 with excess of bearing-pressure, the different proportions being arrived at by varying the pitch and diameter of the rivets, and, in the case of 657, the thickness of the plate also. The margin (or lap minus radius of rivet) was $\frac{3}{4}$ inch in each case. The following table will show how far these intentions were realized, and further comments will be deferred till later.

SERIES VIII.—GENERAL RESULTS.

No. of Test.	No. of Specimens.	Total Breadth.	Diameter of Drilled Holes.	Pitch of Holes.	Thickness of Plate.	Ratio of Shearing to Tearing Area.	Ratio of Bearing to Tearing Area.	Tearing Area.	Stress per Sq. In. of Tearing Area when Joint broke.	Shearing Area.	Stress per Sq. In. of Shearing Area when Joint broke.	Bearing Area.	Bearing Pressure when Joint Broke. Pounds per Square Inch.	Proportion of Breaking-Load at which Visible Slip occurred.	Joint broke by—	Proportional Strength of Joint Per Cent of Solid Plate.
		in.	in.	in.	in.			sq. in.	lbs.	sq. in.	lbs.	sq. in.		per ct.		
652	3	11.34	0.79	1.62	0.400	1.46	0.94	2.337	70240	3.417	48030	2.209	74300	19.3	Shearing.	55.1
653	3	11.84	0.86	1.68	0.399	1.75	1.03	2.320	73030	4.064	41690	2.402	70520	25.2	Two by shearing, one tearing.	54.8
654	3	10.90	0.75	1.56	0.401	1.36	0.93	2.270	69420	3.080	51170	2.103	74950	21.9	Shearing.	54.9
655	3	12.20	0.78	1.74	0.386	1.44	0.83	2.572	66190	3.440	49450	2.137	79660	20.4	Shearing.	54.9
656	3	10.45	0.78	1.49	0.393	1.72	1.10	1.956	80920	3.371	46950	2.156	73420	27.0	Two by tearing, one by shearing.	58.6
657	3	12.32	0.79	1.77	0.347	1.47	0.82	2.347	72930	3.452	49480	1.924	88950	27.0	Shearing.	60.8
											Highest Single Result.					
657								2.212	76220	3.442	48980	1.811	93100	24.0	Shearing.	63.7

Series IX was made on twenty-one joints in $\frac{3}{4}$-inch plate (each containing only two rivets) designed in a manner similar to series VIII, while three were afterwards made from some of the broken plates, with as heavy rivets as it was deemed possible to make tight.

From these tests Kennedy thinks it fair to conclude —

1°. That the efficiency of a single-riveted lap-joint in a $\frac{3}{4}$-inch plate cannot be greater than 50 per cent, unless rivets larger than 1.1 inch are used; and he also calls attention to the fact that, as he claims, strength is gained by putting more metal in the heads and ends of the rivets, claiming that it will make also a tighter joint for boiler-work.

Series X was made on eight single-riveted lap-joints in $\frac{1}{4}$-inch and $\frac{3}{8}$-inch plate, made from the broken specimens of series V and VA; they also had only two rivets each. These joints were made with a view of investigating the effect of more or less bearing-pressure. He claims that high bearing-pressure induces a low shearing-strength in the rivets, and that the bearing-pressure should not exceed about 96000 pounds per square inch; also, that when a large bearing-pressure is used, the "margin" should be extra large to prevent distortion, and consequent local inequalities of stress; also, that smaller bearing-pressures do not much affect the strength of the joint one way or the other.

Series XI was made upon twelve specimens of double-riveted joints; three being lap-joints in $\frac{3}{8}$-inch plate, three lap-joints in $\frac{3}{4}$-inch plate, three butt-joints with two equal covers in $\frac{3}{8}$-inch plate, and three butt-joints with two equal covers in $\frac{3}{4}$-inch plate. Kennedy designed these joints with a view to their being equally likely to fail by tearing or by shearing. His assumptions and the results of the tests are all given in the following table:

SERIES XI. DOUBLE-RIVETED LAP AND BUTT JOINTS—AVERAGES.

Thickness of Plate. In.	Diameter of Rivet. In.	Assumed Tensile Strength of Plate, per Square Inch. Lbs.	Assumed Shearing Strength of Rivet, per Square Inch. Lbs.	Transverse Pitch. In.	Diagonal Pitch. In.	Excess of Diagonal Area over Transverse. %	Bearing Pressure per Square Inch assumed to be less than. Lbs.	Actual Tensile Stress per Square Inch of Net Section of Plate at Fracture. Lbs.	Actual Shearing Stress per Square Inch of Rivet Area at Fracture Lbs.	Actual Bearing Pressure per Square Inch at Fracture. Lbs.	Efficiency of the Joint. %
LAP-JOINTS.											
3/8	0.8	70560	51970	2.9	2.15	29	89609	75150	53920	91530	80.8
3/4	1.1	70560	51970	3.1	2.45	35	Low	69910	49710	58910	70.8
BUTT-JOINTS WITH TWO COVERS.											
3/8	0.7	70560	34720	2.75	2.00	27		68000	33780	94710	80.2
3/4	1.1	67200	42560	4.4	3.18	26	100800	59290	37650	88460	71.3

Series XII contains the same joints as series XI, the strained ends having been cut off, and the rest redrilled and riveted by means of Mr. Twedell's hydraulic riveter; and series XIII contained the same joints treated a second time in the same way. These experiments, so far as they went, showed no gain in ultimate strength to result from hydraulic as compared with hand-riveting; but it was found that, through a misunderstanding, they had been riveted up at a pressure much lower than that intended by Mr. Twedell.

On the other hand, the load at which visible slips occurred was about twice as much greater with hydraulic as with hand riveting.

KENNEDY'S CONCLUSIONS.

The following are a portion of what he gives as his conclusions:

1°. The metal between the rivet-holes had a considerably greater tensile resistance per square inch than the unperforated metal.

2°. In single-riveted joints, with the metal that he used, he assumed about 22 tons (49280 lbs.) per square inch as the shearing-strength of the rivet-steel when the bearing-pressure is below 40 tons (89600 lbs.) per square inch. In double-riveted joints with rivets of about $\frac{3}{4}$-inch diameter we can generally assume 24 tons (53760 lbs.) per square inch, though some fell to 22 tons (49280 lbs.).

3°. He advises large rivet heads and ends.

4°. For ordinary joints the bearing-pressure should not exceed 42 or 43 tons (94000 or 96000 lbs.) per square inch. For double-riveted butt-joints a higher bearing-pressure may be allowed; the effect of a high bearing-pressure is to lower the shearing-strength of the steel rivets.

5°. He advises for margin the diameter of the hole, except in double-riveted butt-joints, where it should be somewhat larger.

6°. In a double-riveted butt-joint the net metal, measured zigzag, should be from 30 to 35 per cent greater than that measured straight across, i.e., the diagonal pitch should be $\frac{2}{3}p + \frac{d}{3}$, where $p =$ transverse pitch and $d =$ diameter of rivet-hole.

7°. Visible slip occurs at a point far below the breaking-load, and in no way proportional to that load.

Kennedy thinks that these tests enable him to deduce rules for proportioning riveted joints, and the following are his rules, viz.:

(*a*) For single-riveted lap-joints the diameter of the hole should be $2\frac{1}{3}$ times the thickness of the plate, and the pitch of the rivets $2\frac{3}{8}$ times the diameter of the hole, the plate-area being thus 71 per cent of the rivet-area. If smaller rivets are used, as is generally the case, he recommends the use of the following formula:

$$p = a\frac{d^2}{t} + d,$$

where $t =$ thickness of plate, $d =$ diameter of rivet, and $p =$ pitch.

For 30-ton (67200 lbs.) plate, and 22-ton (49280 lbs.) rivets, $a = 0.524$
For 28-ton (62720 lbs.) plate, and 22-ton (49280 lbs.) rivets, $a = 0.558$
For 30-ton (67200 lbs.) plate, and 24-ton (53760 lbs.) rivets, $a = 0.570$
For 28-ton (62720 lbs.) plate, and 24 ton (53760 lbs.) rivets, $a = 0.606$

Or, as a mean, $a = 0.56$.

(*b*) For double-riveted lap-joints he claims that it would be desirable to have the diameter of the rivet $2\frac{1}{3}$ times the thickness of the plate, and that the ratio of pitch to diameter of hole should be 3.64 for 30-ton (67200 lbs.) plate and 22-ton (49280 lbs.) or 24-ton (53760 lbs.) rivets, and 3.82 for 28-ton (62720 lbs.) plate.

Here, however, it is specially likely that this size of rivet may be inconveniently large, and then he says they should be made as large as possible, and the pitch should be determined from the formula to

$$p = a\frac{d^2}{t} + d,$$

where,

For 30-ton (67200 lbs.) plate, and 24-ton (53760 lbs.) rivets, $a = 1.16$
For 28-ton (62720 lbs.) plate, and 22-ton (49280 lbs.) rivets, $a = 1.16$
For 30-ton (67200 lbs.) plate, and 22-ton (49280 lbs.) rivets, $a = 1.06$
For 28-ton (62720 lbs.) plate, and 24-ton (53760 lbs.) rivets, $a = 1.24$

(*c*) For double-riveted butt-joints he recommends that the diameter of the hole should be about 1.8 times the thickness of the plate, and the pitch 4.1 times the diameter of the hole, and that this latter ratio be maintained even when the former cannot be.

Two of the principal participants in the discussion of the report were Mr. R. Charles Longridge and Prof. W. C. Unwin.

Mr. Longridge was of the opinion that wider strips with more rivets should have been used ; that holding the specimens in the machine by means of a central pin at each end was not the best method; that the results obtained from specimens which had been made from the remnants of other fractured specimens were at least questionable, for, even if the plate had not been injured, the ratio of the length to the width of the narrowest part was different after the strained ends were cut off from what it was before; that machine-riveting should have been adopted throughout instead of hand-riveting, as it is not possible to secure uniformity with the latter even were it all done by the same man, as he would be more tired at one time than at another; that experiments should be made to determine the effect of different sizes and different shapes of heads, as well as of different pressures upon the load causing visible slip; and that experiments should be made upon chain-riveting, as he thought the chain-riveted joint would show a greater efficiency than the staggered.

Professor Unwin said :

1°. In examining the results to ascertain how far a variation from the best proportions was likely to affect the strength of the joint, he found that while the ratio of rivet diameter to thickness of plate varied 21 per cent, the ratio of shearing to tearing area 30 per cent, and the ratio of crushing to tearing area 34 per cent, the efficiency of the weakest joint was only six per cent less than that of the strongest, or, in other words, the whole variation of strength was only 11 per cent of the strength of the weakest joint.

2°. With reference to the effect which the crushing-pressure on the rivet produced upon the strength of the joint, there were some old experiments, which showed that, when the bearing-pressure on the rivet became very large there was a great diminution in the apparent tenacity of the plate in the case of riveted joints in iron. Why should the crushing-pressure affect either the tenacity of the plate or the shearing resistance of the rivet? He believed that it did not really affect either. What happened was that, if the crushing-pressure exceeded a certain limit, there was a flow of the metal, and the section which was resisting the load was diminished. Either the section of the plate in front of the rivet, if the plate was soft, or the section of the rivet itself, if the rivet was soft, became reduced.

3°. He thought that the point at which visible slip began was the initial point at which the friction of the plates was overcome, and of course was greater the greater the grip upon the plates, and hence greater in machine than in hand riveting. In some cases with hydraulic riveting loads were got as high as 10 tons (22400 lbs.) per square inch of rivet section before slipping began.

4°. In regard to the rules for proportioning riveted joints, he preferred to distinguish the joints as single-shear and double-shear joints, and then we have the following three equations: one by equating the load to the tearing-resistance of the plates, a second by equating it to the shearing-resistance of the rivets, and a third by equating it to the crushing resistance; these three determining the thickness of the plate, the diameter of the rivet, and the pitch.

By taking the crushing as double the tenacity, we should obtain for single shear $d = 2.57t$, and for double-shear, $d = 1.27t$.

In a single-shear joint the rivet cannot generally be made so big, and in the double-shear it could not always be made so small, hence the rivet diameter is chosen arbitrarily, and then the single-shear joint is proportioned by the equations for shearing and tearing, no attention being paid to the crushing, while the double-shear joint is proportioned by the equations for crushing and tearing, no attention being paid to the shearing.

5°. The general drift of the report was to advocate the use of larger rivets. Whether this could be done or not, he could not say. For lap-joints it would increase the strength, whereas for double-shear joints he was not sure that it would not be better to diminish the size of the rivet, and hence the crushing-pressure.

This report has been given so fully because it emanates from a committee of the British Institution of Mechanical Engineers; but inasmuch as series VIII is the only one where wide strips were used, it seems to the writer that any conclusions which may be drawn from any of the other tests given in the report require confirmation by tests on wide strips with more rivets, before being accepted as true.

Government Experiments.—The references to these experiments have been mentioned on page 000.

Those included in the first five of the volumes mentioned may be divided into three parts:—

1°. Those contained in the first two Executive Documents mentioned above.

2°. Those contained in the third and fourth.

3°. Those contained in the fifth.

Summaries of these sets of tests will be given here in their order, as each set was made with certain special objects in view, and, if not all, at any rate the 1° and 2°, form, as has been already stated, the first portion of a systematic series; and it seems to the author that, although the series are not yet completed, yet these tests themselves furnish more reliable information in regard to the behavior and the strength of joints than any other experiments that have been made, and that the figures themselves furnish the engineer with the means of using his judgment in many cases where he had no reliable data before.

A perusal of the tables will give a good idea of the shearing-strength per square inch of the rivet iron, which is seen to be less than the tensile strength of the solid plate; also the effect on strength of the plates due to the entire process of riveting, punching, drilling, and driving the rivets; also the efficiencies of the joints tested.

One of the strongest single-riveted joints tested was a single-riveted lap-joint with a single covering-strip.

The apparent anomaly of the punched plates in a few cases, showing a greater strength than the drilled plates, is explained by Mr. Howard to be due to the strengthening effect of cold-punching combined with smallness of pitch, inasmuch as then the masses of hardened metal on the two sides re-enforce each other.

Further than this, the student is left to study the figures themselves as to the effect of different proportions, etc.

In regard to the first series, i.e., those contained in the first two Executive Documents mentioned, it is stated in the report that —

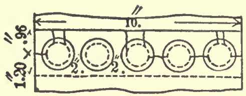
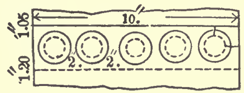
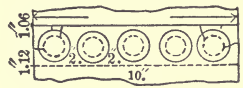
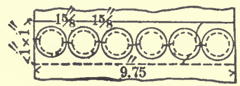
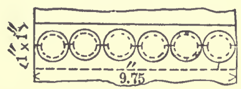
1°. "The wrought-iron plate was furnished by one maker out of one quality of stock."

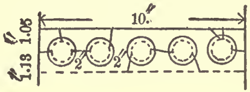
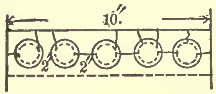
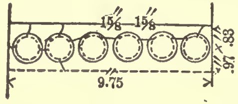
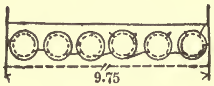
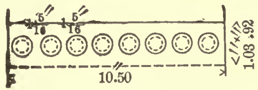
2° "The steel plates were supplied from one heat, cast in ingots of the same size; the thin plates differing from the thicker plates only in the amount of reduction given by the rolls."

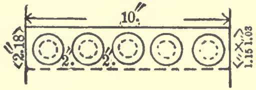
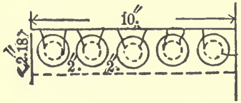
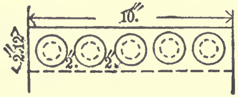
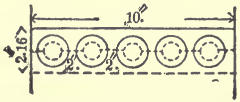
The modulus of elasticity of the metal was, iron plate, 31970000 lbs.; steel plate, 28570000 lbs.

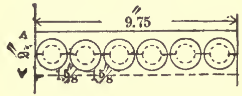
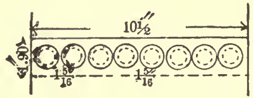
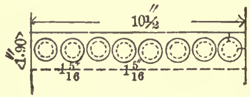
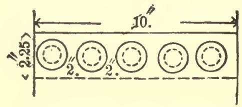
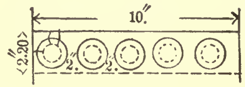
In the tabulated results, the manner of fracture is shown by sketches of the joints, and is further indicated by heavy figures in columns headed "**Maximum Strains on Joints, in lbs., per Square Inch.**"

SINGLE-RIVETED LAP-JOINTS, $\frac{1}{4}$ INCH IRON PLATE.

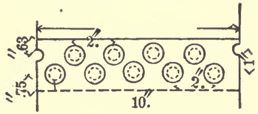
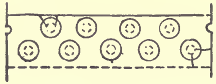
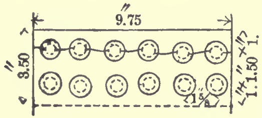
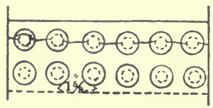
No. of Test.	Style of Joint.	Size and Kind of Rivets and Holes.	Tensile Strength, in lbs., per Square Inch.	Maximum Strains on Joints, in lbs., per Square Inch.				Efficiency of Joint.
				Tension on Gross Area of Plate.	Tension on Net Area of Plate.	Compression on Bearing Surface of Rivets.	Shearing of Rivets.	
35		$\frac{5}{8}$-in. iron rivets, $\frac{11}{16}$-in. punched holes	47925	27630	43230	76140	34900	57.7
36		$\frac{5}{8}$-in. iron rivets, $\frac{11}{16}$-in. punched holes	47925	29444	45520	82910	38640	61.4
37		$\frac{5}{8}$-in. iron rivets, $\frac{11}{16}$-in. drilled holes	47925	25270	38580	73260	34870	52.8
38		$\frac{5}{8}$-in. iron rivets, $\frac{11}{16}$-in. drilled holes	47925	27380	41790	79360	38660	57.1
39		$\frac{5}{8}$-in. iron rivets, $\frac{11}{16}$-in. punched holes	47925	29020	52160	65420	33420	60.6
40		$\frac{5}{8}$-in. iron rivets, $\frac{11}{16}$-in. punched holes	47925	30680	54930	68890	35200	64.0

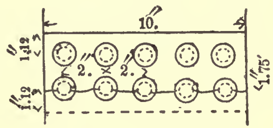
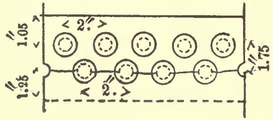
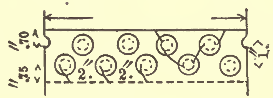
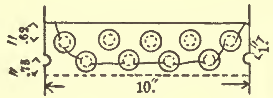
No. of Test.	Style of Joint.	Size and Kind of Rivets and Holes.	Tensile Strength, in lbs., per Square Inch.	Maximum Strains on Joints, in lbs., per Square Inch.				Efficiency of Joint.
				Tension on Gross Area of Plate.	Tension on Net Area of Plate.	Compression on Bearing Surface of Rivets.	Shearing of Rivets.	
41		$\frac{5}{8}$-in. steel rivets; $\frac{11}{16}$-in. punched holes	47925	31580	49420	87670	39640	65.9
42		$\frac{5}{8}$-in. steel rivets; $\frac{11}{16}$-in. punched holes	47925	30220	47260	83940	40610	63.1
43		$\frac{1}{2}$-in. steel rivets; $\frac{9}{16}$-in. punched holes	47925	28875	45890	78220	45300	60.3
44		$\frac{1}{2}$-in. steel rivets; $\frac{9}{16}$-in. punched holes	47925	31390	49720	84660	48420	65.5
45		$\frac{7}{16}$-in. iron rivets; $\frac{1}{2}$-in. drilled holes	47925	25440	41095	66778	44204	53.1

No. of Test.	Style of Joint.	Size and Kind of Rivets and Holes.	Tensile Strength, in lbs., per Square Inch.	Maximum Strains on Joints, in lbs., per Square Inch.				Efficiency of Joint.
				Tension on Gross Area of Plate.	Tension on Net Area of Plate.	Compression on Bearing Surface of Rivets.	Shearing of Rivets.	
46		$\frac{7}{16}$-in. iron rivets, $\frac{1}{2}$-in. drilled holes	47925	23158	**37500**	60886	**42038**	48.3
		$\frac{1}{4}$-Inch Steel Plates.						
426		$\frac{5}{8}$-in. iron rivets, $\frac{11}{16}$-in. punched holes	55765	29690	46340	82480	**37890**	53.2
427		$\frac{5}{8}$-in. iron rivets, $\frac{11}{16}$-in. punched holes	55765	29430	46010	81780	**37860**	52.8
436		$\frac{5}{8}$-in. steel rivets, $\frac{11}{16}$-in. punched holes	55765	38610	60250	**107260**	49270	69.2
437		$\frac{5}{8}$-in. steel rivets, $\frac{11}{16}$-in. punched holes	55765	37895	59240	**105290**	48750	68.0
428		$\frac{5}{8}$-in. iron rivets, $\frac{11}{16}$-in. drilled holes	55765	26870	40950	77870	**36350**	48.2
429		$\frac{5}{8}$-in. iron rivets, $\frac{11}{16}$-in. drilled holes	55765	27680	42370	80200	**36710**	49.6
438		$\frac{5}{8}$-in. steel rivets, $\frac{11}{16}$-in. drilled holes	55765	41460	**63190**	120160	56100	74.3
439		$\frac{5}{8}$-in. steel rivets, $\frac{11}{16}$-in. drilled holes	55765	40040	**61310**	116090	52460	71.8

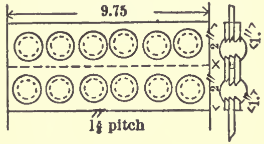
No. of Test.	Style of Joint.	Size and Kind of Rivets and Holes.	Tensile Strength, in lbs., per Square Inch.	Maximum Strains on Joints, in lbs., per Square Inch.				Efficiency of Joint.
				Tension on Gross Area of Plate.	Tension on Net Area of Plate.	Compression on Bearing Surface of Rivets.	Shearing of Rivets.	
430		$\frac{5}{8}$-in. steel rivets $\frac{11}{16}$-in. drilled holes	55765	38360	66860	90000	41790	68.8
431		$\frac{5}{8}$-in. steel rivets $\frac{11}{16}$-in. drilled holes	55765	40160	70000	94230	43750	72.0
47		$\frac{7}{16}$-in. steel rivets $\frac{1}{2}$-in. drilled holes	55765	38496	62496	101180	65220	69.0
48		$\frac{7}{16}$-in. steel rivets $\frac{1}{2}$-in. drilled holes	55765	36114	58338	94800	60382	64.8
49		$\frac{5}{8}$-in. steel rivets $\frac{11}{16}$-in. drilled holes	55765	39400	60184	114603	52742	70.6
50		$\frac{5}{8}$-in. steel rivets $\frac{11}{16}$-in. drilled holes	55765	37680	57439	109650	50645	67.6

DOUBLE-RIVETED LAP-JOINTS, $\frac{1}{4}$-INCH IRON AND STEEL PLATE.

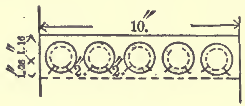
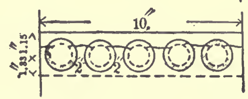
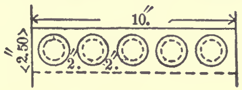
No. of Test.	Style of Joint.	Size and Kind of Rivets and Holes.	Tensile Strength, in lbs., per Square Inch.	Maximum Strains on Joints, in lbs., per Square Inch.				Efficiency of Joint.
				Tension on Gross Area of Plate.	Tension on Net Area of Plate.	Compression on Bearing Surface of Rivets.	Shearing of Rivets.	
85		*Iron Plate.* $\frac{7}{16}$-in. iron rivets, $\frac{1}{2}$-in. drilled holes	47925	28900	38535	64120	43110	60.3
86		*Iron Plate.* $\frac{7}{16}$-in. iron rivets, $\frac{1}{2}$-in. drilled holes	47925	31314	41750	69710	41750	65.3
617		*Iron Plate.* $\frac{1}{2}$-in. iron rivets, $\frac{9}{16}$-in. punched holes	47925	31550	50592	42118	28691	65.8
618		*Iron Plate.* $\frac{1}{2}$-in. iron rivets, $\frac{9}{16}$-in. punched holes	47925	31282	49950	41660	28660	65.3

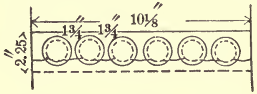
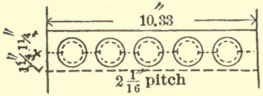
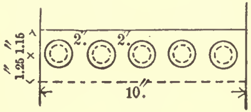
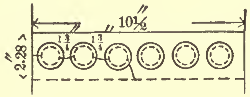
No. of Test.	Style of Joint.	Size and Kind of Rivets and Holes.	Tensile Strength, in lbs., per Square Inch.	Maximum Strains on Joints, in lbs., per Square Inch.				Efficiency of Joint.
				Tension on Gross Area of Plate.	Tension on Net Area of Plate.	Compression on Bearing Surface of Rivets.	Shearing of Rivets.	
432		*Steel Plate.* $\frac{5}{8}$-in. iron rivets, $\frac{11}{16}$-in. punched holes	55765	39280	61510	54640	25400	70.4
433		$\frac{5}{8}$-in. iron rivets, $\frac{11}{16}$-in. punched holes	55765	38700	60300	53715	25530	69.4
434		*Steel Plate.* $\frac{5}{8}$-in. iron rivets, $\frac{11}{16}$-in. punched holes	55765	41785	65400	64600	30430	74.9
435		$\frac{5}{8}$-in. iron rivets, $\frac{11}{16}$-in. punched holes	55765	41460	64600	63430	30430	74.3
87		$\frac{7}{16}$-in. steel rivets, $\frac{1}{2}$-in. drilled holes	55765	42530	56944	94910	57910	76.3
88		$\frac{7}{16}$-in. steel rivets, $\frac{1}{2}$-in. drilled holes	55765	44344	59130	98360	61130	79.5

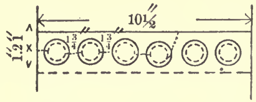
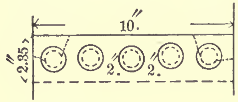
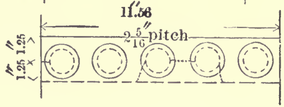
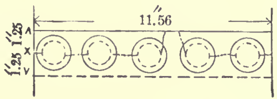
DOUBLE-WELT BUTT-JOINTS, $\frac{1}{4}$-INCH IRON PLATE.

No. of Test.	Style of Joint.	Size and Kind of Rivets and Holes.	Tensile Strength, in lbs., per Square Inch.	Maximum Strains on Joints, in lbs., per Square Inch. Tension on Gross Area of Plate.	Tension on Net Area of Plate.	Compression on Bearing Surface of Rivets.	Shearing of Rivets.	Efficiency of Joint.
615		$\frac{5}{8}$-in. iron rivets, $\frac{11}{16}$-in. punched holes	47925	29800	53475	67321	16944	62.2
616		$\frac{5}{8}$-in. iron rivets, $\frac{11}{16}$-in. punched holes	47925	28397	50959	64138	16719	59.3

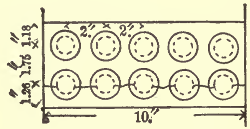
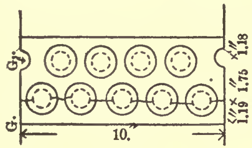
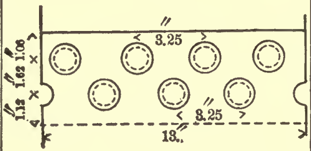
SINGLE-RIVETED LAP-JOINTS, $\frac{3}{8}$-INCH IRON PLATE.

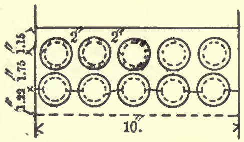
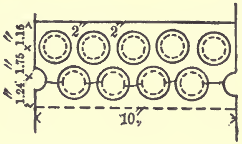
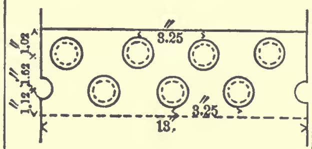
No. of Test.	Style of Joint.	Size and Kind of Rivets and Holes.	Tensile Strength, in lbs., per Square Inch.	Tension on Gross Area of Plate.	Tension on Net Area of Plate.	Compression on Bearing Surface of Rivets.	Shearing of Rivets.	Efficiency of Joint.
62		$\frac{11}{16}$-in. iron rivets, $\frac{3}{4}$-in. punched holes	47180	23110	37460	60340	38280	49.0
63		$\frac{11}{16}$-in. iron rivets, $\frac{3}{4}$-in. punched holes	47180	22280	36130	58150	35520	47.2
64		$\frac{11}{16}$-in. iron rivets, $\frac{3}{4}$-in. drilled holes	47180	23445	38190	60730	37530	49.7
65		$\frac{11}{16}$-in. iron rivets, $\frac{3}{4}$-in. drilled holes	47180	22220	36210	57530	36050	47.1

No. of Test.	Style of Joint.	Size and Kind of Rivets and Holes.	Tensile Strength, in lbs., per Square Inch.	Maximum Strains on Joints, in lbs., per Square Inch. Tension on Gross Area of Plate.	Tension on Net Area of Plate.	Compression on Bearing Surface of Rivets.	Shearing of Rivets.	Efficiency of Joint.
66		$\frac{11}{16}$-in. iron rivets, $\frac{3}{4}$-in. punched holes	47180	23570	41750	54130	34230	50.0
67		$\frac{11}{16}$-in. iron rivets, $\frac{3}{4}$-in. punched holes	47180	23280	41290	53400	34150	49.3
720		1-in. iron rivets, $1\frac{1}{16}$-in. punched holes	47180	38500	61700	52970	26180	60.4
721		1-in. iron rivets, $1\frac{1}{16}$-in. punched holes	47180	26960	58510	50220	24830	57.1
Single-Riveted Lap-Joints, $\frac{3}{8}$-Inch Steel Plate.								
51		$\frac{11}{16}$-in. iron rivets, $\frac{3}{4}$-in. punched holes	53330	24180	39220	63210	39740	45.4
52		$\frac{11}{16}$-in. iron rivets, $\frac{3}{4}$-in. punched holes	53330	23240	37700	60760	38190	43.6
53		$\frac{11}{16}$-in. steel rivets, $\frac{3}{4}$-in. punched holes	53330	34160	55215	89580	56430	64.1
54		$\frac{11}{16}$-in. steel rivets, $\frac{3}{4}$-in. punched holes	53330	33840	54740	88660	55460	63.5
55		$\frac{11}{16}$-in. steel rivets, $\frac{3}{4}$-in. drilled holes	53330	35590	63650	80930	50650	66.7

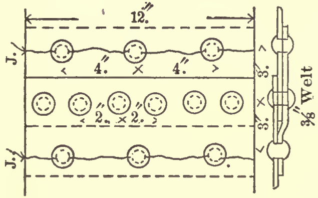
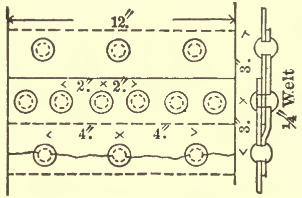
No. of Test.	Style of Joint.	Size and Kind of Rivets and Holes	Tensile Strength, in lbs., per Square Inch.	Maximum Strains on Joints, in lbs., per Square Inch.				Efficiency of Joint.
				Tension on Gross Area of Plate.	Tension on Net Area of Plate.	Compression on Bearing Surface of Rivets.	Shearing of Rivets	
56		{ $\frac{11}{16}$-in. steel rivets $\frac{3}{4}$-in. drilled holes }	53330	35860	63976	81600	50900	67.2
238		{ $\frac{3}{4}$-in. steel rivets $\frac{13}{16}$-in. punched holes }	53330	37810	65460	89490	53560	70.9
239		{ $\frac{3}{4}$-in. steel rivets $\frac{13}{16}$-in. punched holes }	53330	37630	65210	88990	53600	70.6
718		{ 1 -in. iron rivets $1\frac{1}{16}$-in. punched holes }	53330	38075	73394	79510	36614	71.4
719		{ 1 -in. iron rivets $1\frac{1}{16}$-in. punched holes }	53330	38390	73970	80200	36590	72.0

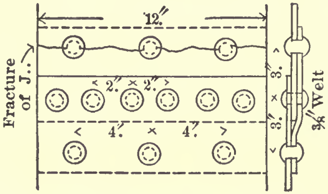
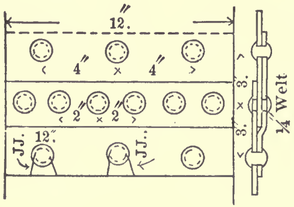
DOUBLE-RIVETED LAP-JOINTS, $\frac{3}{8}$-INCH IRON AND STEEL PLATE.

No. of Test.	Style of Joint.		Size and Kind of Rivets and Holes.	Tensile Strength, in lbs., per Square Inch.	Maximum Strains on Joints, in lbs., per Square Inch. Tension on Gross Area of Plate.	Tension on Net Area of Plate.	Compression on Bearing Surface of Rivets.	Shearing of Rivets.	Efficiency of Joint.
68		Iron plate.	$\frac{11}{16}$-in. iron rivets, $\frac{3}{4}$-in. punched holes	47180	29970	48450	39160	24760	63.5
69			$\frac{11}{16}$-in. iron rivets, $\frac{3}{4}$-in. punched holes	47180	31320	50730	41070	26150	66.4
58			$\frac{11}{16}$-in. iron rivets, $\frac{3}{4}$-in. punched holes	47180	31000	50220	40640	25330	65.7
70		Iron plate.	$\frac{11}{16}$-in. iron rivets, $\frac{3}{4}$-in. punched holes	47180	28530	46255	41480	27550	60.5
71			$\frac{11}{16}$-in. iron rivets, $\frac{3}{4}$-in. punched holes	47180	28475	46110	41270	27010	60.4
81		Iron plate.	$\frac{11}{16}$-in. iron rivets, $\frac{3}{4}$-in. drilled holes	47180	23750	30920	58700	39130	50.4
82			$\frac{11}{16}$-in. iron rivets, $\frac{3}{4}$-in. drilled holes	47180	23180	30130	57340	38410	49.1

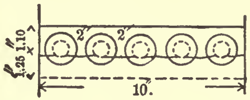
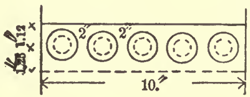
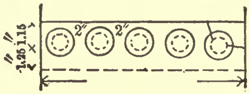
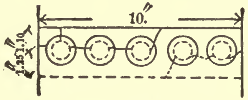
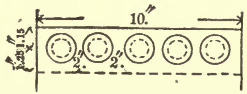
No. of Test.	Style of Joint.	Size and Kind of Rivets and Holes.	Tensile Strength, in lbs., per Square Inch.	Maximum Strains on Joints, in lbs., per Square Inch. Tension on Gross Area of Plate.	Tension on Net Area of Plate.	Compression on Bearing Surface of Rivets.	Shearing of Rivets.	Efficiency of Joint.
57		*Steel plate.* $\frac{11}{16}$-in. iron rivets, $\frac{3}{4}$-in. punched holes	53330	39020	**62800**	50760	32310	73.2
59		*Steel plate.* $\frac{11}{16}$-in. iron rivets, $\frac{3}{4}$-in. punched holes	53330	40080	**64720**	52450	32930	75.2
60		*Steel plate.* $\frac{11}{16}$-in. iron rivets, $\frac{3}{4}$-in. punched holes	53330	39010	**63210**	56860	34710	73.2
61		*Steel plate.* $\frac{11}{16}$-in. iron rivets, $\frac{3}{4}$-in. punched holes	53330	34000	**54930**	49530	30830	63.8
83		*Steel plate.* $\frac{11}{16}$-in. steel rivets, $\frac{3}{4}$-in. drilled holes	53330	34316	44660	84460	**52750**	64.4
84		*Steel plate.* $\frac{11}{16}$-in. steel rivets, $\frac{3}{4}$-in. drilled holes	53330	33585	43650	83000	**51845**	63.0

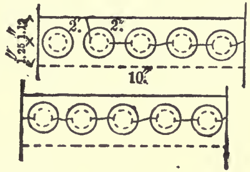
RIVETED JOINTS, $\frac{3}{8}$-INCH IRON AND STEEL PLATE, RE-ENFORCED LAP-JOINTS.

No. of Test.	Style of Joint.		Size and Kind of Rivets and Holes.	Tensile Strength, in lbs., per Square Inch.	Maximum Strains on Joints, in lbs., per Square Inch.				Efficiency of Joint.
					Tension on Gross Area of Plate.	Tension on Net Area of Plate.	Compression on Bearing Surface of Rivets.	Shearing of Rivets.	
244		Iron plate.	$\frac{5}{8}$-in. iron rivets, 0.72-in. drilled holes	47180	31900	38870	59080	40360	67.0
245			$\frac{3}{4}$-in. iron rivets, $\frac{13}{16}$-in. drilled holes	47180	34900	43770	56640	34460	74.0
296		Iron plate.	$\frac{3}{4}$-in. iron rivets, $\frac{13}{16}$-in. drilled holes	47180	35700	44840	57910	33890	75.7
297			$\frac{3}{4}$-in. iron rivets, $\frac{13}{16}$-in. drilled holes	47180	33950	42680	55350	31810	71.9

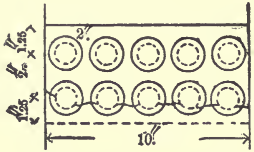
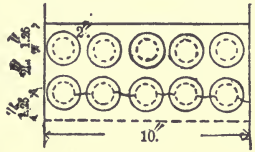
No. of Test.	Style of Joint.	Size and Kind of Rivets and Holes.	Tensile Strength, in lbs., per Square Inch.	Maximum Strains on Joints, in lbs., per Square Inch.				Efficiency of Joint.
				Tension on Gross Area of Plate.	Tension on Net Area of Plate.	Compression on Bearing Surface of Rivets.	Shearing of Rivets.	
246		Steel Plate. { $\frac{7}{8}$-in. steel rivets / $\frac{15}{16}$-in. drilled holes }	53330	47465	62050	67320	32960	89.0
247		Steel Plate. { $\frac{7}{8}$-in. steel rivets / $\frac{15}{16}$-in. drilled holes }	53330	48050	62880	68135	33900	90.1
298		Steel Plate. { $\frac{7}{8}$-in. iron rivets / $\frac{15}{16}$-in. drilled holes }	53330	46820	61020	67300	34250	87.8
299		Steel Plate. { $\frac{7}{8}$-in. iron rivets / $\frac{15}{16}$-in. drilled holes }	53330	47390	61710	68040	34750	88.9

SINGLE-RIVETED LAP-JOINTS, ½-INCH IRON AND STEEL PLATE.

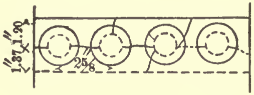
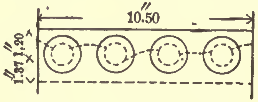
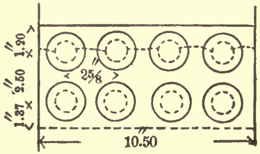
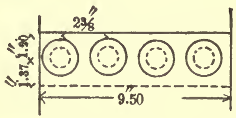
No. of Test.	Style of Joint.		Size and Kind of Rivets and Holes.	Tensile Strength, in lbs., per Square Inch.	Maximum Strains on Joints, in lbs., per Square Inch. Tension on Gross Area of Plate.	Tension on Net Area of Plate.	Compression on Bearing Surface of Rivets.	Shearing of Rivets.	Efficiency of Joint.
240		*Iron Plate.*	¾-in. iron rivets, $\frac{13}{16}$-in. punched holes	44615	17760	31100	41500	34280	39.8
241			¾-in. iron rivets, $\frac{13}{16}$-in. punched holes	44615	17690	31395	41955	34960	39.7
292		*Iron Plate.*	¾-in. iron rivets, $\frac{13}{16}$-in. drilled holes	44615	19130	32376	47850	38020	42.9
293		*Iron Plate.*	¾-in. iron rivets, $\frac{13}{16}$-in. drilled holes	44615	19765	33180	48890	39220	44.3
327		*Iron Plate.*	¾-in. steel rivets, $\frac{13}{16}$-in. drilled holes	44615	23740	39900	58880	47020	52.2
328		*Iron Plate.*	¾-in. steel rivets, $\frac{13}{16}$-in. drilled holes	44615	24200	40590	59900	47830	54.2
242		*Steel Plate.*	¾-in. iron rivets, $\frac{13}{16}$-in. punched holes	57215	21830	38204	50940	41100	38.2
243			¾-in. iron rivets, $\frac{13}{16}$-in. punched holes	57215	20523	35915	47890	38636	35.9

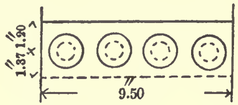
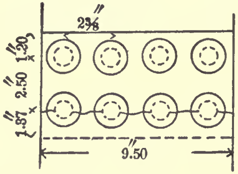
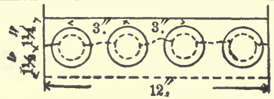
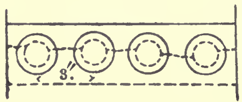
No. of Test.	Style of Joint.	Size and Kind of Rivets and Holes.	Tensile Strength, in lbs., per Square Inch.	Maximum Strains on Joints, in lbs., per Square Inch. Tension on Gross Area of Plate.	Tension on Net Area of Plate.	Compression on Bearing Surface of Rivets.	Shearing of Rivets.	Efficiency of Joint.
294		Steel plate. $\frac{15}{16}$-in. iron rivets, 1-in. punched holes	57215	29310	60210	56980	36770	51.2
295		Steel plate. $\frac{15}{16}$-in. iron rivets, 1-in. punched holes	57215	24145	49590	47060	30540	42.2

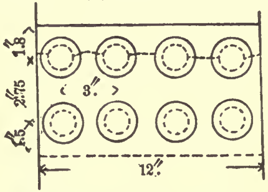
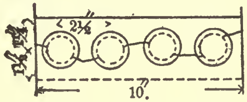
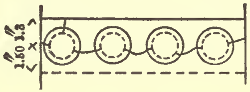
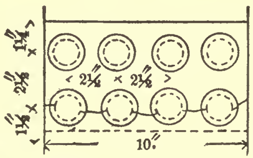
DOUBLE-RIVETED LAP-JOINTS, $\frac{1}{2}$-INCH IRON AND STEEL PLATE.

No. of Test.	Style of Joint.	Size and Kind of Rivets and Holes.	Tensile Strength	Tension on Gross Area of Plate.	Tension on Net Area of Plate.	Compression on Bearing Surface of Rivets.	Shearing of Rivets.	Efficiency of Joint.
329		Iron plate. $\frac{3}{4}$-in. iron rivets, $\frac{13}{16}$-in. punched holes	44615	25430	44320	59640	25280	57.0
635		Iron plate. $\frac{3}{4}$-in. iron rivets, $\frac{13}{16}$-in. punched holes	44615	24660	42920	57950	24560	55.2
619		Steel plate. $\frac{15}{16}$-in. iron rivets, 1-in. punched holes	57215	30757	64602	29354	19670	53.8
620		Steel plate. $\frac{15}{16}$-in. iron rivets, 1-in. punched holes	57215	30778	64519	29371	19644	53.8

SINGLE AND DOUBLE RIVETED LAP-JOINTS, $\frac{5}{8}$-INCH IRON AND STEEL PLATE.

No. of Test.	Style of Joint.	Size and Kind of Rivets and Holes.	Tensile Strength, in lbs., per Square Inch.	Maximum Strains on Joints, in lbs., per Square Inch.				Efficiency of Joint.
				Tension on Gross Area of Plate.	Tension on Net Area of Plate.	Compression on Bearing Surface of Rivets.	Shearing of Rivets.	
730		1 -in. iron rivets; $1\frac{1}{16}$-in. punched holes. *Iron plate.*	44635	20020	34680	47510	35460	44.9
731		1 -in. iron rivets; $1\frac{1}{16}$-in. punched holes. *Iron plate.*	44635	19750	34230	46790	34930	42.0
732		1 -in. iron rivets; $1\frac{1}{16}$-in. punched holes. *Iron plate.*	44635	25150	43580	29740	22990	56.3
733		1 -in. iron rivets; $1\frac{1}{16}$-in. punched holes. *Iron plate.*	44635	26470	45850	31310	23670	59.3
734		1 -in. steel rivets; $1\frac{1}{16}$-in. punched holes. *Steel plate.*	52445	26480	49650	56760	43490	50.5

No. of Test.	Style of Joint.	Size and Kind of Rivets and Holes.	Tensile Strength, in lbs., per Square Inch.	Maximum Strains on Joints, in lbs., per Square Inch.				Efficiency of Joint.
				Tension on Gross Area of Plate.	Tension on Net Area of Plate.	Compression on Bearing Surface of Rivets.	Shearing of Rivets.	
735		1 -in. steel rivets; $1\frac{1}{16}$-in. punched holes. *Steel plate.*	52445	28110	52770	60150	46080	58.6
736		1 -in. steel rivets; $1\frac{1}{16}$-in. punched holes. *Steel plate.*	52445	37180	69680	39780	30470	70.9
737		1 -in. steel rivets; $1\frac{1}{16}$-in. punched holes. *Steel plate.*	52445	35800	67100	38300	29340	68.3
	Single and Double Riveted Lap-Joints, $\frac{3}{4}$-Inch Iron and Steel Plate.							
722		*Iron plate.* $1\frac{1}{8}$ -in. iron rivets; $1\frac{3}{16}$-in. punched holes	46590	17230	29290	41980	32600	37.0
723		*Iron plate.* $1\frac{1}{8}$ -in. iron rivets; $1\frac{3}{16}$-in. punched holes	46590	18080	30730	44040	34200	38.8

No. of Test.	Style of Joint.	Size and Kind of Rivets and Holes.	Tensile Strength, in lbs., per Square Inch.	Maximum Strains on Joints, in lbs., per Square Inch.				Efficiency of Joint.
				Tension on Gross Area of Plate.	Tension on Net Area of Plate.	Compression on Bearing Surface of Rivets.	Shearing of Rivets.	
724		*Iron plate.* $1\frac{1}{8}$-in. iron rivets, $1\frac{3}{16}$-in. punched holes	46590	24720	**42000**	30040	23460	53.1
725		*Iron plate.* $1\frac{1}{8}$-in. iron rivets, $1\frac{3}{16}$-in. punched holes	46590	25870	**43950**	31460	23860	55.5
726		*Steel Plates.* $1\frac{1}{8}$-in. steel rivets, $1\frac{3}{16}$-in. punched holes	51545	20280	**39970**	41180	33840	39.3
727		*Steel plate.* $1\frac{1}{8}$-in. steel rivets, $1\frac{3}{16}$-in. punched holes	51545	23940	**47370**	48540	39890	46.4
728		*Steel plate.* $1\frac{1}{8}$-in. steel rivets, $1\frac{3}{16}$-in. punched holes	51545	24720	**48970**	24990	20450	48.0
729		*Steel plate.* $1\frac{1}{8}$-in. steel rivets, $1\frac{3}{16}$-in. punched holes	51545	24050	**47510**	24340	20060	46.7

No. of Test.	Style of Joint.	Kind and Thickness of Plate.	Size and Kind of Rivets and Holes.	Tensile Strength per Square Inch.	Maximum Stresses on Joints, in lbs., per Square Inch.				Efficiency of Joint.
					Tension on Gross Area of Plate.	Tension on Net Area of Plate.	Compression on Bearing Surface of Rivets.	Shearing of Rivets	
4418	Single-riveted lap	$\frac{3}{8}$-in. iron	$\frac{11}{16}$-in. iron rivets, $\frac{3}{4}$-in. punched holes	47180	23170	39300	50850	33710	47.0
4419		$\frac{3}{8}$-in. iron	$\frac{11}{16}$-in. iron rivets, $\frac{3}{4}$-in. punched holes	47180	23130	41000	53050	35170	49.0
4428		$\frac{1}{2}$-in. iron	$\frac{3}{4}$-in. iron rivets, $\frac{13}{16}$-in. punched holes	44615	20340	35650	47350	37300	45.6
4429		$\frac{3}{8}$-in. iron	$\frac{3}{4}$-in. iron rivets, $\frac{13}{16}$-in. punched holes	44615	20050	35150	46640	36780	44.9
764	Single-riveted butt	$\frac{3}{8}$-in. iron	$\frac{11}{16}$-in. iron rivets, $\frac{3}{4}$-in. punched holes	47180	28260	46360	72390	25380	59.9
765		$\frac{3}{8}$-in. iron	$\frac{11}{16}$-in. iron rivets, $\frac{3}{4}$-in. punched holes	47180	28550	46875	73050	25450	60.5
768		$\frac{1}{2}$-in. iron	$\frac{3}{4}$-in. iron rivets, $\frac{13}{16}$-in. punched holes	44615	26530	46400	61940	24630	59.4
769		$\frac{1}{2}$-in. iron	$\frac{3}{4}$-in. iron rivets, $\frac{13}{16}$-in. punched holes	44615	26400	46140	61740	24310	59.2
772		$\frac{3}{8}$-in. iron	1 -in. iron rivets, $1\frac{1}{16}$-in. punched holes	44635	25530	44260	60330	23010	57.2
773		$\frac{3}{8}$-in. iron	1 -in. iron rivets, $1\frac{1}{16}$-in. punched holes	44635	24490	42350	58080	22310	54.9
776		$\frac{3}{4}$-in. iron	$1\frac{1}{8}$-in. iron rivets, $1\frac{3}{16}$-in. punched holes	46590	24290	42310	57000	21870	52.1
777		$\frac{3}{4}$-in. iron	$1\frac{1}{8}$-in. iron rivets, $1\frac{3}{16}$-in. punched holes	46590	24073	41920	56540	22140	51.7

No. of Test.	Style of Joint.	Kind and Thickness of Plate.	Size and Kind of Rivets and Holes.	Tensile Strength per Square Inch.	Maximum Stresses on Joints, in lbs., per Square Inch.				Efficiency of Joint.
					Tension on Gross Area of Plate.	Tension on Net Area of Plate.	Compression on Bearing Surface of Rivets.	Shearing of Rivets.	
4420	Single-riveted lap	$\frac{3}{8}$-in. steel	$\frac{3}{4}$-in. iron rivets $\frac{13}{16}$-in. punched holes	53330	31720	61270	65760	40390	59.5
4427	Single-riveted lap	$\frac{3}{8}$-in. steel	$\frac{3}{4}$-in. iron rivets $\frac{13}{16}$-in. punched holes	53330	31500	60830	65320	39990	59.1
4430	Single-riveted lap	$\frac{1}{2}$-in. steel	$\frac{15}{16}$-in. iron rivets 1-in. punched holes	57215	33000	47530	44590	29390	40.2
4431	Single-riveted lap	$\frac{3}{8}$-in. steel	$\frac{15}{16}$-in. iron rivets 1-in. punched holes	57215	24180	49840	46960	31070	42.3
766	Single-riveted butt	$\frac{3}{8}$-in. steel	$\frac{11}{16}$-in. iron rivets $\frac{3}{4}$-in. punched holes	53330	38240	62770	97940	31240	71.7
767	Single-riveted butt	$\frac{3}{8}$-in. steel	$\frac{11}{16}$-in. iron rivets $\frac{3}{4}$-in. punched holes	53330	37260	61210	95210	31020	69.8
770	Single-riveted butt	$\frac{1}{2}$-in. steel	$\frac{15}{16}$-in. steel rivets 1-in. punched holes	57215	32690	68920	62220	20370	57.1
771	Single-riveted butt	$\frac{1}{2}$-in. steel	$\frac{15}{16}$-in. steel rivets 1-in. punched holes	57215	31470	66710	59580	19890	55.0
774	Single-riveted butt	$\frac{5}{8}$-in. steel	1-in. steel rivets $1\frac{1}{16}$-in. punched holes	52445	33250	62180	71450	27750	63.4
775	Single-riveted butt	$\frac{5}{8}$-in. steel	1-in. steel rivets $1\frac{1}{16}$-in. punched holes	52445	33470	62590	71930	27940	63.8
778	Single-riveted butt	$\frac{3}{4}$-in. steel	$1\frac{1}{8}$-in. steel rivets $1\frac{3}{16}$-in. punched holes	51545	27810	54650	55610	23190	54.0
779	Single-riveted butt	$\frac{3}{4}$-in. steel	$1\frac{1}{8}$-in. steel rivets $1\frac{1}{16}$-in. punched holes	51545	27500	54200	55840	22810	53.4

GOVERNMENT TESTS OF GROOVED SPECIMENS.

Tensile Tests of ¼-in. Grooved Specimens Wrought-Iron Punched.

Width at Bottom of Groove.	Thickness of Plate.	Ultimate Strength per Sq. Inch, in lbs.
Inch.	Inch.	
0.48	0.240	48090
0.46	0.235	46940
0.46	0.241	49280
0.49	0.240	55340
0.44	0.239	51520
0.47	0.241	49910
0.97	0.247	49540
0.98	0.247	49960
0.94	0.249	50128
0.96	0.248	46900
0.98	0.250	46980
0.96	0.251	46350
1.47	0.250	37636
1.50	0.252	37326
1.48	0.249	41030
1.48	0.247	39480
1.47	0.250	37446
1.45	0.251	39533
1.96	0.281	43194
1.95	0.274	47490
1.95	0.282	41360
1.92	0.279	43080
2.03	0.250	41140
1.99	0.248	39575
2.42	0.280	36210
2.40	0.245	42245
2.47	0.243	42233
2.46	0.285	42712
2.48	0.245	38125
2.44	0.248	41620
2.97	0.247	38964
2.98	0.241	41540
2.96	0.241	39972
2.92	0.240	41712
2.98	0.250	40430
2.95	0.247	40850

Tensile Tests of ¼-in. Grooved Specimens Wrought-Iron Drilled.

Width at Bottom of Groove.	Thickness of Plate.	Ultimate Strength per Sq. Inch, in lbs.
Inch.	Inch.	
0.51	0.249	55787
0.52	0.245	55905
0.52	0.275	57480
0.52	0.276	56000
0.49	0.248	49600
0.50	0.248	56700
0.47	0.275	54880
0.51	0.276	57800
1.00	0.276	54300
1.02	0.273	57700
1.00	0.276	53800
1.00	0.280	52430
1.00	0.252	49400
1.02	0.275	54060
1.01	0.247	52770
1.00	0.278	54600
1.50	0.276	49130
1.52	0.273	51300
1.48	0.251	47220
1.51	0.273	53400
1.52	0.275	54180
1.50	0.276	54600
1.48	0.274	56250
1.50	0.249	46260
2.01	0.275	45900
2.05	0.279	46820
2.00	0.275	47950
2.00	0.278	49640
2.00	0.286	44650
2.00	0.275	50780
2.02	0.279	48850
2.00	0.277	49840
2.51	0.244	44980
2.52	0.280	40150
2.51	0.282	43150
2.50	0.244	45500
2.51	0.285	46500
2.49	0.242	49520
2.49	0.242	–
2.50	0.280	44780
3.02	0.250	45700
3.02	0.249	44870
3.00	0.240	46760
3.00	0.250	45700
2.93	0.242	47950
2.99	0.250	48740
2.98	0.279	45900
3.01	0.281	44410

Tensile Tests of ¼-in. Grooved Specimens Steel Plate Punched.

Width at Bottom of Groove.	Thickness of Plate.	Ultimate Strength per Sq. Inch, in lbs.
Inch.	Inch.	
0.49	0.250	65120
0.47	0.249	67010
0.48	0.249	63420
0.48	0.248	66550
0.48	0.247	67060
0.47	0.248	65300
0.99	0.249	59840
1.00	0.250	62160
1.01	0.249	68246
0.96	0.250	67330
0.96	0.248	65966
0.95	0.245	62700
1.45	0.248	64080
1.45	0.252	64000
1.45	0.249	61025
1.51	0.251	59420
1.96	0.250	59900
1.93	0.252	63500
1.98	0.250	59350
1.96	0.251	59060
2.49	0.249	58100
2.47	0.249	63900
2.43	0.250	61640
2.95	0.251	56530
3.01	0.249	58780
3.04	0.253	55500
2.97	0.252	60060
2.98	0.251	54050
2.97	0.249	56040

Tensile Tests of ¼-in. Grooved Specimens Steel Plate Drilled.

Width at Bottom of Groove.	Thickness of Plate.	Ultimate Strength per Sq. Inch, in lbs.
Inch.	Inch.	
0.52	0.246	67890
0.54	0.248	67160
0.53	0.247	66870
0.50	0.247	65610
0.51	0.249	66370
0.51	0.250	67420
0.52	0.248	67750
0.52	0.252	61910
1.03	0.247	57090
1.02	0.250	66390
1.02	0.246	66770
1.02	0.250	67730
1.01	0.247	66020
1.00	0.251	67010
1.00	0.247	64450
1.01	0.250	66090
1.54	0.250	64390
1.52	0.251	63350
1.50	0.253	64370
1.54	0.248	64895
2.02	0.252	64320
2.00	0.251	62970
2.00	0.251	60910
2.50	0.248	59260
2.50	0.252	63250
2.53	0.248	59390
3.03	0.251	61577
3.00	0.249	59080
3.02	0.251	59550
3.02	0.250	59700
3.00	0.250	63370
3.00	0.251	58630
3.03	0.252	63940

IRON PUNCHED.

Tensile Tests of Grooved Wrought-Iron Plates.		
Width.	Thickness.	Ultimate Strength per Sq. Inch, in lbs.
Inch.	Inch.	
1.01	0.373	47000
0.98	0.370	47520
2.00	0.382	39760
2.02	0.383	36630
2.39	0.390	37600
2.98	0.395	36340
2.98	0.392	39210
3.47	0.390	37680
3.47	0.389	38340
0.97	0.467	50820
1.48	0.506	45090
1.49	0.506	45050
1.91	0.513	42500
1.97	0.512	43430
2.47	0.516	39410
2.41	0.513	39720
3.00	0.515	38950
2.90	0.517	37290
3.50	0.520	37800
3.49	0.513	37770
4.00	0.515	35730
4.03	0.516	36690
3.99	0.511	37000
4.03	0.508	37420
0.97	0.614	49770
1.01	0.619	52960
1.48	0.618	46320
1.52	0.620	46750
2.99	0.614	40140
3.50	0.615	37480
3.50	0.616	36940
4.04	0.619	37310
0.98	0.678	50840
1.01	0.682	46590
1.49	0.688	45970
3.48	0.691	40350
3.53	0.692	39380

IRON DRILLED.

Tensile Tests of Grooved Wrought-Iron Plates.		
Width.	Thickness.	Ultimate Strength per Sq. Inch, in lbs.
Inch.	Inch.	
0.98	0.376	50870
0.98	0.377	52660
1.98	0.379	49710
2.00	0.380	49830
2.50	0.390	50250
3.00	0.392	45150
3.00	0.393	47540
3.50	0.392	43940
3.49	0.390	46490
0.99	0.477	47140
1.00	0.479	48370
1.49	0.510	51240
1.49	0.512	51510
1.98	0.514	50050
1.98	0.516	47790
2.51	0.520	45580
2.52	0.516	44960
3.00	0.515	44980
3.01	0.519	47030
3.51	0.513	46170
3.49	0.514	44760
3.99	0.510	45330
3.98	0.513	45000
4.00	0.506	46100
0.97	0.628	47220
1.00	0.626	48350
1.52	0.625	47170
1.49	0.629	46530
2.98	0.613	48220
3.46	0.616	47770
3.47	0.617	44900
3.91	0.625	44840
3.96	0.626	45100
0.99	0.695	47500
0.99	0.691	52780
1.51	0.692	48470
3.44	0.700	47750
3.49	0.692	46350

STEEL PUNCHED.

Tensile Tests of Grooved Steel Plates.		
Width.	Thickness.	Ultimate Strength per Sq. Inch, in lbs.
Inch.	Inch.	
1.99	0.365	61890
0.99	0.494	70080
1.00	0.492	68130
1.50	0.497	66340
1.51	0.494	63810
1.99	0.499	55930
1.97	0.500	64260
2.43	0.502	52050
2.51	0.504	64360
3.00	0.503	60320
2.99	0.503	62430
3.50	0.503	49430
3.50	0.505	48270
4.00	0.497	48010
4.00	0.499	55190
3.99	0.501	55780
3.99	0.498	46250
1.01	0.613	66720
1.52	0.612	64800
1.50	0.615	64400
2.50	0.618	58060
2.52	0.619	58780
2.99	0.617	57180
3.46	0.615	58410
3.51	0.615	57190
4.04	0.612	54450
4.03	0.614	57380
1.01	0.721	67930
1.00	0.718	67620
1.50	0.719	62890
3.50	0.735	56730
3.51	0.733	54220

STEEL DRILLED.

Tensile Tests of Grooved Steel Plates.		
Width.	Thickness.	Ultimate Strength per Sq. Inch, in lbs.
Inch.	Inch.	
1.97	0.369	63620
1.00	0.498	66220
0.99	0.495	66800
1.00	0.500	67000
1.53	0.497	65930
1.50	0.498	66270
1.98	0.504	67510
2.03	0.502	66730
2.50	0.497	67950
2.52	0.501	67440
3.01	0.502	66310
3.01	0.503	66190
3.49	0.504	64920
3.50	0.502	65210
3.99	0.499	64470
4.00	0.498	64810
4.00	0.503	64690
4.00	0.498	64140
0.99	0.619	60290
1.49	0.614	63610
1.49	0.616	63450
2.49	0.620	59170
2.50	0.619	59600
3.01	0.617	59270
3.50	0.614	61610
3.49	0.617	62060
4.00	0.615	60330
4.01	0.617	61120
0.96	0.726	58480
1.01	0.727	58790
1.51	0.726	59290
3.50	0.736	58700
3.49	0.729	59180

Next will be given the two series of tests already referred to, with Mr. Howard's analysis of them.

TENSILE TESTS OF RIVETED JOINTS.

"Earlier experiments on this subject made with single and double riveted lap and butt joints in different thicknesses of iron and steel plate, together with the tests of specimens prepared to illustrate the strength of constituent parts of joints, are recorded in the report of tests for 1882 and 1883.

From the results thus obtained it appeared desirable to institute a synthetical series of tests, beginning with the most elementary forms of joints in which the stresses are found in their least complicated state. To meet these conditions, a series of joints have been prepared which may be designated as single-riveted butt-joints, in which the covers are extended so as to be grasped in the testing-machine; thereby enabling one plate of the joint to be dispensed with, and securing the test of one line of riveting.

Such a joint, made with carefully annealed mild steel plate of superior quality, with drilled holes, seems well adapted to demonstrate the influence on the tensile strength of the metal taken across the line of riveting, of variations in the width of the net section between rivets, and variations in the compression stress on the bearing-surface of the rivets: elements which are believed to be fundamental in all riveted construction.

This series comprises 216 specimen joints, the thickness of the plate ranging from $\frac{1}{4}''$ to $\frac{3}{4}''$, advancing by eighths. The covers are from $\frac{3}{16}''$ to $\frac{7}{16}''$. The rivets are wrought-iron, and range from $\frac{9}{16}''$ to $1\frac{3}{16}''$ diameter; they are machine-driven in drilled holes $\frac{1}{16}''$ larger in diameter than the nominal size of the rivets. Tensile tests of the material accompany the tests of the joints.

From each sheet of steel two test-strips were sheared, one lengthwise and one crosswise. The strips were $2\frac{1}{2}''$ wide and $24''$ to $36''$ long; they were annealed with the specimen plates, and had their edges planed, reducing their widths to $1\frac{1}{2}''$ before testing.

Micrometer readings were taken in 10″ along the middle of the length of each.

The strength and ductility appear to be substantially the same in each direction. But the practice of the rolling-mill where these sheets were rolled is such that nearly the same amount of work may have been given the steel in each direction; that is, lengthwise and crosswise the finished sheet.

The ingots of open-hearth metal are first rolled down to slabs about 6″ thick, then reheated and rolled either lengthwise or crosswise their former direction, as best suits the required finished dimensions.

The tensile tests show among the thinner plates a relatively high elastic limit as compared with the tensile strength; in the $\frac{3}{16}''$ plate the percentage is 72.2, while with the $\frac{3}{4}''$ plate the percentage is found to be 53.3.

It is noticeable that the thinner plates particularly exhibit a large stretch immediately following the elastic limit, and the stretching is continued at times under a load lower than that which has been previously sustained. It is characteristic of all the thicknesses that a considerable stretch takes place under loads approaching the tensile strength—in some cases the stretch increases 5 to 6 per cent, while the stress advances 1000 pounds per square inch or less. Herein is found a valuable property of this metal as a material for riveted construction. The stress from the bearing-surface of the rivets is distributed over the net section of plate between the rivets, due to the large stretch of the metal, with little elevation of the stress, and a nearer approximation of uniform stress in this section attained than is found in a brittle or less ductile metal. The joints were held for testing in the hydraulic jaws of the testing-machine, having 24″ exposure between them. A loose piece of steel the same thickness as the plate was placed between the covers to receive the grip of the jaws, and avoid bending the covers.

Elongations were measured in a gauged length of 5″, the micrometer covering the joint at the middle of its width. Loads were applied in increments of 1000 pounds per square inch of the gross section of the plate, the effect of each increment determined by the micrometer, and permanent sets observed at intervals.

The progress of the test of a joint is generally marked by three well-defined periods. In the first period greatest rigidity is found, and it is thought that the joint is now held entirely by the friction of the rivet-heads, and the movement of the joint is principally that due to the elasticity of the metal.

The second period is distinguished by a rapid increase in the stretch of the joint; attributed to the overcoming of the friction under the rivet-heads and closing up any clearance about the rivets, bringing them into bearing condition against the fronts of the rivet-holes. Rivets which are said to fill the holes can hardly do so completely, on account of the contraction of the metal of the rivet from a higher temperature than that of the plate, after the rivet is driven.

After a brief interval the movement of the joint is retarded, and the third period is reached. The stretch of the joint is now believed to be due to the distortion of the rivet-holes and the rivets themselves. The movement begins slowly, and so continues till the elastic limit of the metal about the rivet-holes is passed, and general flow takes place over the entire cross-section, and rupture is reached. These stages in the test of a joint are well defined, except when the plates are in a warped condition initially, when abnormal micrometer readings are observed.

The difference in behavior of a joint and the solid metal suggests the propriety of arranging tension joints in boiler construction and elsewhere as nearly in line as practicable.

The efficiencies of the joints are computed on the basis of the tensile strength of the lengthwise strips, this being the direction in which the metal of the joints is strained. The efficiencies here found are undoubtedly lowered somewhat by the contraction in width of the specimens, causing in most cases fractures to begin at the edges and extend towards the middle of the joint. Of the entire series, 88 joints have been tested; the $\frac{1}{2}''$, $\frac{5}{8}''$, and $\frac{3}{4}''$ plates yet remain."

TABULATION OF O. H. STEEL STRIPS REPRESENTING METAL EMPLOYED IN RIVETED JOINTS.

No. of test.	Sheet letter.	Nominal thickness.	Direction of grain.	Actual dimensions. Width.	Actual dimensions. Thickness.	Sectional area.	Elastic limit. Per square inch.	Elastic limit. Mean.	Tensile strength. Per square inch.	Tensile strength. Mean.	Elongation in 10 inches.	Contraction of area.	Appearance of fracture.
		inch.		inches.	inch.	sq. in.	lbs.		lbs.		per ct.	per ct.	
1200..	A	$\frac{3}{16}$	Lengthwise	1.497	.184	.275	42180		63050		16.7	51.6	Fine silky, slight lamination.
1201..	B	$\frac{3}{16}$	do.	1.497	.183	.274	43070		59160		18.7	53.7	Silky, lamellar.
1202 .	C	$\frac{3}{16}$	do.	1.498	.193	.289	43600		61280		20.9	49.8	Do.
1203..	D	$\frac{3}{16}$	do.	1.499	.186	.279	48030	43740	63400	63015	19.0	57.0	Fine silky.
1204..	E	$\frac{3}{16}$	do.	1.500	.180	.270	43700		63370		20.0	58.1	Do.
1205..	......	$\frac{3}{16}$	do.	1.502	.172	.258	41860		67830		25.1	50.0	Do.
1210..	A	$\frac{3}{16}$	Crosswise	1.497	.182	.272	49260		63270		18.1	44.1	Silky seam.
1206..	B	$\frac{3}{16}$	do.	1.498	.187	.280	43570		61570		15.4	32.9	Silky, lamellar.
1207..	C	$\frac{3}{16}$	do.	1.495	195	.292	47950	46712	61710	62220	22.9	35.6	Do.
1208..	D	$\frac{3}{16}$	do.	1.497	.189	.283	45940		61130		23.2	45.9	Do.
1209..	E	$\frac{3}{16}$	do.	1.497	.180	.269	46840		63420		22.1	47.2	Silky.
1211..	F	$\frac{1}{4}$	Lengthwise	1.493	.246	.367	38150		61740		24.4	48.2	Do.
1212..	G	$\frac{1}{4}$	do.	1.499	.241	.361	41550		62660		24.3	50.4	Fine silky.
1213..	H	$\frac{1}{4}$	do.	1.498	.243	.364	42860		59180		21.2	53.8	Silky, lamellar.
1214..	I	$\frac{1}{4}$	do.	1.482	.251	.372	37660		58660		25.2	56.5	Fine silky.
1215..	J	$\frac{1}{4}$	do.	1.499	.257	.385	38960		61000		25.0	53.8	Fine silky, slight lamination.
1216	K	$\frac{1}{4}$	do.	1.499	.252	.378	36500	40158	58150	60295	23.9	58.5	Silky, lamellar.
1217..	L	$\frac{1}{4}$	do.	1.498	.250	.375	45310		61470		23.8	51.5	Do.
1218..	M	$\frac{1}{4}$	do.	1.498	.244	.366	42620		58170		24.7	62.3	Fine silky.
1219..	N	$\frac{1}{4}$	do.	1.499	.244	.366	39340		55740		20.4	59.8	Do.
1220..	I	$\frac{1}{4}$	do.	1.498	.250	.375	38450		62300		22.9	48.0	Do.
1221..	......	$\frac{1}{4}$	do.	1.500	.238	.357	40340		64170		24.2	46.2	Do.
1222..	F	$\frac{1}{4}$	Crosswise	1.494	.248	.371	39870		61730		23.0	48.8	Silky, slight lamination.
1223..	G	$\frac{1}{4}$	do.	1.496	.242	.362	38680		62870		17.0	43.6	Fine silky, surface blister.
1224..	H	$\frac{1}{4}$	do.	1.481	.250	.370	45950		60840		21.8	47.6	Silky, slight lamination.
1225..	I	$\frac{1}{4}$	do.	1.497	.252	.377	38700		61230		22.4	43.8	Silky, stratified.
1226 .	I	$\frac{1}{4}$	do.	1.475	.252	.372	37100	39869	60890	59876	25.3	47.8	Silky.
1227..	J	$\frac{1}{4}$	do.	1.491	.253	.377	38700		55380		27.3	54.1	Fine silky.
1228..	K	$\frac{1}{4}$	do.	1.498	.255	.382	36650		53900		24.8	51.0	Silky, lamellar.
1229..	L	$\frac{1}{4}$	do.	1.492	.250	.373	38070		57400		19.9	49.1	Silky, slight lamination.
1230..	M	$\frac{1}{4}$	do.	1.498	.246	.369	41190		62470		24.0	47.1	Silky, laminated, stratified.
1231..	N	$\frac{1}{4}$	do.	1.498	.247	.370	43780		62050		22.1	43.0	Silky, laminated.

1235..	O	$\frac{5}{16}$	Lengthwise	1.498	.301	.451	33260		54350		30.0	60.8	Fine silky.
1236..	P	$\frac{5}{16}$	do.	1.497	.304	.455	37140		59300		25.4	54.5	Do.
1234..	Q	$\frac{5}{16}$	do.	1.499	.311	.466	33480	36532	56760	57708	24.7	56.9	Fine silky, slight lamination.
1232..	R	$\frac{5}{16}$	do.	1.499	.308	.462	42420		61130		27.2	51.9	Fine silky.
1233..	S	$\frac{5}{16}$	do.	1.499	.308	.462	36360		57000		26.3	58.2	Do.
1240..	O	$\frac{5}{16}$	Crosswise	1.499	.307	.460	36090		63430		22.7	48.5	Silky.
1241..	P	$\frac{5}{16}$	do.	1.499	.307	.460	34350		57280		22.1	48.9	Silky, slight lamination.
1239..	Q	$\frac{5}{16}$	do.	1.500	.308	.462	34200	35280	57140	59776	24.3	50.0	Silky, lamellar.
1237..	R	$\frac{5}{16}$	do.	1.497	.310	.464	36210		60600		21.8	48.9	Silky, slight lamination.
1238..	S	$\frac{5}{16}$	do.	1.499	.312	.467	35550		60430		24.1	49.7	Do.
1252.	A	$\frac{3}{8}$	Lengthwise	1.498	.368	.551	33030		54260		25.9	57.7	Silky, lamellar.
1253..	B	$\frac{3}{8}$	do.	1.493	.390	.582	34020		59730		26.1	52.6	Silky.
1254 .	C	$\frac{3}{8}$	do.	1.498	.376	.563	34810		57870		27.6	53.5	Silky, lamellar.
1255..	D	$\frac{3}{8}$	do.	1.498	.383	.574	28920		53730		28.2	55.7	Silky, very lamellar.
1256..	E	$\frac{3}{8}$	do.	1.498	.385	.577	33970	33027	58340	56489	27.1	55.1	Silky, slight lamination.
1257..	F	$\frac{3}{8}$	do.	1.499	.378	.567	34920		54290		27.1	57.8	Silky, lamellar.
1258..	G	$\frac{3}{8}$	do.	1.498	.365	.547	32910		53840		23.1	57.4	Do.
1259..	H	$\frac{3}{8}$	do.	1.498	.367	.550	30550		56670		28.5	57.6	Fine silky, slight lamination.
1260..	I	$\frac{3}{8}$	do.	1.495	.372	.557	33390		59030		24.7	50.4	Silky, stratified.
1261..	J	$\frac{3}{8}$	do.	1.496	.372	.557	33750		57130		26.0	56.2	Silky, slight lamination.
1242..	A	$\frac{3}{8}$	Crosswise	1.498	.368	.551	28310		55170		25.0	38.8	Silky, exceedingly lamellar.
1243..	B	$\frac{3}{8}$	do.	1.497	.390	.584	29110		57850		26 6	52.9	Silky, slight lamination.
1244..	C	$\frac{3}{8}$	do.	1.497	.375	.561	34050		58000		26.0	49.7	Silky, lamellar.
1245..	D	$\frac{3}{8}$	do.	1.498	.380	.569	30580		55890		21.9	51.7	Do.
1246..	E	$\frac{3}{8}$	do.	1.498	.383	.574	31360	30814	54430	56510	26.5	48.4	Do.
1247..	F	$\frac{3}{8}$	do.	1.499	.380	.570	28420		54370		26.1	51.4	Silky, very lamellar.
1248..	G	$\frac{3}{8}$	do.	1.496	.366	.548	31390		55420		26.4	44.9	Silky, lamellar.
1249..	H	$\frac{3}{8}$	do.	1.498	.372	.557	30520		57000		25.2	50.2	Silky, slight lamination.
1250..	I	$\frac{3}{8}$	do.	1.498	.371	.556	32730		58130		20.7	51.6	Do.
1251..	J	$\frac{3}{8}$	do.	1.498	.375	.562	31670		58840		27.0	51.6	Silky,
1262..	K	$\frac{7}{16}$	Lengthwise	1.496	.423	.633	33490		59000		24.6	49.6	Silky, slight lamination.
1263..	L	$\frac{7}{16}$	do.	1.496	.438	.655	31450		59390		22.5	48.7	Silky, stratified.
1264..	M	$\frac{7}{16}$	do.	1.496	.425	.636	33960		61650		24.1	53.8	Fine silky.
1265..	N	$\frac{7}{16}$	do.	1.497	.411	.615	34150	32460	61140	58245	20.2	51.7	Silky, stratified, laminated.
1266..	O	$\frac{7}{16}$	do.	1.500	.419	.629	31480		52910		29.0	59.5	Fine silky.
1267..	P	$\frac{7}{16}$	do.	1.496	.433	.648	33020		58090		27.5	54.6	Do.
1268..	Q	$\frac{7}{16}$	do.	1.498	.423	.634	30280		56960		30.7	56.9	Do.
1269..	......	$\frac{7}{16}$	do.	1.496	.420	.628	31850		56820		23.8	49.7	Silky, slight lamination.

TABULATION OF O. H. STEEL STRIPS REPRESENTING METAL EMPLOYED IN RIVETED JOINTS—*Continued.*

No. of test.	Sheet letter.	Nominal thickness.	Direction of grain.	Actual dimensions. Width.	Actual dimensions. Thickness.	Sectional area.	Elastic limit. Per square inch.	Elastic limit. Mean.	Tensile strength. Per square inch.	Tensile strength. Mean.	Elongation in 10 inches.	Contraction of area.	Appearance of fracture.
		inch.		inches.	inch.	sq. in.	lbs.		lbs.		per ct.	per ct.	
1270..	K	7/16	Crosswise	1.498	.424	.635	30000		57400		24.0	43.3	Silky, coarse lamination
1271..	L	7/16	do.	1.497	.438	.656	36280		59910		26.4	54.0	Silky.
1272..	M	7/16	do.	1.497	.421	.630	30790		59560		24.4	48.9	Do.
1273..	N	7/16	do.	1.497	.410	.614	33220	32450	62050	58790	20.8	46.1	Do.
1274..	O	7/16	do.	1.497	.415	.621	32530		60480		27.5	44.4	Silky, stratified.
1275..	P	7/16	do.	1.497	.435	.651	31640		57710		22.5	49.3	Silky, slight lamination.
1276..	Q	7/16	do.	1.497	.417	.624	32690		54420		29.9	58.3	Silky.
1281..	R	1/2	Lengthwise	1.497	.488	.731	30100		57180		21.3	53.6	Fine silky.
1282..	S	1/2	do.	1.496	.470	.703	34990	31602	59050	58558	26.0	46.4	Silky lamellar.
1283..	T	1/2	do.	1.498	.480	.719	32270		60000		24.7	48.0	Fine silky.
1284..	U	1/2	do.	1.496	.483	.723	29050		58000		24.6	52.1	Fine silky, slight lamination.
1277..	R	1/2	Crosswise	1.498	.482	.722	32960		60480		24.6	47.8	Silky, slight lamination.
1278..	S	1/2	do.	1.498	.466	.698	28370	30270	57420	58378	26.5	42.0	Silky, very lamellar.
1279..	T	1/2	do.	1.496	.482	.721	28850		58200		24.9	53.0	Silky.
1280..	U	1/2	do.	1.498	.484	.725	30900		57410		27.1	45.4	Silky, slight lamination.
1285..	V	5/8	Lengthwise	1.497	.622	.931	28790		55000		31.1	55.7	Silky, lamellar.
1286..	W	5/8	do.	1.498	.610	.914	29540	28227	57290	56077	23.6	35.9	Silky, stratified.
1287..	X	5/8	do.	1.498	.618	.926	26350		55940		26.1	48.7	Very rough on one side.
1288..	V	5/8	Crosswise	1.498	.624	.935	34650		58160		28.7	51.4	Silky, lamellar.
1289..	W	5/8	do.	1.496	.609	.911	28760	30793	56870	56677	23.8	40.3	Silky, slight lamination.
1290..	X	5/8	do.	1.496	.619	.925	28970		55000		24.0	54.6	Do.
1294..	Y	3/4	Lengthwise	1.498	.728	1.090	27160		60420		28.8	51.9	Do.
1293..	Z	3/4	do.	1.495	.722	1.079	27800	28320	59000	58313	27.9	55.1	Silky.
1295..	......	3/4	do.	1.501	.743	1.115	30000		55520		24.0	36.5	Silky, stratified.
1291..	Y	3/4	Crosswise	1.497	.746	1.117	37610	34845	63690	60185	24.1	44.9	Silky, slight lamination.
1292..	Z	3/4	do.	1.496	.763	1.141	32080		56680		27.8	52.2	Silky.

RIVET METAL USED IN THE CONSTRUCTION OF RIVETED JOINTS.

No. of Test.	Nominal Diameter.	Actual Diameter.	Sectional Area.	Elastic Limit.		Tensile Strength.		Elongation in 10 Inches.		Area at Fracture.		Contraction of Area.	Elongation of Inch Sections.
				Total.	Per Square Inch.	Total.	Per Square Inch.			Diameter.	Area.		
	in.	in.	sq. in.	lbs.	lbs.	lbs.	lbs.	in.	per ct.	in.	sq. in.	per ct.	
1296	$\frac{9}{16}$	.54	.229	8950	39080	12310	53760	2.23	22.3	.41	.132	42.4	″.21, ″.17, ″.19, ″.19, ″.18, ″.22, ″.21, ″.23, ″.43*, ″.20.
1297	$\frac{9}{16}$	.54	.229	8940	39040	12190	53230	2.15	21.5	.42	.139	39.3	″.21, ″.18, ″.20, ″.17, ″.17, ″.20, ″.18, ″.23, ″.40*, ″.21.
1298	$\frac{11}{16}$	.66	.342	14700	42980	20160	58950	2.48	24.8	.53	.221	35.4	″.22, ″.20, ″.24, ″.43*, ″.24, ″.23, ″.24, ″.24. ″.23, ″.21.
1299	$\frac{11}{16}$	.66	.342	13450	39330	18850	55120	3.13	21.3	.51	.204	40.4	.18, .43*, .21, .19, .21, .19, .18, .18, .19, .17.
1300	$\frac{13}{16}$	.79	.490	20240	41310	27620	56370	2.05	20.5	.67	.353	28.0	.24, .24, .21, .21, .20, .20, .19, .19, .19, .18.
1301	$\frac{13}{16}$	.79	.490	20210	41240	27670	56470	2.51	25.1	.62	.302	38.4	.21, .22, .26, .42*, .28, .24, .22, .21, .21, .24.
1302	$\frac{15}{16}$	.92	.665	23320	35070	34820	52360	2.86	28.6	.69	.374	43.8	.23, .25, .23, .29, .52*, .31, .24, .24, .29, .26.
1303	$\frac{15}{16}$	.92	.665	23980	36060	34880	52450	2.62	26.2	.72	.407	38.8	.25, .24, .48*, .32, .23, .26, .22, .21, .20, .21.
1304	$1\frac{1}{16}$	1.06	.882	30650	34750	46280	52470	2.78	27.8	.80	.503	42.9	.22, .21, .23, .25, .31, .59*, .29, .24, .23, .21.
1305	$1\frac{1}{16}$	1.06	.882	30700	34810	46610	52850	2.81	28.1	.80	.503	42.9	.26, .43*, .45*, .26, .24, .24, .24, .26, .22, .21.
1306	$1\frac{3}{16}$	1.17	1.075	36400	33860	57620	53600	2.74	27.4	.92	.665	38.1	″.24, ″.25, ″.36, ″.47*, ″.25, ″.23, ″.24, ″.24, ″.23, ″.23.
1307	$1\frac{3}{16}$	1.17	1.075	36550	34000	57300	53300	2.18	21.8	.97	.739	31.2	.21, .19, .22, .42*, .29, .20, .17, .15, .16, .17.

* Appearance of fractures fibrous.

TABULATION OF SINGLE-

¼″ STEEL

No. of Test.	Sheet Letters.			Pitch.	No. of Rivets.	Width of Joint.	Nominal Thickness.		Size of Rivets and Holes.	Actual Thick-ness of Plate.	Lap.
	Plate.	Covers.					Plate.	Covers.			
				in.		in.	in.	in.	in.	in.	in.
1308	F	A	A	$1\frac{5}{8}$	6	9.75	1/4	3/16	$\frac{9}{16}$ & $\frac{5}{8}$	.242	2
1309	F	A	A	"	6	"	"	"	"	.242	2
1310	F	A	A	$1\frac{3}{4}$	6	10.50	"	"	"	.242	2
1311	F	A	A	"	6	"	"	"	"	.249	2
1312	F	A	A	$1\frac{7}{8}$	6	11.25	"	"	"	.244	2
1313	F	A	A	"	6	"	"	"	"	.243	2
1314	F	A	A	$1\frac{3}{4}$	6	10.49	"	"	$\frac{11}{16}$ & $\frac{3}{4}$	.248	2
1315	F	A	A	"	6	"	"	"	"	.242	2
1316	F	A	A	$1\frac{7}{8}$	6	11.27	"	"	"	.244	2
1317	F	A	A	"	6	"	"	"	"	.246	2
1318	F	B	B	2	6	12.01	"	"	"	.245	2
1319	G	B	B	"	6	"	"	"	"	.240	2
1320	G	B	B	$2\frac{1}{8}$	6	12.76	"	"	"	243	2
1321	G	B	B	"	6	"	"	"	"	.243	2
1322	H	D	D	$2\frac{1}{4}$	6	13.51	"	"	"	.245	2
1323	H	D	C	"	6	"	"	"	"	.247	2
1324	F	A	A	$1\frac{7}{8}$	6	11.26	"	"	$\frac{13}{16}$ & $\frac{7}{8}$	.248	2
1325	F	A	A	"	6	"	"	"	"	.245	2
1326	G	B	B	2	6	12.00	"	"	"	.241	2
1327	G	B	B	"	6	"	"	"	"	.242	2
1328	G	B	C	$2\frac{1}{8}$	6	12.76	"	"	"	.241	2
1329	G	C	C	"	6	"	"	"	"	.242	2
1330	L	C	D	$2\frac{1}{4}$	6	13.50	"	"	"	.248	2
1331	H	C	D	"	6	"	"	"	"	.246	2
1332	H	D	E	$2\frac{3}{8}$	6	14.25	"	"	"	.248	2
1333	H	D	E	"	6	"	"	"	"	.248	2
1334	M	E	E	$2\frac{1}{2}$	6	15.00	"	"	"	.243	2
1335	O	....	....	"	6	"	"	"	"	.245	2
1336	H	C	C	$2\frac{5}{8}$	5	13.13	"	"	"	238	2
1337	L	C	C	"	5	"	"	"	"	.252	2
1338	G	B	B	2	6	12.00	"	"	$\frac{15}{16}$ & 1	.238	2
1339	F	B	B	"	6	"	"	"	"	.248	2
1340	G	B	C	$2\frac{1}{8}$	6	12.75	"	"	"	.240	2
1341	G	C	C	"	6	"	"	"	"	.242	2
1342	L	C	D	$2\frac{1}{4}$	6	13.51	"	"	"	.250	2
1343	L	D	D	"	6	"	"	"	"	.250	2

RIVETED BUTT-JOINTS.

PLATE.

Sectional Area of Plate. Gross.	Sectional Area of Plate. Net.	Bearing Surface of Rivets.	Shearing Area of Rivets.	Tensile Strength of Plate per Sq. In.	Maximum Stress on Joint per Sq. In. Tension on Gross Section of Plate.	Tension on Net. Section of Plate.	Comp. on Bearing Surface of Rivets.	Shearing of Rivets.	Efficiency of Joint.
sq. in.	sq. in.	sq in.	sq. in.	lbs.	lbs.	lbs.	lbs.	lbs.	
2.360	1.452	.908	3.682	61740	41690	67770	108370	26720	67.5
2.360	1.452	.908	3.682	61740	42180	68560	109640	27040	68.3
2.541	1.634	.907	3.682	61740	42540	66160	119180	29360	68.9
2.615	1.681	.934	3.682	61740	43170	67160	119810	30660	69.9
2.745	1.830	.915	3.682	61740	44920	67380	134750	33490	72.8
2.739	1.827	.912	3.682	61740	44520	66750	133720	33120	72.1
2.602	1.486	1.116	5.300	61740	40700	71270	94890	19980	65.9
2.541	1.452	1.089	5.300	61740	40000	70000	93330	19180	64.8
2.750	1.652	1.098	5.300	61740	40980	68210	102630	21260	66.4
2.772	1.665	1.107	5.300	61740	41430	68980	103750	21670	67.1
2.942	1.840	1.102	5.300	61740	42180	67450	112610	23420	68.3
2.882	1.802	1.080	5.300	62660	43000	68770	114750	23380	68.6
3.100	2.007	1.093	5.300	62660	43060	66520	122140	25190	68.7
3.100	2.007	1.093	5.300	62660	44030	68000	124880	25750	70.3
3.310	2.207	1.103	5.300	59180	43040	64540	129150	26880	72.7
3.337	2.225	1.112	5.300	59180	43810	65700	131460	27580	74.0
2.792	1.490	1.302	7.216	61740	38650	72480	82870	14950	62.0
2.756	1.470	1.286	7.216	61740	39430	73930	84510	15010	63.8
2.892	1.627	1.265	7.216	62660	40340	71700	92210	16170	64.4
2.909	1.638	1.271	7.216	62660	41280	73310	94480	16640	65.9
3.075	1.810	1.265	7.216	62660	42290	71850	102810	18020	67.5
3.088	1.817	1.271	7.216	62660	42750	72650	103860	18290	68.2
3.348	2.046	1.302	7.216	61470	43100	70530	110830	20000	70.1
3.321	2.029	1.291	7.216	59180	41450	67840	106620	19080	70.0
3.534	2.232	1.302	7.216	59180	41820	66210	113500	20480	70.7
3.534	2.232	1.302	7.216	59180	42760	67710	116070	20940	72.3
3.645	2.369	1.276	7 216	58170	44650	68700	127550	22550	76.8
3.675	2.389	1.286	7.216	64170	43050	66230	123030	21930	67.1
3.125	2.084	1.041	6.013	59180	41310	61960	124020	21470	69.8
3 309	2.206	1.103	6.013	61470	42000	62990	125990	23110	68.3
2.856	1.428	1.428	9.425	62660	40620	81230	81230	12310	64.8
2.976	1.488	1.488	9.425	61740	36290	72580	72580	11460	58.8
3.060	1.620	1.440	9.425	62660	38660	73020	82150	12550	61.7
3.088	1.636	1.452	9.425	62660	38000	71730	80820	12450	60.6
3.378	1.878	1.500	9.425	61470	37800	68000	85130	13550	61.5
3.375	1.875	1.500	9.425	61470	39000	70200	87750	13970	63.4

TABULATION OF SINGLE-

¼″ STEEL

No. of Test.	Sheet Letters.			Pitch.	No. of Rivets.	Width of Joint.	Nominal Thickness.		Size of Rivets and Holes.	Actual Thickness of Plate.	Lap.
	Plate.	Covers.					Plate.	Covers.			
				in.		in.	in.	in.	in.	in.	in.
1344	L	D	E	2⅜	6	14.24	1/4	3/16	$\frac{15}{16}$ & 1	.250	2
1345	H	D	E	"	6	"	"	"	"	.247	2
1346	I	E	E	2½	6	15.00	"	"	"	.251	2
1347	I	E	E	"	6	"	"	"	"	.252	2
1348	H	C	C	2⅝	5	13.13	"	"	"	.244	2
1349	G	C	C	"	5	"	"	"	"	.239	2
1350	N	D	D	2¾	5	13.77	"	"	"	.250	2
1351	N	D	D	"	5	"	"	"	"	.249	2
1352	H	D	D	2⅞	5	14.39	"	"	"	.247	2
1353	H	E	E	"	5	"	"	"	"	.248	2
1354	I	E	E	3	5	15.00	"	"	"	.252	2
1355*	I	E	E	"	5	"	"	"	"	.251	2

⅜″ STEEL

No. of Test.	Plate.	Covers.		Pitch.	No. of Rivets.	Width of Joint.	Nominal Plate.	Nominal Covers.	Size of Rivets and Holes.	Actual Thickness of Plate.	Lap.
1356	A	I	I	1⅝	6	9.75	3/8	1/4	$\frac{9}{16}$ & ⅝	.365	2
1357	A	I	I	"	6	"	"	"	"	.364	2
1358	A	I	J	1¾	6	10.49	"	"	"	.365	2
1359	A	J	J	"	6	"	"	"	"	.366	2
1360	A	I	J	"	6	10.50	"	"	$\frac{11}{16}$ & ¾	.366	2
1361	A	I	I	"	6	"	"	"	"	.367	2
1362	A	J	J	1⅞	6	11.25	"	"	"	.366	2
1363	A	J	J	"	6	"	"	"	"	.365	2
1364	B	K	K	2	6	12.00	"	"	"	.388	2
1365	B	K	K	"	6	"	"	"	"	.390	2
1366	C	K	K	2⅛	6	12.76	"	"	"	.367	2
1367	B	K	L	"	6	"	"	"	"	.387	2
1368	A	J	J	1⅞	6	11.25	"	"	$\frac{13}{16}$ & ⅞	.369	2
1369	A	J	J	"	6	"	"	"	"	.366	2
1370	B	K	K	2	6	12.00	"	"	"	.389	2
1371	B	J	J	"	6	"	"	"	"	.388	2
1372	B	L	L	2⅛	6	12.77	"	"	"	.385	2
1373	C	K	L	"	6	"	"	"	"	.367	2
1374	D	H	N	2¼	6	13.50	"	"	"	.376	2
1375	E	N	N	"	6	"	"	"	"	.380	2
1376	D	L	M	2⅜	6	14.23	"	"	"	.383	2
1377	H	L	M	"	6	"	"	"	"	.371	2

* Fractured two outside sections of plate at each edge along line

RIVETED BUTT-JOINTS—*Continued.*

PLATE—*Continued.*

Sectional Area of Plate. Gross.	Sectional Area of Plate. Net.	Bearing Surface of Rivets.	Shearing Area of Rivets.	Tensile Strength of Plate per Sq. In.	Maximum Stress on Joint per Sq. In. Tension on Gross Section of Plate.	Tension on Net Section of Plate.	Comp. on Bearing Surface of Rivets.	Shearing of Rivets.	Efficiency of Joint.
sq. in.	sq. in.	sq. in.	sq. in.	lbs.	lbs.	lbs.	lbs.	lbs.	
3.560	2.060	1.500	9.425	61470	39130	**67640**	92870	14780	63.6
3.520	2.038	1.482	9.425	59180	40450	**69860**	96070	15110	68.3
3.765	2.259	1.506	9.425	60480	43590	**72640**	108960	17410	72.1
3.780	2.268	1.512	9.425	60480	41420	**69030**	103540	16610	68.5
3.204	1.984	1.220	7.854	59180	38700	**62490**	101620	15790	65.4
3.131	1.936	1.195	7.854	62660	42890	**69370**	112380	17100	68.4
3.442	2.192	1.250	7.854	55740	42960	**67460**	118300	18830	77.1
3.426	2.181	1.245	7.854	55740	41780	**65640**	114980	18230	74.9
3.554	2.319	1.235	7.854	59180	43510	**66690**	125220	19590	73.5
3.569	2.329	1.240	7.854	59180	43830	**67170**	126160	19920	74.1
3.780	2.520	1.260	7.854	60480	44580	**66870**	133730	21450	73.7
3.765	2.510	1.255	7.854	60480	44410	66610	133230	21290	73.4

PLATE.

3.559	2.190	1.369	3.682	54260	40460	**65740**	105170	39100	74.6
3.549	2.184	1.365	3.682	54260	39420	**64060**	102490	38000	72.6
3.829	2.460	1.369	3.682	54260	39780	61910	111250	**41360**	73.3
3.843	2.471	1.372	3.682	54260	39060	**60740**	109400	**40770**	72.0
3.843	2.196	1.647	5.300	54260	37000	**64750**	86330	26830	68.2
3.854	2.202	1.652	5.300	54260	37050	**64840**	86430	26940	68.3
4.118	2.471	1.647	5.300	54260	37450	**62400**	93620	29090	69.0
4.106	2.464	1.642	5.300	54260	38040	**63390**	95130	29470	70.1
4.656	2.910	1.746	5.300	59730	41820	**66910**	111510	36730	70.0
4.680	2.925	1.755	5.300	59730	42000	**67200**	112000	37090	70.3
4.683	3.031	1.652	5.300	57870	41040	**63410**	116340	36260	70.9
4.938	3.197	1.741	5.300	59730	40910	**63180**	116030	**38110**	68.5
4.151	2.214	1.937	7.216	54260	35000	**65620**	75000	20130	64.5
4.114	2.192	1.922	7.216	54260	34180	**64140**	73150	19480	63.0
4.668	2.626	2.042	7.216	59730	36870	**65540**	84280	23850	61.7
4.656	2.619	2.037	7.216	59730	38940	**69220**	89000	25120	65.2
4.916	2.895	2.021	7.216	59730	38730	**65770**	94210	26390	64.8
4.672	2.745	1.927	7.216	57870	39010	**65660**	94580	25260	67.4
5.076	3.102	1.974	7.216	53730	37960	**62120**	97620	26700	70.6
5.130	3.135	1.995	7.216	58340	39810	**65140**	102360	28300	68.2
5.450	3.439	2.011	7.216	53730	38920	**61670**	105470	29390	72.4
5.290	3.342	1.948	7.216	56670	39870	**63110**	108260	29230	70.4

of riveting; the two middle sections sheared in front of rivets.

TABULATION OF SINGLE-

$\frac{3}{8}''$ STEEL

No. of Test.	Sheet Letters.			Pitch.	No. of Rivets.	Width of Joint.	Nominal Thickness.		Size of Rivets and Holes.	Actual Thickness of Plate.	Lap.
	Plate.	Covers.					Plate.	Covers.			
				in.		in.	in.	in.	in.	in.	in.
1378	D	M	M	$2\frac{1}{2}$	6	15.00	3/8	1/4	$\frac{13}{16}$ & $\frac{7}{8}$	.383	2
1379*	D	M	M	"	6	"	"	"	"	.385	2
1380	B	J	K	2	6	12.00	"	"	$\frac{15}{16}$ & 1	.388	2
1381	E	K	K	"	6	"	"	"	"	.381	2
1382	B	K	K	$2\frac{1}{8}$	6	12.75	"	"	"	.388	2
1383†	E	G	K	"	6	"	"	"	"	.383	2
1384	F	H	H	$2\frac{1}{4}$	6	13.49	"	"	"	.381	2
1385	E	N	N	"	6	"	"	"	"	.380	2
1386‡	H	L	L	$2\frac{3}{8}$	6	14.25	"	"	"	.368	2
1387	G	L	M	"	6	"	"	"	"	.365	2
1388	D	M	M	$2\frac{1}{2}$	6	15.00	"	"	"	.385	2
1389	D	M	M	"	6	"	"	"	"	.386	2
1390	C	G	G	$2\frac{5}{8}$	5	13.12	"	"	"	.372	2
1391§	C	H	L	"	5	"	"	"	"	.369	2
1392	C	L	L	$2\frac{3}{4}$	5	13.75	"	"	"	.374	2
1393	C	L	N	"	5	"	"	"	"	.372	2
1394§	D	I	M	$2\frac{7}{8}$	5	14.39	"	"	"	.386	2
1395	D	M	M	"	5	"	"	"	"	.383	2

* Test discontinued soon after passing maximum load.
† Test discontinued at maximum load.
‡ Test discontinued after passing maximum load.
§ Test discontinued before fracture was complete.

RIVETED BUTT-JOINTS—*Continued.*

PLATE—*Continued.*

Sectional Area of Plate.		Bearing Surface of Rivets.	Shearing Area of Rivets.	Tensile Strength of Plate per Square Inch.	Maximum Stress on Joint per Square Inch.				Efficiency of Joint.
Plate.	Covers.				Tension on Gross Section of Plate.	Tension on Net Section of Plate.	Compression on Bearing Surface of Rivets.	Shearing of Rivets.	
sq. in.	sq. in.	sq. in.	sq. in.	lbs.	lbs.	lbs.	lbs.	lbs.	
5.745	3.734	2.011	7.216	53730	40560	**62400**	115860	32290	75.5
5.775	3.754	2.021	7.216	53730	40700	**62620**	116280	32570	75.7
4.656	2.328	2.328	9.425	59730	34500	**69010**	69010	17050	57.8
4.572	2.286	2.286	9.425	58340	33440	**66880**	66880	16220	57.3
4.947	2.619	2.328	9.425	59730	35590	**67230**	75640	18680	59.6
4.883	2.585	2.298	9.425	58340	34730	65610	73800	17990	59.5
5.140	2.854	2.286	9.425	54290	35490	**63930**	79810	19360	65.4
5.126	2.846	2.280	9.425	58340	35840	**64550**	80570	19490	61.4
5.244	3.036	2.208	9.425	56670	37010	63930	87910	20590	65.3
5.205	3.015	2.190	9.425	53840	36750	**63450**	87350	20300	68.2
5.775	3.465	2.310	9.425	53730	37490	**62480**	93710	22970	69.8
5.790	3.474	2.316	9.425	53730	37360	**62260**	93390	21890	69.5
4.881	3.021	1.860	7.854	57870	39000	**63010**	102340	24240	67.4
4.841	2.996	1.845	7.854	57870	39520	**63850**	103690	24360	68.3
5.143	3.273	1.870	7.854	57870	39840	**62590**	109540	26080	68.9
5.111	3.251	1.860	7.854	57870	40420	**63550**	111070	26310	69.8
5.555	3.625	1.930	7.854	53730	39340	**60290**	113240	27830	73.2
5.496	3.581	1.915	7.854	53730	40300	**61850**	115660	28200	75.0

SINGLE-RIVETED BUTT-JOINTS, STEEL PLATE.

DESCRIPTION OF TESTS AND DISCUSSION OF RESULTS.

"The following tests complete a series of two hundred and sixteen single-riveted butt-joints in steel plates, in which the thickness of the plates ranged from $\frac{1}{4}''$ to $\frac{3}{4}''$, and the size of the rivets from $\frac{9}{16}''$ to $1\frac{3}{16}''$ diameter.

The plates were annealed after shearing to size, the edges opposite the joint milled to the finished width; the holes were drilled and rivets machine-driven. Iron rivets were used throughout, except in some of the $\frac{3}{4}''$ joints.

Tensile tests of the plates and rivet-metal, together with the tests of the joints in $\frac{1}{4}''$ and $\frac{3}{8}''$ plate, are contained in the Report of Tests of 1885, Senate Document No. 36, Forty-ninth Congress, first session.

The tests herewith presented comprise the details and tabulation of joints in $\frac{1}{2}''$, $\frac{5}{8}''$, and $\frac{3}{4}''$ thickness of plate, a portion of which were tested hot.

The gauged length in which elongations and sets were measured was $5''$; $2\frac{1}{2}''$ each side of the centre line of the joint.

During the progress of testing the same characteristics were displayed which were referred to in the previous report. The joints were very rigid under the early loads. This rigidity is overcome by loads which exceed the friction between the plate and covers, after which the stretching proceeded slowly with some fluctuations till elongation of the metal of the net section became general; the metal under compression in front of the rivets yielding, also the rivets themselves.

The behavior of joints in different thicknesses of plate is substantially the same, and an examination of the results shows that when exposed to similar conditions the strength per unit of fractured metal is nearly the same, whether $\frac{1}{4}''$ or $\frac{3}{4}''$ plate is used.

It will not be understood from this, however, that as a consequence the same efficiency may be obtained in different thicknesses of plate for single-riveted work, because it will be seen that certain essential conditions change as we approach the stronger joints in different thicknesses of plate.

A riveted joint of the maximum efficiency should fracture the plate along the line of riveting, for it is clear that if failure occurs in any other manner, as by shearing the rivets or tearing out the plate in front of the rivet-holes, there remains an excess of strength along the line of riveting, or in other words along the net section of metal—if in a single-riveted joint—which has not been made use of; but when fracture occurs along the net section an excess of strength in other directions is immaterial.

If the strength per unit of metal of the net section was constant, it would be a very simple matter to compute the efficiency of any joint, as it would merely be the ratio of the net to the gross areas of the plates.

The tenacity of the net section, however, varies, and this variation extends over wide limits.

In the present series there is an excess in strength of the net section over the strength of the tensile test-pieces in all joints.

Special tables have been prepared showing this behavior.

The efficiencies shown in Table No. 1 are obtained by dividing the tensile stress on the gross area of plate by the tensile strength of the plate as represented by the strength of the tensile test-strip, stating the values in per cent of the latter.

Table No. 2 exhibits the differences between the efficiencies of the joints and the ratios of net to gross areas of plate. If the tenacity of net section remained constant per unit of area, the efficiencies in Table No. 1 would, as above explained, be identical with the ratios of net to gross areas of plate, and the values in this table reduced to zero.

Table No. 3 shows the excess in strength of the net section of the joint over the strength of the tensile test-strip in per cent of the latter.

Table No. 4 exhibits the compression on the bearing-surface of the rivets in connection with the excess in tensile strength of the net section of plate.

Table No. 1 is valuable in showing at once the value of different joints wherein the pitch of the rivets and their diameters vary.

It is seen there is considerable latitude allowed in the choice of rivets and pitch without materially changing the efficiency of the joint; thus in $\frac{1}{4}''$ plate,

$\frac{5}{8}''$ rivets (driven), $1\frac{7}{8}''$ pitch, 72.4 per cent efficiency,
$\frac{3}{4}''$ rivets (driven), $2\frac{1}{4}''$ pitch, 73.3 per cent efficiency,
$\frac{7}{8}''$ rivets (driven), $2\frac{3}{8}''$ pitch, 71.5 per cent efficiency,
$1''$ rivets (driven), $2\frac{1}{2}''$ pitch, 70.3 per cent efficiency,
$1''$ rivets (driven), $2\frac{7}{8}''$ pitch, 73.8 per cent efficiency,

give nearly the same results.

In these examples the ratios of net to gross areas of plate range from 60 to 67 per cent, while the rivet-areas range from .3067 square inch to .7854 square inch. The actual areas of net sections of plate and rivets are as follows:

	$\frac{5}{8}''$ rivets.	$\frac{3}{4}''$ rivets.	$\frac{7}{8}''$ rivets.	$1''$ rivets.
	sq. in.	sq. in.	sq. in.	sq. in.
Rivets.	.3067	.4418	.6013	.7854
Plate	1.486	2.207	2.232	{ 2.259 { 2.319

The areas of the rivets stand to each other as the following numbers:

100 144 196 256

and the net areas of the plate to each other as

100 149 150 $\begin{cases} 152 \\ 156 \end{cases}$

From these illustrations it appears that to attain the same degree of efficiency in this quality of metal, although that efficiency is probably not the highest attainable, a fixed ratio between rivet metal and net section of plate is not essential.

In $\frac{1}{2}''$ plate with $\frac{7}{8}''$ rivets the efficiencies of the joints tested cold are nearly constant over the range of pitches tested.

The efficiencies and the ratio of net to gross areas of plate are as follows:

	Pitch.			
	$1\frac{7}{8}''$	$2''$	$2\frac{1}{8}''$	$2\frac{1}{4}''$
	per cent.	per cent.	per cent.	per cent.
Efficiency. . . .	64.5	66.3	66.3	66.4
Ratio of areas . .	53.4	56.3	58.9	61.1

In this we have illustrated a case which, in passing from the widest pitch, having 61.1 per cent of the solid plate left, to the narrowest pitch, which had 53.4 per cent of the solid plate, the gain or excess in strength in the net section almost exactly compensated for the loss of metal.

In Table No. 3 the average of all the joints shows the highest per cent of excess of strength in the narrowest pitch, and a tendency to lose this excess as the pitch increases.

Tests of detached grooved specimens show the same kind of behavior, but as they are not subject to all the conditions found in a joint, the analogy does not extend very far.

The maximum gain in strength on the net section, not for the time being regarding the hot joints, and disregarding the exceptionally high value of joint No. 1339, $\frac{1}{4}''$ plate, was 21.2 per cent, the minimum value 2.5 per cent of the tensile test-strip. In other forms of joints, and with punched holes in both iron and steel plate, illustrations are numerous in which there have been large deficiencies, the metal of the net section falling far below the strength of the plate.

It is believed to have been amply shown that increasing the net width diminishes the apparent tenacity of the plate, although other influences may tend to counteract this tendency in some joints.

In order to compare the excess in strength of one thickness of plate with another having the same net widths, we have the following table, rejecting those joints that failed otherwise than along the line of riveting in making these averages:—

Thickness of Plate.	Width of plate between rivet-holes.								
	$1''$	$1\frac{1}{8}''$	$1\frac{1}{4}''$	$1\frac{3}{8}''$	$1\frac{1}{2}''$	$1\frac{5}{8}''$	$1\frac{3}{4}''$	$1\frac{7}{8}''$	$2''$
	P. ct.	P. ct.	P. ct.	P. ct.	P. ct.	P. ct.	P. ct.	P. ct.	P. ct.
$\frac{1}{4}''$	16.7	12.6	11.4	12.0	13.4	8.9	11.5	13.1	10.6
$\frac{3}{8}''$	18.4	13.7	12.7	13.5	14.6	12.9	9.0	13.6	....
$\frac{1}{2}''$	16.7	14.3	9.3	10.7	9.1	8.8	8.2	12.2	....
$\frac{5}{8}''$	17.7	16.3	14.2	14.5	14.6	12.7	9.9	9.8	....
$\frac{3}{4}''$	11.4	15.1	13.8	14.1	7.6	11.8	10.0	10.1	3.5
Average of all thicknesses	16.2	14.4	12.3	12.9	11.9	11.0	9.7	11.8	7.0

The excess in strength is generally well maintained in each of the several thicknesses, and were it possible to retain the same ratio of net to gross areas of plate, and at the same time equal net widths between rivets, it would seem from this point of view feasible to obtain the same degree of efficiency in thick as in thin plates.

The following causes, however, tend to prevent such a consummation.

For equal net widths thick plates require larger rivets to avoid shearing than thin ones, the diameters of the rivets being somewhat increased for this cause, and again because it has become necessary to increase the metal of the net section in order to retain a suitable ratio of net to gross areas of plate.

There results from these considerations such an increase in net width of plate that the excess in strength displayed by narrower sections is lost, and consequently the result is a joint of lower efficiency.

The data relating to the influence of compression on the bearing-surface of the rivets, on the tensile strength of the plate, as shown by Table No. 4 are more or less conflicting. However, in the $\frac{1}{4}''$ plate, in which the most intense pressures are found, there is seen a pronounced increase in tensile strength as the pressures diminish in intensity.

It is probable that the effects of intense compression would be more conspicuous in a less ductile metal, or one in which the ductility had been impaired by punched holes or otherwise.

A number of joints were tested at temperatures ranging between 200° and 700° Fahr.

The heating was done after the joints were in position for testing, by means of Bunsen burners, arranged in a row parallel to and under the line of riveting.

The temperature was determined with a mercurial thermometer, the bulb of which was immersed in a bath of oil, contained in a pocket drilled in the middle rivet of the joint.

When at the required temperature the thermometer was removed from the joint, a dowel was driven into the pocket to compensate for the metal of the rivet which had been removed by the drill, and then loads applied and gradually increased up to the time of rupture.

Three joints, Nos. 1423, 1426, and 1430, were tested without dowels in the oil-pockets.

The method of heating was to raise the temperature of the joint, as shown by the thermometer, a few degrees above the temperature at which the test was made, shut off the gas-burners, and allow the temperature to fall to the required limit. The temperature fell slowly, draughts of cold air being excluded from the under side of the joint by the hood which covered the gas-burners; the upper side and edges of the joint were covered with fine dry coal-ashes.

The results show an increase in tensile strength when heated over the duplicate cold joints at each temperature except 200° Fahr.

From 200° there was a gain in strength up to 300°, when the resistance fell off some at 350°, increased again at 400°, and reached the maximum effect observed at 500° Fahr.; from this point the strength fell rapidly at 600° and 700°.

In per cent of the cold joint there was a loss at 200° of 3.2 per cent, the average of three joints; at 500° the gain was 22.6 per cent, the average of four joints. The maximum and minimum joints at this temperature showed gains of 27.6 per cent, and 18.3 per cent, respectively.

The highest tensile strength on the net section of plate was found in joint No. 1433, tested at 500° Fahr., where 81050 pounds per square inch was reached against a strength of 58000 pounds per square inch in the cold tensile test-strip.

The hot joints showed less ductility than the cold ones, those tested at 200° Fahr. not being exempt from this behavior, although there was no near approach to brittleness in any.

Three joints, Nos. 1418, 1420, and 1424, were heated;

strained when hot with loads exceeding the ultimate strength of their duplicate cold joints; the loads were released, and after having cooled to the temperature of the testing-room (No. 1424 cooled to 150° Fahr.) were tested to rupture, and were found to have retained substantially the strength due their temperature when hot.

In order to ascertain that the time intervening between hot straining and final rupture did not contribute towards the elevation in strength, joint No. 1434 was strained in a similar manner with a load approaching rupture, after which a period of rest was allowed and then ruptured without material gain in strength.

A peculiarity of the joints fractured at 400° and higher temperatures was the comparatively smooth surface of the fractured sections, and which took place in planes making angles of about 50° with the rolled surface of the plate.

The shearing-strength of the iron rivets was also increased by an elevation of temperature.

The rivets in joint No. 1410 at the temperature of 350° sheared at 43060 pounds per square inch, while in the duplicate cold joint No. 1411 they sheared at 38530 pounds per square inch, and the rivets in joint No. 1398 at 300° Fahr. were loaded with 46820 pounds per square inch and did not shear.

Other examples, where some of the rivets sheared and the plate fractured in part, showed corresponding gains in shearing-strength.

The almost entire absence of granular fractures in these tests is a feature too important to pass by without special mention."

TABULATION OF SINGLE-

STEEL PLATE.

No. of Test.	Sheet Letters.			Pitch.	No. of Rivets.	Width of Joint.	Nominal Thickness.		Size of Rivets and Holes.	Actual Thickness of Plate.	Lap.
	Plate.	Covers.					Plate.	Covers.			
				in.		in.	in.	in.	in.	in.	in.
1396	R	R	R	1¾	6	10.50	1/2	5/16	$\frac{11}{16}$ & ¾	.481	2
1397	R	R	R	"	6	"	"	"	"	.484	2
1398	R	R	R	1⅞	6	11.25	"	"	"	484	2
1399	R	R	R	"	6	"	"	"	"	.483	2
1400	R	R	R	2	6	12.00	"	"	"	.486	2
1401	R	S	T	"	6	"	"	"	"	.483	2
1402	R	R	R	1⅞	6	11.25	"	"	$\frac{13}{16}$ & ⅞	.481	2
1403	R	R	R	"	6	"	"	"	"	.486	2
1404	R	R	R	2	6	12.00	"	"	"	.486	2
1405	R	S	S	"	6	"	"	"	"	.487	2
1406	S	S	S	2⅛	6	12.75	"	"	"	.470	2
1407	S	S	S	"	6	"	"	"	"	.471	2
1408	T	Q	Q	2¼	6	13.50	"	"	"	.486	2
1409	T	Q	Q	"	6	"	"	"	"	.482	2
1410	T	O	O	2⅜	6	14.25	"	"	"	.481	2
1411	T	O	O	"	6	"	"	"	"	.485	2
1412	R	R	R	2	6	12.00	"	"	$\frac{15}{16}$ & 1	.484	2
1413	R	R	R	"	6	"	"	"	"	.481	2
1414	S	S	S	2⅛	6	12.75	"	"	"	.472	2
1415	S	S	S	"	6	"	"	"	"	.468	2
1416	S	Q	Q	2¼	6	13.50	"	"	"	.468	2
1417	T	Q	Q	"	6	"	"	"	"	.482	2
1418	T	O	O	2⅜	6	14.25	"	"	"	.481	2
1419	T	O	O	"	6	"	"	"	"	.482	2
1420	U	P	P	2½	6	15.00	"	"	"	.479	2
1421	U	P	P	"	6	"	"	"	"	.483	2
1422	S	S	S	2⅝	5	13.13	"	"	"	.469	2
1423	S	S	S	"	5	"	"	"	"	.473	2
1424	T	O	O	2¾	5	13.75	"	"	"	.484	2
1425	T	O	O	"	5	"	"	"	"	.483	2
1426	S	S	S	2⅛	6	12.75	"	"	$1\frac{1}{16}$ & 1⅛	.474	2
1427	R	S	S	"	6	"	"	"	"	.475	2
1428	T	Q	Q	2¼	6	13.50	"	"	"	.479	2
1429	S	O	Q	"	6	"	"	"	"	.465	2
1430	U	O	O	2⅜	6	14.25	"	"	"	.484	2
1431	U	O	O	"	6	"	"	"	"	.483	2

RIVETED BUTT-JOINTS.

STEEL PLATE.

Sectional Area of Plate. Gross.	Sectional Area of Plate. Net.	Bearing Surface of Rivets.	Shearing Area of Rivets.	Tensile Strength of Plate per Sq. In.	Max. Stress on Joint per Sq. In. Tension on Gross Section of Plate.	Max. Stress on Joint per Sq. In. Tension on Net Section of Plate.	Max. Stress on Joint per Sq. In. Comp. on Bearing Surface of Rivets.	Max. Stress on Joint per Sq. In. Shearing of Rivets.	Efficiency of Joint.	Temp. of Joint in Deg. Fahr.
sq in.	sq. in.	sq. in.	sq. in.	lbs.	lbs.	lbs.	lbs.	lbs.		
5.051	2.886	2.165	5.301	57180	37750	**66070**	88080	35970	66.1	200
5.082	2.904	2.178	5.301	57180	38980	**68220**	90960	37370	68.1	....
5.445	3.267	2.178	5.301	57180	45560	75960	113960	**46820**	79.6	300
5.439	3.266	2.173	5.301	57180	39260	**65400**	98290	**40290**	68.6	....
5.842	3.655	2.187	5.301	57180	37000	59140	98830	**40770**	64.7	....
5.796	3.622	2.174	5.301	57180	39420	63080	105100	**43100**	68.9	350
5.416	2.891	2.525	7.216	57180	36890	**69110**	79130	27690	64.5	...
5.477	2.926	2.551	7.216	57180	38250	**71600**	81730	29030	66.9	250
5.832	3.281	2.551	7.216	57180	37910	**67380**	86670	30640	66.3	....
5.854	3.297	2.557	7.216	57180	43730	**77650**	100120	35480	76.4	300
5.997	3.529	2.468	7.216	59050	44790	**76110**	108830	37220	76.0	400
6.010	3.537	2.473	7.216	59050	39210	**66630**	95310	32660	66.3	....
6.561	4.010	2.551	7.216	60000	39850	**65210**	102500	36230	66.4	....
6.512	3.982	2.530	7.216	60000	46610	**76220**	119980	**42060**	77.6	500
6.859	4.334	2.525	7.216	60000	45300	71690	123050	**43060**	75.5	350
6.916	4.370	2.546	7.216	60000	40050	63390	108800	**38530**	66.7	....
5.813	2.909	2.904	9.425	57180	35920	**71770**	71900	22150	62.8	250
5.772	2.886	2.886	9.425	57180	34390	**68780**	68780	21060	60.1	....
6.023	3.191	2.832	9.425	59050	35000	**66020**	74430	22360	59.2	....
5.967	3.159	2.808	9.425	59050	40250	**76030**	85540	25480	68.1	300
6.327	3.519	2.808	9.425	59050	34660	**62320**	78090	23160	60.3	200
6.512	3.620	2.892	9.425	60000	36950	**66480**	83220	25530	61.5	..
6.859	3.973	2.886	9.425	60000	43710	**75460**	103880	31810	72.8	(*)
6.883	3.991	2.892	9.425	60000	38720	**66770**	92150	28270	64.5	....
7.194	4.320	2.874	9.425	58000	44840	**74670**	112250	34230	77.3	(†)
7.245	4.347	2.898	9.425	58000	38740	**64570**	96850	29780	66.9	...
6.167	3.822	2.345	7.854	59050	39730	**64110**	104490	31200	67.2	....
6.220	3.855	2.365	7.854	59050	45420	**73250**	119450	**35840**	76.9	400
6.660	4.240	2.420	7.854	60000	48950	**76890**	137110	**41510**	81.5	(‡)
6.632	4.217	2.415	7.854	60000	40600	**63860**	111490	34280	67.6	....
6.053	2.853	3.200	11.928	59050	35070	**74410**	66340	18630	59.3	300
6.042	2.836	3.206	11.928	59050	30420	**64810**	57330	15410	51.5	....
6.471	3.238	3.233	11.928	60000	40330	**80620**	80730	21880	67.2	350
6.278	3.139	3.139	11.928	59050	33420	**66840**	66840	17590	56.5	....
6.897	3.620	3.277	11.928	58000	36390	**65150**	76590	21040	62.7	700
6.892	3.632	3.260	11.928	58000	33660	**63870**	71160	19450	58.0	....

* Strained while at temperature of 400° Fahr., and allowed to cool before rupture.

† Strained while at temperature of 500° Fahr., and allowed to cool before rupture.

‡ Strained while at temperature of 500° Fahr., then cooled to 150° Fahr., and ruptured.

TABULATION OF SINGLE-

STEEL PLATE—*Continued.*

No. of Test.	Sheet Letters.			Pitch.	No. of Rivets.	Width of Joint.	Nominal Thickness.		Size of Rivets and Holes.	Actual Thickness of Plate.	Lap.
	Plate.	Covers.					Plate.	Covers.			
				in.		in.	in.	in.	in.	in.	in.
1432	U	P	P	$2\frac{1}{2}$	6	15.00	1/2	5/16	$1\frac{1}{16}$ & $1\frac{1}{8}$	.484	2
1433	U	P	P	"	6	"	"	"	"	.481	2
1434	R	S	Q	$2\frac{5}{8}$	5	13.13	"	"	"	.472	2
1435	S	S	S	"	5	"	"	"	"	.475	2
1436	T	O	O	$2\frac{3}{4}$	5	13.75	"	"	"	.482	2
1437	T	O	O	"	5	"	"	"	"	.482	2
1438	U	P	P	$2\frac{7}{8}$	5	14.38	"	"	"	.484	2
1439	U	P	P	"	5	"	"	"	"	.485	2
1440	U	P	P	3	5	15.00	"	"	"	.482	2
1441	U	P	P	"	5	"	"	"	"	.483	2
1442	U	P	P	$3\frac{1}{8}$	5	15.68	"	"	"	.483	2
1443	U	P	P	"	5	"	"	"	"	.484	2
1444	V	E	E	$1\frac{7}{8}$	6	11.25	5/8	3/8	$\frac{13}{16}$ & $\frac{7}{8}$	.621	2
1445	V	B	E	"	6	"	"	"	"	.624	2
1446	V	E	E	2	6	12.00	"	"	"	.616	2
1447	V	E	E	"	6	"	"	"	"	.624	2
1448	V	E	E	$2\frac{1}{8}$	6	12.75	"	"	"	.621	2
1449	V	E	E	"	6	"	"	"	"	.624	2
1450	W	F	G	$2\frac{1}{4}$	6	13.50	"	"	"	.610	2
1451	W	G	G	"	6	"	"	"	"	.611	2
1452	V	E	E	2	6	12.00	"	"	$\frac{15}{16}$ & 1	.624	2
1453	V	E	E	"	6	"	"	"	"	.620	2
1454	V	E	E	$2\frac{1}{8}$	6	12.75	"	"	"	.622	2
1455	V	E	E	"	6	"	"	"	"	.618	2
1456	W	G	F	$2\frac{1}{4}$	6	13.50	"	"	"	.612	2
1457	W	G	G	"	6	"	"	"	"	.611	2
1458	W	I	I	$2\frac{3}{8}$	6	14.25	"	"	"	.610	2
1459	W	H	H	"	6	"	"	"	"	.608	2
1460	X	I	I	$2\frac{1}{2}$	6	15.00	"	"	"	.617	2
1461	X	J	I	"	6	"	"	"	"	.618	2
1462	V	F	F	$2\frac{5}{8}$	5	13.13	"	"	"	.630	2
1463	W	F	F	"	5	"	"	"	"	.608	2
1464	V	E	E	$2\frac{1}{8}$	6	12.75	"	"	$1\frac{1}{16}$ & $1\frac{1}{8}$	.624	2
1465	V	D	E	"	6	"	"	"	"	.623	2
1466	W	G	G	$2\frac{1}{4}$	6	13.50	"	"	"	.613	2
1467	W	N $\frac{7}{16}''$	G	"	6	"	"	"	"	.606	2

RIVETED BUTT-JOINTS—*Continued.*

STEEL PLATE—*Continued.*

Sectional Area of Plate.		Bearing Surface of Rivets.	Shearing Area of Rivets.	Tensile Strength of Plate per Square Inch.	Maximum Stress on Joint per Square Inch.				Efficiency of Joint.	Temperature of Joint in Degrees Fahrenheit.
Gross.	Net.				Tension on Gross Section of Plate.	Tension on Net Section of Plate.	Compression on Bearing Surface of Rivets.	Shearing of Rivets.		
sq. in	sq. in.	sq. in.	sq. in.	lbs.	lbs.	lbs.	lbs.	lbs.		
7.270	4.043	3.227	11.928	58000	36140	**65000**	81430	22030	62.3	....
7.215	3.968	3.247	11.928	58000	44490	**81050**	99040	26960	76.7	500
6.193	3.538	2.655	9.940	59050	36670	**64190**	85540	22850	62.0	....
6.232	3.560	2.672	9.940	59050	36200	**63370**	84420	22690	61.2	....
6.632	3.921	2.711	9.940	60000	42430	**71610**	103580	28250	70.5	600
6.632	3.921	2.711	9.940	60000	38720	**65440**	94730	25840	64.5	....
6.965	4.243	2.722	9.940	58000	46630	**76550**	119320	32680	80.3	500
6.974	4.246	2.728	9.940	58000	38900	**63890**	99400	27290	67.0	....
7.230	4.519	2.711	9.940	58000	39180	**62690**	104500	28500	67.5	200
7.250	4.528	2.722	9.940	58000	40360	**65060**	108220	29630	70.0	....
7.564	4.837	2.717	9.940	57410	38570	60450	**107610**	29410	67.2	....
7.565	4.843	2.722	9.940	57410	39160	61170	**108830**	29800	68.2	....
6.986	3.726	3.260	7.216	55000	33750	**63280**	72330	32670	60.1	....
7.020	3.744	3.276	7.216	55000	34530	**64740**	74000	33590	62.7	....
7.392	4.158	3.234	7.216	55000	36760	**65340**	84010	**37650**	66.0	....
7.488	4.212	3.276	7.216	55000	35120	**62440**	80280	36440	63.8	....
7.918	4.658	3.260	7.216	55000	41930	71270	101840	**46010**	76.2	300
7.956	4.680	3.276	7.216	55000	36800	62560	89370	**40570**	66.9	....
8.241	5.039	3.202	7.216	57290	39320	64290	101180	**44900**	68.6	400
8.249	5.042	3.207	7.216	57290	36850	60290	94790	**42130**	64.3	600
7.488	3.744	3.744	9.425	55000	32080	**64150**	64150	25480	58.3	....
7.440	3.720	3.720	9.425	55000	32060	**64110**	64110	25300	58.3	....
7.931	4.199	3.732	9.425	55000	34120	**64440**	72510	28710	60.0	...
7.880	4.172	3.708	9.425	55000	34000	**64220**	72250	28420	61.8	....
8.262	4.590	3.672	9.425	57290	36490	**65680**	82110	32000	63.6	....
8.249	4.583	3.666	9.425	57290	36020	**64830**	81040	31520	62.8	....
8.662	5.002	3.660	9.425	57290	37720	**65310**	89260	38490	65.8	....
8.664	5.016	3.648	9.425	57290	37540	**64850**	89170	34510	65.5	....
9.255	5.553	3.702	9.425	55940	37300	**62160**	93250	**36630**	66.6	...
9.282	5.574	3.708	9.425	55940	37000	**61610**	92620	**36440**	66.1	....
8.259	5.109	3.150	7.854	55000	35780	57840	93810	**37620**	65.0	....
7.965	4.925	3.040	7.854	57290	36960	59770	96840	**37500**	64.5	....
7.950	3.738	4.212	11.928	55000	31090	**66130**	58690	20720	56.5	....
7.949	3.744	4.205	11.928	55000	31090	**66020**	58780	20720	56.5	....
8.269	4.131	4.138	11.928	57290	33150	**66350**	66240	22980	57.8	....
8.181	4.090	4.091	11.928	55940	33240	**66250**	66240	22720	58.0	....

TABULATION OF SINGLE-

STEEL PLATE—*Continued.*

No. of Test.	Sheet Letters. Plate	Covers.		Pitch.	No. of Rivets.	Width of Joint.	Nominal Thickness. Plate.	Covers.	Size of Rivets and Holes.	Actual Thickness of Plate.	Lap.
				in.		in.	in.	in.	in.	in.	in.
1468	W	H	I	$2\frac{3}{8}$	6	14.25	5/8	3/8	$1\frac{1}{16}$ & $1\frac{1}{8}$	.613	2
1469	W	I	N $\frac{7}{10}''$	"	6	"	"	"	"	.609	2
1470	X	J	I	$2\frac{1}{2}$	6	15.00	"	"	"	.619	2
1471	X	J	I	"	6	"	"	"	"	.616	2
1472	V	K	....	$2\frac{5}{8}$	5	13.13	"	"	"	.628	2
1473	W	F	F	"	5	"	"	"	"	.609	2
1474	W	H	D	$2\frac{3}{4}$	5	13.75	"	"	"	.609	2
1475	W	G	....	"	5	"	"	"	"	.610	2
1476	W	I	I	$2\frac{7}{8}$	5	14.38	"	"	"	.610	2
1477	W	I	I	"	5	"	"	"	"	.609	2
1478	X	I	J	3	5	15.00	"	"	"	.616	2
1479	X	I	J	"	5	"	"	"	"	.623	2
1480	G	E	E	$3\frac{1}{8}$	4	12.50	"	"	"	.625	2
1481	H	E	E	"	4	"	"	"	"	.621	2
1482	Z	K	K	2	6	12.00	3/4	7/16	$\frac{15}{16}$ & 1	.736	2
1483	Z	K	K	"	6	"	"	"	"	.757	2
1484	Z	K	K	$2\frac{1}{8}$	6	12.75	"	"	"	.742	2
1485	Z	P	P	"	6	"	"	"	"	.762	2
1486	Z	L	M	$2\frac{1}{4}$	6	13.50	"	"	"	.749	2
1487	Z	L	M	"	6	"	"	"	"	.764	2
14 8	Z	N	N	$2\frac{3}{8}$	6	14.25	"	"	"	.745	2
1489	Z	O	N	"	6	"	"	"	"	.735	2
1490	Z	K	....	$2\frac{1}{8}$	6	12.75	"	"	$1\frac{1}{16}$ & $1\frac{1}{8}$	.723	2
1491	Z	P	P	"	6	"	"	"	"	752	2
1492	Z	M	L	$2\frac{1}{4}$	6	13.50	"	"	"	.736	2
1493	Z	M	M	"	6	"	"	"	"	.754	2
1494	Z	N	O	$2\frac{3}{8}$	6	14.25	"	"	"	.760	2
1495	Z	O	O	"	6	"	"	"	"	.760	2
1496	Y	Q	P	$2\frac{1}{2}$	6	15.00	"	"	"	.745	2
1497	Y	P	P	"	6	"	"	"	"	.725	2
1498	Z	L	L	$2\frac{5}{8}$	5	13.13	"	"	"	.733	2
1499	Z	L	L	"	5	"	"	"	"	.744	2
1500	Z	N	M	$2\frac{3}{4}$	5	13.75	"	"	"	.762	2
1501	Z	N	N	"	5	"	"	"	"	.727	2
1502	Z	O	P	$2\frac{7}{8}$	5	14.38	"	"	"	.722	2
1503	Z	O	O	"	5	"	"	"	"	.741	2

RIVETED BUTT-JOINTS—*Continued.*

STEEL PLATE—*Continued.*

Sectional Area of Plate. Gross.	Sectional Area of Plate. Net.	Bearing Surface of Rivets.	Shearing Area of Rivets.	Tensile Str'gth of Plate per Sq. In.	Max. Stress on Joint per Sq. In. Tension on Gross Section of Plate.	Tension on Net Section of Plate.	Comp. on Bearing Surface of Rivets.	Shearing of Rivets.	Efficiency of Joint.	Temperature of Joint in Deg. Fahr.
sq. in.	sq. in.	sq. in.	sq. in.	lbs.	lbs.	lbs.	lbs.	lbs.		
8.735	4.597	4.138	11.928	57250	34260	**65110**	72330	25090	59.8	
8.690	4.579	4.111	11.928	"	34790	**66030**	73540	25350	60.7	
9.285	5.107	4.178	11.928	55940	34980	**63600**	77740	27230	62.5	
9.240	5.082	4.158	11.928	"	34770	**63220**	77270	26930	62.1	
8.239	4.706	3.533	9.940	55000	36350	**63640**	84770	30130	66.1	
7.978	4.552	3.426	9.940	57290	37100	**65020**	86400	29780	64.7	
8.362	4.936	3.426	9.940	"	38150	**64630**	93110	32090	66.5	
8 381	4.950	3.431	9.940	"	38120	**64540**	93120	32140	66.5	
8.833	5.402	3.431	9.940	"	38620	**63140**	99410	34310	67.4	
8.739	5.313	3.426	9.940	"	38180	**62830**	97430	33580	66.6	
9.240	5.775	3.465	9.940	55940	38480	**61570**	102630	**35770**	68.7	
9.345	5.841	3.504	9.940	"	38410	**61430**	102440	36110	68.6	
7.813	5.000	2.813	7.952	55000	37340	58360	103730	**36690**	67.9	
7.763	4.968	2.795	7.952	"	38440	60060	106760	**37520**	69.9	
8.847	4.431	4.416	9.425	59000	31990	**63870**	64090	30030	54.2	
9.099	4.557	4.542	9.425	"	31980	**63860**	64070	30870	54.2	
9.475	5.023	4.452	9.425	"	34440	**64960**	73920	34620	58.3	
9.723	5.151	4.572	9.425	"	34700	**67340**	73790	**35800**	58.8	
10.112	5.618	4.494	9.425	"	35000	63000	78750	**37550**	59.3	
10.329	5.745	4.584	9.425	"	36780	**66130**	82870	40310*	62.3	
10.624	6.154	4.470	9.425	"	38120	**65810**	90600	42970*	64.6	
10.488	6.078	4.410	9.425	"	34000	58670	80860	**37830**	57.6	
9.233	4.353	4.880	11.928	"	31050	**65860**	58750	24030	52.6	
9.596	4.520	5.076	11.928	"	32000	**67940**	65340	25740	54.2	
9.951	4.983	4.968	11.928	"	34270	**68430**	68640	28590	58.0	
10.179	5 082	5.090	11.928	"	33770	**67540**	67520	28810	57.2	
10 845	5.715	5.130	11.928	"	34900	**66230**	73780	31730	59.1	
10.838	5.708	5.130	11.928	"	35810	**67990**	75650	32540	60.6	
11.175	6.146	5.029	11.928	60420	38470	**69940**	85480	36040	63.6	
10.890	5.996	4.894	11.928	"	37740	**68650**	83980	34460	62.4	
9.624	5.501	4.123	9.940	59000	35000	**61230**	81700	**33890**	57.0	
9 776	5.572	4.204	10.030	"	36470	**63990**	84810	35550	61.7	
10.478	6.192	4.286	9.940	"	38760	**65590**	94750	40850*	65.7	
10.004	5.915	4.089	9.940	"	36740	**62130**	89880	**36970**	62.2	
10.390	6.329	4.061	9.940	"	37930	**62270**	97050	39650*	64.3	
10.663	6.495	4.168	9.940	"	40630	**65810**	90600	42970*	68.8	

* Steel rivets.

TABULATION OF SINGLE-

STEEL PLATE—*Continued.*

No. of Test.	Sheet Letters.			Pitch.	No. of Rivets.	Width of Joint.	Nominal Thickness.		Size of Rivets and Holes.	Actual Thickness of Plate.	Lap.
	Plate.	Covers.					Plate.	Covers.			
				in.		in.	in.	in.	in.	in.	in.
1504	Z	M	L	$2\frac{1}{4}$	6	13.50	3/4	7/16	$1\frac{3}{16}$ & $1\frac{1}{4}$	.722	2
1505	Z	M	M	"	6	"	"	"	"	.762	2
1506	Y	N	O	$2\frac{3}{8}$	6	14.25	"	"	"	.727	2
1507	Y	N	O	"	6	"	"	"	"	.735	2
1508	Y	Q	Q	$2\frac{1}{2}$	6	15.00	"	"	"	.737	2
1509	Y	Q	Q	"	6	"	"	"	"	.753	2
1510	Y	L	L	$2\frac{5}{8}$	5	13.13	"	"	"	.748	2
1511	Y	L	L	"	5	"	"	"	"	.755	2
1512	Z	...	....	$2\frac{3}{4}$	5	13.75	"	"	"	.750	2
1513	Z	N	N	"	5	"	"	"	"	.764	2
1514	Y	O	O	$2\frac{7}{8}$	5	14.13	"	"	"	.760	2
1515	Y	O	O	"	5	"	"	"	"	.746	2
1516	....	P	P	3	5	15.00	"	"	"	.749	2
1517	Y	P	Q	"	5	"	"	"	"	.741	2
1518	Z	K	K	$3\frac{1}{8}$	4	12.50	"	"	"	.756	2
1519	Z	K	K	"	4	"	"	"	"	.741	2
1520	Z	K	K	$3\frac{1}{4}$	4	13.00	"	"	"	.763	2
1521	Z	K	K	"	4	"	"	"	"	.718	2
1522	Z	M	M	$3\frac{3}{8}$	4	13.50	"	"	"	.742	2
1523	Z	M	M	"	4	"	"	"	"	.754	2

RIVETED BUTT-JOINTS—*Continued.*

STEEL PLATE—*Continued.*

Sectional Area of Plate.		Bearing Surface of Rivets.	Shearing Area of Rivets.	Tensile Str'gth of Plate per Sq. In.	Max. Stress on Joint per Sq. In.				Efficiency of Joint.	Temperature of Joint in Deg. Fahr.
Gross.	Net.				Tension on Gross Section of Plate.	Tension on Net Section of Plate.	Comp. on Bearing Surface of Rivets.	Shearing of Rivets.		
sq. in.	sq. in.	sq. in.	sq. in.	lbs.	lbs.	lbs.	lbs.	lbs.		
6.761	4.346	5.415	14.726	59000	29090	**65350**	52460	19280	49.3	
10.287	4.572	5.715	14.726	59000	30010	**67520**	54010	20960	50.8	
10.367	4.914	5.453	14.726	60420	33610	**70900**	63890	23660	55.6	
10.474	4.961	5.513	14.726	60420	33660	**71070**	63960	23940	55.7	
11.070	5.542	5.528	14.726	60420	34780	**69470**	69650	26140	57.5	
11.310	5.662	5.648	14.726	60420	34670	**69250**	69420	26620	57.4	
9.918	5.243	5.675	12.272	60420	36120	**68380**	76680	29210	59.7	
9.928	5.209	4.719	12.272	60420	36940	**70400**	77710	29880	61.1	
10.328	5.640	4.688	12.272	59000	33730	**61770**	74320	28390	57.0	
10.505	5.730	4.775	12.272	59000	35260	**64640**	77570	30100	59.7	
10.929	6.179	4.750	12.272	60420	37930	**67080**	87260	36220	62.7	
10.735	6.072	4.663	12.272	60420	38720	**68460**	89150	33870	64.0	
11.205	6.524	4.681	12.272	55520	36530	**62740**	87440	36610	65.8	
11.108	6.477	4.631	12.272	60430	38740	66440	92920	**35060**	64.1	
9.465	5.685	3.780	9.818	59000	37560	62360	93780	**36110**	63.6	
9.277	5.572	3.705	9.818	59000	39000	**64930**	97650	36850*	66.1	
9.934	6.119	3.815	9.818	59000	37600	**61040**	97900	38040*	63.7	
9.348	5.758	3.590	9.818	59000	36000	58440	93740	**34280**	61.0	
10.032	6.322	3.710	9.818	59000	40040	**63540**	**108270**	40910*	67.7	
10.187	6.417	3.770	9.818	59000	39720	63050	**107320**	41210*	67.3	

* Steel rivets.

TABLE

TABLE OF EFFICIENCIES OF

STEEL PLATE.

Plate.	No. of Test.	Pitch of Rivets.				
		1⅝″	1¾″	1⅞″	2″	2⅛″
		per cent.	per cent.	per cent.	per cent.	per cent.
¼″	1308	67.5	68.9	72.8	....	....
	1313	68.3	69.9	72.1	....	....
	1314	....	65.9	66.4	68.3	68.7
	1323	....	64.8	67.1	68.6	70.3
	1324	....	....	62.6	64.4	67.5
	1337	....	....	63.9	65.9	68.2
	1338	....	....	....	64.8	61.7
	1355	....	....	....	58.7	60.6
⅜″	1356	74.6	**73.3**	....	....	....
	1359	72.7	**72.0**	....	....	....
	1360	....	68.2	69.0	70.0	70.9
	1367	....	68.3	70.1	70.3	**68.5**
	1368	....	....	64.5	61.7	64.8
	1379	....	....	63.0	65.2	67.4
	1380	....	....	....	58.7	59.6
	1395	....	....	....	57.3	59.5
½″	1396	....	^{200}66.1	^{300}79.6	64.7	....
	1401	....	68.1	**68.6**	^{350}68.9	....
	1402	....	....	64.5	66.3	^{400}76.0
	1411	....	....	^{250}66.9	^{300}76.4	66.3
	1412	....	....	....	^{250}62.8	59.2
	1425	...	....	....	60.1	^{300}68.1
	1426	....	....	....	....	^{300}59.3
	1443	....	....	....	....	51.5
⅝″	1444	....	....	60.1	**66.0**	^{300}76.2
	1451	....	....	62.7	63.8	**66.9**
	1452	....	....	....	58.3	60.0
	1463	....	....	...	58.3	61.8
	1464	....	...	...	....	56.5
	1481	....	....	....	....	56.5
¾″	1482	....	....	...	54.2	58.3
	1489	....	....	....	54.2	58.8
	1490	....	....	....	....	52.6
	1503	....	....	....	....	54.2
	1504	....	....	....	....	....
	1523	....	....	....	....	....

NOTES.—Figures in heavy-face type denote that
Super numbers state the temperature of

NO. 1.

SINGLE-RIVETED BUTT-JOINTS.

STEEL PLATE.

Pitch of Rivets.										Diameter of Rivet-holes.
2¼″	2⅜″	2½″	2⅝″	2¾″	2⅞″	3″	3⅛″	3¼″	3⅜″	
per ct.	per ct.	per ct.	per ct.	per ct.	per ct.	per ct.	per ct.	per ct.	per ct.	in.
....	....	....	....	....	....	....	....	....	....	⅝
....	....	....	....	....	....	....	....	....	....	⅝
72.7	....	....	....	....	....	....	....	....	....	¾
74.0	....	....	....	....	...	....	....	....	....	¾
70.1	70.7	76.8	69.8	....	....	....	....	....	....	⅞
70.0	72.3	67.1	68.3	....	....	....	....	....	....	⅞
61.5	63.7	72.1	65.4	77.1	73.5	73.7	....	....	....	1
63.4	68.4	68.5	68.4	75.0	74.1	**73.4**	....	....	....	1
....	....	....	....	....	....	....	....	....	....	⅝
....	....	....	....	....	....	....	....	....	....	⅝
....	....	....	....	....	....	....	....	....	....	¾
....	....	....	....	....	....	....	....	....	....	¾
70.7	72.4	75.5	....	....	....	....	....	....	....	⅞
68.2	70.4	75.6	....	....	....	....	....	....	....	⅞
65.4	65.3	69.8	67.4	68.8	73.2	....	....	....	....	1
64.1	68.3	69.5	68.3	69.8	75.0	....	....	....	....	1
....	....	....	....	....	....	....	....	....	....	¾
...	....	....	....	....	....	....	....	....	....	¾
66.4	350 **75.5**	....	....	....	....	....	....	....	....	⅞
500 77.6	**66.7**	....	....	....	....	....	....	...	....	⅞
200 60.3	400 72.8	500 77.3	67.2	500 81.5	...	....	....	....	....	1
61.5	64.5	66.9	400 76.9	67.6	....	....	....	....	.. .	1
350 67.2	700 62.7	62.3	62.0	600 70.5	500 80.3	200 67.5	**67.2**	....	....	1⅛
56.5	58.0	500 76.7	61.2	64.5	67.0	70.0	**68.2**	....	....	1⅛
400 68.6	....	....	....	....	....	....	....	....	....	⅞
600 64.3	...	...	....	....	....	...	....	....	....	⅞
63.6	65.8	**66.6**	**65.0**	....	....	....	....	....	....	1
62.8	65.5	**66.1**	**64.5**	....	....	....	....	....	....	1
57.8	59.8	62.5	66.1	66.5	67.1	68.7	**67.9**	...	....	1⅛
58.0	60.7	62.1	64.7	66.5	66.6	68.6	**69.9**	....	....	1⅛
59.3	64.6	....	...	....	....	....	....	....	....	1
62.3	**57.6**	....	...	...	....	....	....	....	....	1
58.0	59.1	63.6	**57.0**	65.7	64.3	...	....	...	....	1⅛
57.2	60.6	62.4	61.7	**62.2**	68.8	....	....	....	....	1⅛
49.3	55.6	57.5	59.7	57.0	62.7	65.8	**63.6**	63.7	**67.7**	1¼
50.8	55.7	57.4	61.1	59.7	64.0	**64.1**	66.1	**61.0**	**67.3**	1¼

joint did not fracture along line of riveting.

oints tested at temperatures above atmospheric.

TABLE NO. 2.

TABLE OF DIFFERENCES BETWEEN THE EFFICIENCIES AND RATIOS OF NET TO GROSS AREAS.—SINGLE-RIVETED BUTT-JOINTS, STEEL PLATE.

Plate.	No. of Test.	Width of Plate between Rivet Holes.									Diameter of Rivet Holes.
		1″	1⅛″	1¼″	1⅜″	1½″	1⅝″	1¾″	1⅞″	2″	
in.	in.	per ct.	per ct.	per ct.	per ct.	per ct.	per ct.	per ct.	per ct.	per ct.	in.
¼	1308	6.0	4.6	6.1	....	....	....	....	....	....	⅝
	1313	6.8	5.6	5.4	...	....	....	....	....	....	⅝
	1314	8.8	6.3	7.2	4.0	7.5	....	....	....	....	¾
	1323	7.7	7.0	6.1	5.6	8.8	....	....	....	...	¾
	1324	8.0	8.1	8.6	11.7	7.6	11.8	3.1	....	....	⅞
	1337	9.3	9.6	9.4	8.9	9.2	2.1	1.6	....	....	⅞
	1338	14.8	8.8	5.9	5.8	12.1	3.5	13.4	8.2	7.0	1
	1355	18.8	7.6	7.8	10.5	8.5	6.6	11.3	8.8	**6.7**	1
⅜	1356	13.1	**9.1**	...	....	....	....	....	....	....	⅝
	1359	11.2	**7.7**	....	....	....	...	....	....	....	⅝
	1360	11.1	9.0	7.5	6.2	....	....	...	....	....	¾
	1367	11.2	10.1	7.8	**3.8**	....	....	....	....	....	¾
	1368	11.2	5.4	5.9	9.6	9.4	10.5	....	....	....	⅞
	1379	9.7	8.9	8.6	7.1	7.2	10.6	....	....	....	⅞
	1380	6.8	6.7	9.9	7.4	9.8	5.5	5.2	7.9	....	1
	1395	7.3	6.6	5.9	10.4	9.5	6.4	6.2	9.8	....	1
½	1396	^{200} 9.0	^{300} 19.6	2.1	....	....	....	....	....	...	¾
	1401	11.1	**8.6**	^{350} 6.4	....	....	....	....	....	....	¾
	1402	11.1	10.0	^{400} 17.2	5.3	^{350} **12.3**	....	...	....	....	⅞
	1411	^{250} 13.5	^{300} 20.1	7.4	^{500} 16.4	**3.5**	....	....	....	....	⅞
	1412	^{250} 12.8	6.2	^{200} 4.7	^{400} 14.9	^{500} 17.2	5.2	^{500} 17.8	....	....	1
	1425	10.1	^{300} 15.2	5.9	6.5	6.9	^{400} 14.9	4.0	....	....	1
	1426	^{300} 12.2	^{350} 17.2	^{700} 10.2	6.9	4.9	^{600} 11.4	^{500} 19.4	^{200} 5.0	**3.2**	1⅛
	1443	4.6	6.5	5.3	^{500} 21.7	4.1	5.4	6.1	7.5	**4.2**	1⅛
⅝	1444	6.4	**9.7**	^{300} 17.4	^{400} 7.5	...	....	....	....	....	⅞
	1451	9.4	7.5	**8.1**	^{600} 3.2	....	....	....	....	....	⅞
	1452	8.3	7.1	8.0	8.1	**6.6**	**3.1**	....	....	....	1
	1463	8.3	8.9	7.2	7.6	**6.0**	**2.7**	....	....	....	1
	1464	9.5	7.8	7.2	7.5	9.0	7.5	6.2	**6.2**	**3.9**	1⅛
	1481	9.4	8.0	8.0	7.1	7.6	7.4	5.8	6.1	**5.9**	1⅛
¾	1482	4.1	5.3	**3.7**	6.7	....	..	....	....	....	1
	1489	4.1	5.8	6.6	**−0.4**	....	....	....	...	....	1
	1490	5.5	7.9	6.4	8.6	**−.02**	6.6	3.4	....	....	1⅛
	1503	7.1	7.2	7.9	7.3	4.7	**3.1**	7.9	...	....	1⅛
	1504	4.8	8.2	7.4	6.8	2.4	6.2	7.6	**3.5**	2.1	1¼
	1521	6.4	8.3	7.3	8.6	5.1	7.4	**5.8**	6.0	**−0.6**	1¼
										2⅛″	
										per ct.	
	....	....	....	....	....	....	....	....	....	**4.7**	
	....	....	....	....	....	...	....	....	....	**4.3**	

NOTES.—Figures in heavy-faced type denote that joint did not fracture along line of riveting Super numbers state the temperature of joints tested at temperatures above atmospheric.

TABLE NO. 3.

EXCESS IN STRENGTH OF NET SECTION IN JOINT OVER STRENGTH OF TENSILE TEST-STRIP.—SINGLE-RIVETED BUTT-JOINTS, STEEL PLATE.

Plate.	No. of Test.	Width of Plate between Rivet Holes.									Diameter of Rivet Holes.
		1″	1⅛″	1¼″	1⅜″	1½″	1⅝″	1¾″	1⅞″	2″	
in.		per ct.	per ct.	per ct.	per ct.	per ct.	per ct.	per ct.	per ct.	per ct.	in.
¼	1308	9.8	7.2	9.1	....	....	....	....	...	....	⅝
	1313	11.1	8.8	8.1	....	....	....	....	....	....	⅝
	1314	15.4	10.5	9.2	6.2	9.1	...	....	....	....	¾
	1323	13.4	11.7	9.7	8.5	11.0	....	....	....	....	¾
	1324	17.4	14.4	14.7	14.7	11.9	16.1	4.7	....	....	⅞
	1337	19.7	17.0	15.9	14.6	14.4	3.2	2.5	....	....	⅞
	1338	29.6	16.5	10.6	10.0	20.1	5.6	21.0	12.7	10.6	1
	1355	17.6	14.5	14.2	18.0	14.1	10.7	17.7	13.5	**10.1**	1
⅜	1356	21.2	**14.1**	....	....	....	....	....	....	....	⅝
	1359	18.1	**11.9**	....	....	....	....	....	...	....	⅝
	1360	19.3	15.0	12.0	9.6	....	....	....	....	....	¾
	1367	19.5	16.8	12.5	**5.8**	....	....	....	...	....	¾
	1368	20.9	9.7	10.1	15.6	14.9	16.1	....	...	....	⅞
	1379	18.2	15.9	13.5	11.7	11.4	16.5	....	....	....	⅞
	1380	15.5	12.6	17.8	12.8	16.3	8.9	8.2	12.2	....	1
	1395	14.6	12.5	10.6	17.9	15.9	10.3	9.8	15.1	....	1
½	1396	^{200} 15.6	^{300} 32.8	3.4	....	....	...	....	....	....	¾
	1401	15.8	**14.4**	^{350} 10.3	....	....	....	....	....	....	¾
	1402	20.9	17.8	^{400} 28.9	8.7	^{350} **19.5**	....	....	....	....	⅞
	1411	^{250} 25.2	^{300} 35.8	12.8	^{500} 27.0	**5.6**	....	....	....	....	⅞
	1412	^{250} 25.5	11.8	^{200} 5.5	^{400} 25.8	^{500} 28.7	8.6	^{500} 28.1	....	....	1
	1425	20.3	^{300} 28.8	10.8	11.3	11.3	^{400} 24.0	6.4	....	...	1
	1426	^{300} 26.0	^{350} 34.4	^{700} 12.3	12.1	8.7	^{600} 19.4	^{500} 32.0	^{200} 8.1	**5.3**	1⅛
	1443	9.8	13.2	10.1	^{500} 39.7	7.3	9.1	10.1	12.2	**6.5**	1⅛
⅝	1444	15.1	**18.8**	^{300} 29.6	^{400} 12.2	....	....	....	....	....	⅞
	1451	17.7	13.5	**13.7**	^{600} 5.2	....	....	....	....	...	⅞
	1452	16.6	17.2	14.6	14.0	**11.1**	**5.2**	....	....	....	1
	1463	16.6	16.8	13.2	13.2	**10.1**	**4.3**	....	....	....	1
	1464	20.2	15.8	13.6	13.7	15.7	12.8	10.2	**10.1**	**6.1**	1⅛
	1481	20.0	18.4	15.3	13.0	13.5	12.7	9.7	9.8	**9.2**	1⅛
¾	1482	8.3	10.1	**6.8**	11.5	....	....	....	....	....	1
	1489	8.2	**14.1**	12.1	**−0.6**	....	....	....	....	....	1
	1490	11.6	16.0	12.3	15.8	**3.8**	11.2	5.5	....	....	1⅛
	1503	15.2	14.5	15.2	13.6	8.5	**5.3**	11.5	....	....	1⅛
	1504	10.8	17.4	15.0	13.2	4.7	11.0	13.0	**5.7**	3.5	1¼
	1521	14.4	17.6	14.6	16.5	9.6	13.3	**10.0**	10.1	**−0.9**	1¼
										2⅛″	1¼
										per ct.	
	....	....	....	....	....	....	....	....	....	**7.7**	1¼
	....	....	....	....	....	....	....	....	....	**6.9**	1¼
Average of all joints.		16.2	14.4	12.3	12.9	11.9	11.0	9.7	11.8	7.0	

NOTES.—Figures in heavy-faced type denote that joint did not fracture along line of riveting.
Super numbers state the temperature of joints tested at temperatures above atmospheric.

In the Report of Tests made at Watertown Arsenal during the fiscal year ended June 30, 1891, is the following account of another series of tests on riveted joints:

"Comprised in the present report are 113 tests made with steel plates of 1/4", 5/16", 3/8", and 7/16" thickness with iron rivets machine driven in drilled or punched holes.

"The plates used were from material used in earlier tests, the results of which have been published in previous reports.

"In the use of metal once before tested, such plates were selected as had not been overstrained previously, or those in which the elastic limit had been but very slightly exceeded.

SINGLE-RIVETED

STEEL PLATE.

No. of Test.	Sheet Letters.			Pitch.	No. of Rivets.	Width of Joint.	Nominal Thickness.		Size and Kind of Holes.	Actual Thickness of Plate.	Lap.
	Plate.	Covers.					Plate.	Covers.			
				in.		in.	in.	in.	in.	in.	in.
4913	H	C	D	2¾	5	13.72	1/4	3/16	7/8 d	.247	2
4914	L	D	D	"	5	13.69	"	"	" "	.248	"
4915	M	E	E	2⅞	5	14.32	"	"	" "	.247	"
4916	M	D	D	"	5	14.33	"	"	" "	.247	"
4917	M	E	E	3	5	15.00	"	"	" "	.246	"
4918	M	E	E	"	5	14.98	"	"	" "	.247	"
4985	Q	D	D	3½	4	14.00	5/16	"	1 "	.309	1
4987	S	C	. . .	"	4	14.01	"	"	" "	.310	1¼
4991	Q	[illegible]	....	"	4	14.05	"	"	" "	.308	1¾
5125	R	A	A	1	10	10.02	5/16	"	1/2 "	.306	1½
5126	R	A	....	1¼	8	10.02	"	"	" "	.304	"
5127	R	B	....	1½	7	10.51	"	"	" "	.310	"
5143	L	P	....	2	7	14.03	7/16	5/16	7/8 p.	.440	1¾
5144	L	O	O	"	7	14.01	"	"	" d.	.440	"
5145	O	Q	Q	2¼	6	13.50	"	"	" "	.434	"
5146	M	O	....	"	6	13.51	"	"	" p.	.421	"
5147	O	P	P	2½	6	15.02	"	"	" d.	.413	"
5148	N	P	....	"	6	15.02	"	"	" p.	.411	"
5155	K	S	....	2¾	5	13.75	"	"	" d.	.425	"

"The present tests are supplementary to those of earlier reports, and occupy a place intermediate between the elementary forms of joints and the more elaborate types of joints which have been investigated.

"Wide variation has been given the pitches, and rivets of extreme diameters have been used for the purpose of including joints in which these features have been carried to their extreme limits.

"The efficiencies of the joints are stated in per cent of strength of the solid plate."

BUTT-JOINTS.

STEEL PLATE.

Sectional Area of Plate.		Bearing Surface of Rivets.	Shearing Area of Rivets.	Tensile Str'gth of Plate per Sq. In.	Maximum Stress on Joint per Sq. In.				Efficiency of Joint.
Gross.	Net.				Tension on Gross Section of Plate.	Tension on Net Section of Plate.	Compression on Bearing Surface of Rivets.	Shearing of Rivets.	
sq. in.	sq. in.	sq. in.	sq. in.	lbs.	lbs.	lbs.	lbs.	lbs.	
3.39	2.31	1.08	6.01	59180	44180	**65760**	140650	25270	75.7
3.40	2.31	1.09	6.01	61470	43500	**64030**	135690	24610	70.8
3.54	2.46	1.08	6.01	58170	46300	66630	**151780**	27270	79.6
3.54	2.46	1.08	6.01	58170	43500	62590	**142570**	25620	74.8
3.69	2.61	1.08	6.01	58170	46290	65440	**158150**	28420	79.6
3.70	2.62	1.08	6.01	58170	44400	62700	**152110**	27330	76.3
4.33	3.09	1.24	6.28	56760	24040	33690	**83950**	16580	42.3
4.34	3.10	1.23	6.28	57000	26770	37480	**93710**	18500	46.9
4.33	3.10	1.23	6.28	56760	33940	47410	**119500**	23400	59.8
3.07	1.54	1.53	3.92	61130	35930	**71620**	72090	28140	58.8
3.05	1.83	1.22	3.14	61130	41280	**68800**	103200	**40100**	67.5
3.26	2.17	1.09	2.74	61130	39250	58960	117380	**46690**	64.2
6.17	3.38	2.79	8.41	59390	32540	**59410**	71970	23880	54.8
6.16	3.48	2.69	8.41	59390	23360	**41350**	53490	17110	30.3
5.86	3.58	2.28	7.21	52910	42250	**60160**	108600	34340	79.8
5.69	3.40	2.29	7.21	61650	39740	**66500**	98730	31360	64.5
6.20	4.03	2.17	7.21	52910	44150	**67920**	126130	37960	83.4
6.17	3.94	2.23	7.21	61650	36660	**57410**	101430	31370	60.0
5.84	3.98	1.86	6.01	59000	40270	59100	126450	**39130**	68.3

TABULATION OF SINGLE-

STEEL PLATE.

No. of Test.	Sheet Letters.		Pitch.	No. of Rivets.	Width of Joint.	Nominal Thickness.		Size and Kind of Holes.	Actual Thickness of Plate.	Lap.
	Plate.	Plate.				Plate.	Plate.			
			in.		in.	in.	in.	in.	in.	in.
4933	I	J	2⅛	5	10.62	1/4	1/4	⅞ d.	.252	2
4934	J	J	"	5	10.65	"	"	"	.253	2
4939	L	K	2⅞	4	11.50	"	"	1 d.	.250	2
4940*	J	J	"	4	11.50	"	"	"	.256	2
4941	K	J	"	4	11 51	"	"	1⅛ d.	.252	2
4942	K	J	"	4	11.52	"	"	"	.250	2
4943	K	K	"	4	11.50	"	"	1¼ d.	.252	2
4944	E	J	"	4	11.50	"	"	"	.253	2
4945	K	K	3⅛	4	12.52	"	"	"	.248	2
4946	L	G	"	4	12.55	"	"	"	.253	2
4947†	N	H	"	4	13.52	"	"	"	.247	2
4948‡	N	H	"	4	13.52	"	"	"	.247	2
4949	M	L	3⅝	4	14.51	"	"	"	.248	2
4950*	M	L	"	4	14.51	"	"	"	.247	2
4961	B	E	1¾	6	10.52	3/8	3/8	¾ d.	.388	2
4979	E	E	2⅜	5	11.84	"	"	1 d.	.384	2
5131	K	K	1½	8	12.00	7/16	7/16	⅞ d.	.427	1.75
5132	N	O	"	8	12.00	"	"	⅞ p.	.415	1.75
5133*	K	K	1⅝	8	13.00	"	"	⅞ d.	.427	1.75
5134	N	N	"	8	13.00	"	"	⅞ p.	.413	1.75
5135	M	M	1¾	8	14.03	"	"	⅞ d.	.422	1.75
5136	....	....	"	8	13.99	"	"	⅞ p.	.420	1.75
5137	L	M	2	7	14.02	"	"	⅞ d.	.4[illegible]4	1.75
5138	P	M	"	7	14.05	"	"	⅞ p.	.420	1.75
5139	O	K	"	6	12.06	"	"	1¼ d.	.428	2
5140	M	M	2⅜	6	14.28	"	"	"	.421	2
5141*	L	L	2¾	5	13.73	"	"	"	.438	2
5142*	Q	Q	3⅛	5	15.67	"	"	"	.422	2

* Pulled off rivet-heads.
† Pulled off 3 rivet-heads.
‡ Pulled off 2 rivet-heads.

RIVETED LAP-JOINTS.

STEEL PLATE.

Sectional Area of Plate.		Bearing Surface of Rivets.	Shearing Area of Rivets.	Tensile Strength of Plate per Sq. In.	Maximum Stress on Joint per Sq. In.				Efficiency of Joint.
Gross.	Net.				Tension on Gross Section of Plate.	Tension on Net Section of Plate.	Comp. on Bearing Surface of Rivets.	Shearing of Rivets.	
sq. in.	sq. in.	sq. in.	sq. in.	lbs.	lbs.	lbs.	lbs.	lbs.	
2.68	1.57	1.10	3.00	61000	39750	67850	96840	35510	65.1
2.70	1.59	1.11	3.00	61000	39660	67360	96490	35700	65.0
2.87	1.87	1.00	3.14	58150	40560	62250	116400	37070	69.7
2.94	1.92	1.02	3.14	61000	37010	56670	106670	34650	60.6
2.90	1.77	1.13	3.98	61000	43280	70900	111060	31530	70.9
2.88	1.75	1.13	3.98	58150	42770	73880	119010	30950	73.5
2.90	1.64	1.26	4.91	58150	41130	72730	94660	24290	70.7
2.91	1.64	1.27	4.91	58150	40200	71330	92110	23820	69.1
3.10	1.86	1.24	4.91	58150	40030	66720	100080	25270	69.1
3.17	1.91	1.26	4.91	61470	41770	69320	105080	26970	68.0
3.34	2.10	1.24	4.91	55740	42240	67180	112970	28730	75.7
3.34	2.10	1.24	4.91	59180	42600	67760	114760	28980	71.9
3.60	2.36	1.24	4.91	61470	41390	63140	120180	30350	67.3
3.58	2.35	1.23	4.91	58170	42150	64210	122680	30730	72.4
4.08	2.33	1.75	2.65	58340	25950	45440	60500	39950	44.4
4.55	2.63	1.92	3.93	58340	33050	57190	78330	38270	56.6
5.12	2.13	2.99	4.81	59000	31740	76290	54350	33780	53.8
4.99	1.98	3.01	4.81	52910	27820	70100	46110	28860	52.6
5.55	2.56	2.99	4.81	59000	31100	67420	57730	35880	52.7
5.37	2.37	2.99	4.81	61140	30370	68820	54550	33910	49.7
5.90	2.95	2.95	4.81	61650	29240	58490	58490	35870	47.4
5.89	2.95	2.94	4.81	.....	31870	63630	63840	39020	....
5.94	3.35	2.60	4.21	59390	27580	48900	63000	38910	46.4
5.90	3.24	2.66	4.21	52910	28530	51940	63270	39980	53.9
5.16	1.95	3.21	7.36	58090	28190	74610	45320	19770	48.5
6.01	2.85	3.16	7.36	61650	34850	73490	66280	28460	56.5
6.01	3.28	2.74	6.14	59390	33560	61490	73610	32850	56.5
6.61	3.97	2.64	6.14	56960	30420	50650	76170	32750	53.4

TABULATION OF DOUBLE-CHAIN-RIVETING—STEEL PLATE.

No. of Test.	Sheet Letters. Plate.	Sheet Letters. Covers.		Pitch.	Distance Apart of Rows, Centre to Centre.	Total Number of Rivets.	Width of Joint.	Nominal Thickness. Plate.	Nominal Thickness. Covers.	Size and Kind of Holes.	Actual Thickness of Plate.	Lap.
				in.	in.		in.	in.	in.	in.	in.	in.
4911	K	C	C	2 5/8	2 1/8	10	13.10	1/4	3/16	5/8 d.	.253	1 13/16
4912	L	C	C	"	"	10	13.10	"	"	"	.253	"
4919	L	E	D	2 7/8	2 3/8	10	14.32	"	"	7/8 d.	.247	1 15/16
4920	L	E	D	"	"	10	14.32	"	"	"	.249	"
4921	K	C	B	3 1/8	"	8	12.52	"	"	"	.252	"
4922	K	C	C	"	"	8	12.49	"	"	"	.252	"
4923	J	A	A	3 7/8	"	6	11.57	"	"	"	.257	"
4924	J	A	....	"	"	6	11.53	"	"	"	.255	"
4925	L	B	....	4 3/8	"	6	13.09	"	"	"	.251	"
4926	K	C	B	"	"	6	13.10	"	"	"	.230	"
5128	R	C	C	1 3/4	2	14	12.27	5/16	3/16	1/2 d.	.304	1 3/4
5129	Q	C	....	2	"	14	14.00	"	"	"	.305	"
5130	S	D	....	2 1/4	"	12	13.58	"	"	"	.307	"
4993	Q	E	....	3 1/2	1 1/4	8	14.05	"	"	1 d.	.309	1 1/4
4995	Q	E	....	"	1 3/4	8	14.06	"	"	"	.305	1 1/2
4997	Q	E	....	"	2	8	14.08	"	"	"	.308	1 3/4
4951	B	I	I	2 1/8	2 1/4	8	8.52	3/8	1/4	3/4 d.	.392	1 7/8
4952	E	R	....	"	"	8	8.51	"	5/6	"	.383	"
4953	E	R	....	2 5/8	"	8	10.51	"	"	"	.388	"
4954	E	N	H	"	"	8	10.03	"	1/4	"	384	"
4955	E	S	....	3 1/8	"	8	12.50	"	5/16	"	.383	"
4957	H	O	O	3 5/8	"	8	14.51	"	"	"	.369	"
4958	H	M	M	"	"	8	14.52	"	1/4	"	.369	"
4959	B	S	S	4 1/8	"	6	12.42	"	5/16	"	.388	1 7/8
4960	E	L	N	"	"	6	12.42	"	1/4	"	.384	"
4967	C	M	M	2 1/8	2 1/8	10	14.38	"	"	1 d.	.375	2
4969	E	M	...	3 3/8	"	8	13.50	"	"	"	.382	"
4970	F	....	....	"	"	8	13.58	"	5/16	"	.380	"
4971	J	P	P	3 7/8	"	8	15.46	"	"	"	.379	"
4973	E	S	S	4 3/8	"	6	13.50	"	"	"	.385	"
4975	I	O	P	4 7/8	"	6	14.65	"	"	"	.373	"
4977	J	P	P	5 3/8	"	6	16.08	"	"	"	.379	"
4956	K	N	N	3 1/8	2 1/4	8	12.48	7/16	1/4	3/4 d.	.427	1 7/8
4968	N	O	....	2 7/8	2 1/8	10	14.41	"	5/16	1 d.	.409	2

RIVETED BUTT-JOINTS.

CHAIN-RIVETING—STEEL PLATE.

Sectional Area of Plate.		Bearing Surface of Rivets.	Shearing Area of Rivets.	Tensile Strength of Plate per Square Inch.	Maximum Stress on Joint per Sq. In.				Efficiency of Joint.
Gross.	Net.				Tension on Gross Section of Plate.	Tension on Net Section of Plate.	Compression on Bearing Surface of Rivets.	Shearing on Rivets.	
sq. in.	sq. in.	sq. in.	sq. in.	lbs.	lbs.	lbs.	lbs.	lbs.	
3.31	2.52	1.58	6.14	58150	49090	**64480**	102850	26470	84.4
3.31	2.52	1.58	6.14	61470	51960	**68250**	108860	28010	84.5
3.54	2.46	2.16	12.02	61470	46810	**67370**	76720	13790	76.1
3.57	2.48	2.18	12.02	61470	45700	**65790**	74840	13570	74.3
3.16	2.27	1.76	9.62	58150	46330	**64490**	83180	15220	79.6
3.15	2.27	1.76	9.62	58150	46730	**64850**	83640	15300	80.3
2.97	2.30	1.35	7.21	61000	49520	**63940**	108930	20400	81.2
2.94	2.27	1.34	7.21	61000	49460	**64050**	108510	20170	81.1
3.29	2.63	1.32	7.21	61470	51440	**64350**	128210	23470	83.7
3.01	2.41	1.21	7.21	58150	55500	**69320**	138070	23170	95.4
3.73	2.66	2.13	5.49	61130	46690	**66650**	83240	32300	76.4
4.27	3.20	2.14	5.49	56760	49040	**65430**	97850	38140	86.4
4.15	3.23	1.84	4.70	57000	46480	59720	104840	**41040**	81.5
4.34	3.11	2.47	12.57	56760	44740	**62430**	78600	15450	78.8
4.29	3.07	2.44	12.57	56760	45490	**63570**	79980	15530	80.1
4.34	3.10	2.46	12.57	56760	45530	**63740**	80330	15720	80.2
3.34	2.16	2.35	7.07	59730	43290	**66940**	61520	20450	72.4
3.26	2.11	2.30	7.07	58340	42380	**65470**	60070	19540	72.6
4.08	2.91	2.33	7.07	58340	46590	**65330**	81590	26890	79.8
3.85	2.70	2.30	7.07	58340	49130	**70060**	82240	26750	84.2
4.79	3.64	2.30	7.07	58340	48610	**63970**	101250	32940	83.3
5.35	4.25	2.21	7.07	56670	48500	**61060**	117420	36700	85.5
5.36	4.25	2.21	7.07	56670	47700	60160	115700	**36170**	84.1
4.82	3.95	1.75	5.30	59730	43070	52560	118630	**39170**	72.1
4.77	3.91	1.73	5.30	58340	42520	51870	117230	**38260**	72.8
5.39	3.52	3.75	15.71	57870	42890	**65680**	61650	14720	74.1
5.16	3.63	3.06	12.57	58340	44260	**62920**	74640	18170	75.9
5.16	3.64	3.04	12.57	54290	43240	**61290**	73390	17750	79.6
5.86	4.34	3.03	12.57	57130	44910	**60650**	86860	20940	78.6
5.20	4.04	2.31	9.42	58340	45980	**59180**	103510	25380	78.8
5.46	4.35	2.24	9.42	59030	46720	**58640**	113880	27080	79.1
6.09	4.96	2.27	9.42	57130	44650	**54830**	119800	28870	78.1
5.33	4.05	2.56	7.07	59000	48120	63300	100190	**36280**	83.3
5.89	3.85	4.09	15.71	61140	43360	**66340**	62440	16260	70.9

TABULATION OF RIVETED

DOUBLE-RIVETED LAP-JOINTS.

No. of Test.	Sheet Letters of Plates and Covers.	Pitch.	Dist. apart of Rows at rt. ang. to line of Rivets.	No. of Rivets in Front Row.	No. of Rivets in Second Row.	No. of Rivets in Third Row.	Width of Joint.	Nominal Thickness. Plates.	Nominal Thickness. Covers.	Size and Kind of Holes.	Actual Thickness of Plate.	Lap.
		in.	in.				in.	in.	in.	in.	in.	in.
4935	J-J	$2\frac{1}{8}$	$2\frac{5}{8}$	5	5		10.68	1/4		7/8 d.	.257	$1\frac{15}{16}$
4936	I-J	"	"	5	5		10.53	"		" "	.251	"
4937	L-M	$2\frac{7}{8}$	"	5	5		14.38	"		" "	.248	"
4938	I-L	"	"	5	5		14.40	"		" "	.249	"
4999	Q-Q	$3\frac{1}{2}$	$2\frac{1}{4}$	4	4		14.02	5/16		1 "	.305	2
5000	P-P	"	"	4	4		14.00	"		1 p.	.306	"
4963	E-E	$1\frac{3}{4}$	"	6	6		10.50	3/8		3/4 d.	.387	$1\frac{7}{8}$
4965	E-E	2	"	6	6		12.00	"		" "	.384	"
4981	E-D	$2\frac{5}{8}$	$2\frac{1}{2}$	5	5		11.83	"		1 "	.385	2
4983	D-H	$2\frac{7}{8}$	"	5	5		14.36	"		1 "	.370	"
5149	M-L	2	$2\frac{3}{8}$	7	7		14.00	7/16		7/8 "	.425	$1\frac{7}{16}$
5150	M-M	"	"	7	7		14.00	"		" p.	.423	"
5151	K-K	$2\frac{1}{4}$	"	6	6		13.53	"		" d.	.428	"
5152	L-O	"	"	6	6		13.50	"		" p.	.440	$1\frac{1}{4}$
5153	O-O	$2\frac{1}{2}$	"	6	6		15.01	"		" d.	.409	$1\frac{7}{16}$
5154	O-O	"	"	6	6		15.02	"		" p.	.412	"
5156	K-K	$2\frac{3}{4}$	"	5	5		13.77	"		" d.	.422	"

DOUBLE-RIVETED BUTT-JOINTS.

No. of Test.	Sheet Letters of Plates and Covers.	Pitch.	Dist. apart of Rows at rt. ang. to line of Rivets.	No. of Rivets in Front Row.	No. of Rivets in Second Row.	No. of Rivets in Third Row.	Width of Joint.	Nominal Thickness. Plates.	Nominal Thickness. Covers.	Size and Kind of Holes.	Actual Thickness of Plate.	Lap.
4927	MDE	$2\frac{7}{8}$	$2\frac{5}{8}$	5	4		14.36	1/4	3/16	7/8 d.	.250	$1\frac{15}{16}$
4928	LDE	"	"	5	4		14.35	"	"	" "	.247	"
4929	KBC	$3\frac{1}{8}$	"	4	3		12.51	"	"	" "	.255	"
4930	KC	"	"	4	3		12.50	"	"	" "	.251	"
4931	HDD	$4\frac{3}{8}$	"	3	2		13.12	"	"	" "	.248	"
4932	HCC	"	"	3	2		13.12	"	"	" "	.246	"

DOUBLE-RIVETED LAP-JOINTS.

No. of Test.	Sheet Letters of Plates and Covers.	Pitch.	Dist. apart of Rows at rt. ang. to line of Rivets.	No. of Rivets in Front Row.	No. of Rivets in Second Row.	No. of Rivets in Third Row.	Width of Joint.	Nominal Thickness. Plates.	Nominal Thickness. Covers.	Size and Kind of Holes.	Actual Thickness of Plate.	Lap.
5119	O-Q	$3\frac{1}{2}$	$1\frac{1}{2}$	4	3		14.00	5/16		1 d.	.303	$1\frac{1}{4}$
5120	P-P	"	"	4	3		14.03	"		1 p.	.305	"
5121	Q-O	"	$1\frac{3}{4}$	4	3		14.03	"		1 d.	.302	$1\frac{1}{2}$
5122	O-O	"	"	4	3		14.03	"		1 p.	.304	"
5123	O-O	"	2	4	3		14.02	"		1 d.	.302	$1\frac{3}{4}$
5124	P-P	"	"	4	3		14.02	"		1 p.	.307	"

TREBLE-RIVETED LAP-JOINTS.

No. of Test.	Sheet Letters of Plates and Covers.	Pitch.	Dist. apart of Rows at rt. ang. to line of Rivets.	No. of Rivets in Front Row.	No. of Rivets in Second Row.	No. of Rivets in Third Row.	Width of Joint.	Nominal Thickness. Plates.	Nominal Thickness. Covers.	Size and Kind of Holes.	Actual Thickness of Plate.	Lap.
5157	KK	$2\frac{3}{4}$	$2\frac{3}{8}$	5	5	5	13.14	7/16		7/8 d.	.432	$1\frac{11}{16}$
5158	OP	3	"	5	5	5	15.05	"		" "	.412	"
5159	PP	$3\frac{1}{4}$	"	4	4	4	12.78	"		" "	.432	"
5160	LL	$3\frac{1}{8}$	"	4	4	4	13.50	"		" "	.438	"

JOINTS.—STEEL PLATE.

CHAIN-RIVETING.

Sectional Area of Plate.		Bearing Surface of Rivets.	Shearing Area of Rivets.	Tensile Strength of Plate per Sq. In.	Maximum Stress on Joint per Sq. In.				Efficiency of Joint.
Gross.	Net.				Tension on Gross Section of Plate	Tension on Net Section of Plate.	Compression on Bearing Surface of Rivets.	Shearing on Rivets.	
sq. in.	sq. in.	sq. in.	sq. in.	lbs.	lbs.	lbs.	lbs.	lbs.	
2.74	1.62	2.25	6.01	61000	42770	**72350**	52090	19500	70.1
2.64	1.54	2.20	6.01	62300	42350	**72600**	58180	18600	67.9
3.57	2.49	2.17	6.01	61470	47870	**68630**	78760	28440	77.9
3.58	2.49	2.18	6.01	61470	48530	**69780**	79700	28910	78.9
4.28	3.06	2.44	6.28	56760	46070	**64440**	80820	31400	80.1
4.28	3.03	2.53	6.28	59300	43960	**62100**	74370	29960	74.1
4.06	2.32	3.48	5.30	58340	40570	**70900**	47330	31080	69.5
4.61	2.88	3.46	5.30	58340	42150	**67470**	56160	36660	72.2
4.55	2.63	3.85	7.85	53730	38790	**67100**	45840	22480	72.2
5.31	3.46	3.70	7.85	56670	43050	**66070**	61780	29120	76.0
5.95	3.35	5.21	8.41	61650	40620	**72150**	46290	28740	65.8
5.93	3.24	5.36	8.41	61650	37910	**69380**	41940	26730	61.4
5.79	3.54	4.49	7.21	59000	43150	**70570**	55640	34650	73.1
5.94	3.55	4.78	7.21	59390	38870	**65040**	48310	32020	65.4
6.14	3.99	4.29	7.21	52910	43530	66900	62310	**37070**	82.3
6.19	3.95	4.48	7.21	52910	40380	**63290**	55800	34670	76.3
5.81	3.96	3.69	6.01	59000	38850	56990	61170	**37550**	65.8

ZIGZAG-RIVETING.

3.59	2.50	1.97	10.82	58170	48150	**69140**	87740	15980	80.3
3.54	2.46	1.95	10.82	61470	47420	**68240**	86090	15520	77.1
3.19	2.30	1.56	8.41	58150	46610	**64650**	95320	17680	80.2
3.14	2.26	1.56	8.41	58150	47520	**66020**	95640	17740	81.7
3.25	2.60	1.05	6.01	59180	47640	**59550**	147450	25760	80.5
3.23	2.58	1.08	6.01	59180	46720	**58490**	139720	25110	78.9

ZIGZAG-RIVETING.

4.24	3.03	2.12	5.50	56760	42750	**59830**	85510	32960	75.3
4.28	3.02	2.20	5.50	59300	40630	**57580**	79050	31620	68.5
4.23	3.02	2.11	5.50	54350	42990	60220	**86180**	33060	79.1
4.27	3.01	2.19	5.50	54350	44340	62000	**86450**	34420	81.6
4.23	3.02	2.11	5.50	54350	44870	**62850**	89950	34510	82.5
4.33	3.03	2.22	5.50	59300	43490	**62150**	84820	31400	73.3

CHAIN-RIVETING.

5.93	4.04	5.66	9.02	59000	45720	**67100**	47900	30060	77.5
6.20	4.40	5.41	9.02	52910	48710	**68630**	55820	33480	92.1
5.52	4.01	4.54	7.21	58090	48040	**66130**	58410	36780	82.7
5.91	4.38	4.60	7.21	59390	46430	62650	59650	**38060**	78.2

In the design of a riveted tension-joint the problem usually presents itself in the following form:

Given, in all particulars, the two plates to be united, to design the joint; i.e., to determine, 1°, the diameter of rivet to be used; 2°, the spacing of the rivets, centre to centre; and, 3°, the lap.

In regard to the determination of the lap, the common practice has been already explained and very little has been done experimentally.

In order to determine the diameter and the spacing of the rivets by the usual methods of calculation, it becomes necessary to know the three following kinds of resistance of the metals, viz.:

1°. The tensile strength per square inch of the plate along the line or lines of rivet-holes;

2°. The shearing-strength of the rivet metal;

3°. The resistance to compression on the bearing-surface of either plate or rivet.

Hence we need to ascertain what the tests cited show in regard to these three quantities.

Tension.—The tensile strength of the plate used should, of course, be determined by means of tests made on specimens cut from it. Further than this, questions arise as to the excess tenacity due to the grooved specimen form, and as to any injury due to punching when the holes are punched.

The excess tenacity is, of course, greater with small than with large spaces between the rivet-holes; hence, inasmuch as the tendency is toward the use of large rivets, and, consequently, large pitches, the excess tenacity applicable in practical cases becomes small, and would be better disregarded in the design of most riveted joints. In cases where the holes are drilled, therefore, we should use for tensile strength per square inch of the plate along the line of rivet-holes, the tensile strength per square inch of the plate itself.

The better and more ductile the plate the less is the injury done by punching; but, while more or less punching is done, the better class of work is drilled. A study of the results in the cases of punched plates will show approximately what allowance to make for the weakening due to punching different qualities of plate.

Shearing.—A study of the results of the government tests show that it is fair to assume the shearing-strength of the wrought-iron rivets used, to be about 38000 pounds per square inch, which is about two thirds of the tensile strength of the same rivet metal.

For steel rivets, of the kinds now prescribed in most specifications, the shearing-strength appears to be about 45000 pounds per square inch.

Compression.—To determine what we should estimate as the ultimate compression on the bearing-surface is a more difficult problem; for if a joint fails in consequence of too great compression on the bearing-surface the cause of the failure does not exhibit itself directly, but in some indirect manner—probably by decreasing the resisting properties of either the plate or the rivets, and hence by causing either the joint to break by tearing the plate or by shearing either the rivets or the plate in front of the rivets, but at a lower load than that at which it would have broken had the compression not been excessive; and hence when such breakage occurs it is difficult to say whether it is due to excessive compression reducing the tensile or the shearing strength, or whether its full tensile or shearing strength was really reached.

Observe, moreover, that in the tables of Government tests the heavy numbers in the column marked "Compression on the bearing-surface of the rivets" indicate that the plate broke out in front of the rivets, which might be due to excessive compression or to a deficiency of lap.

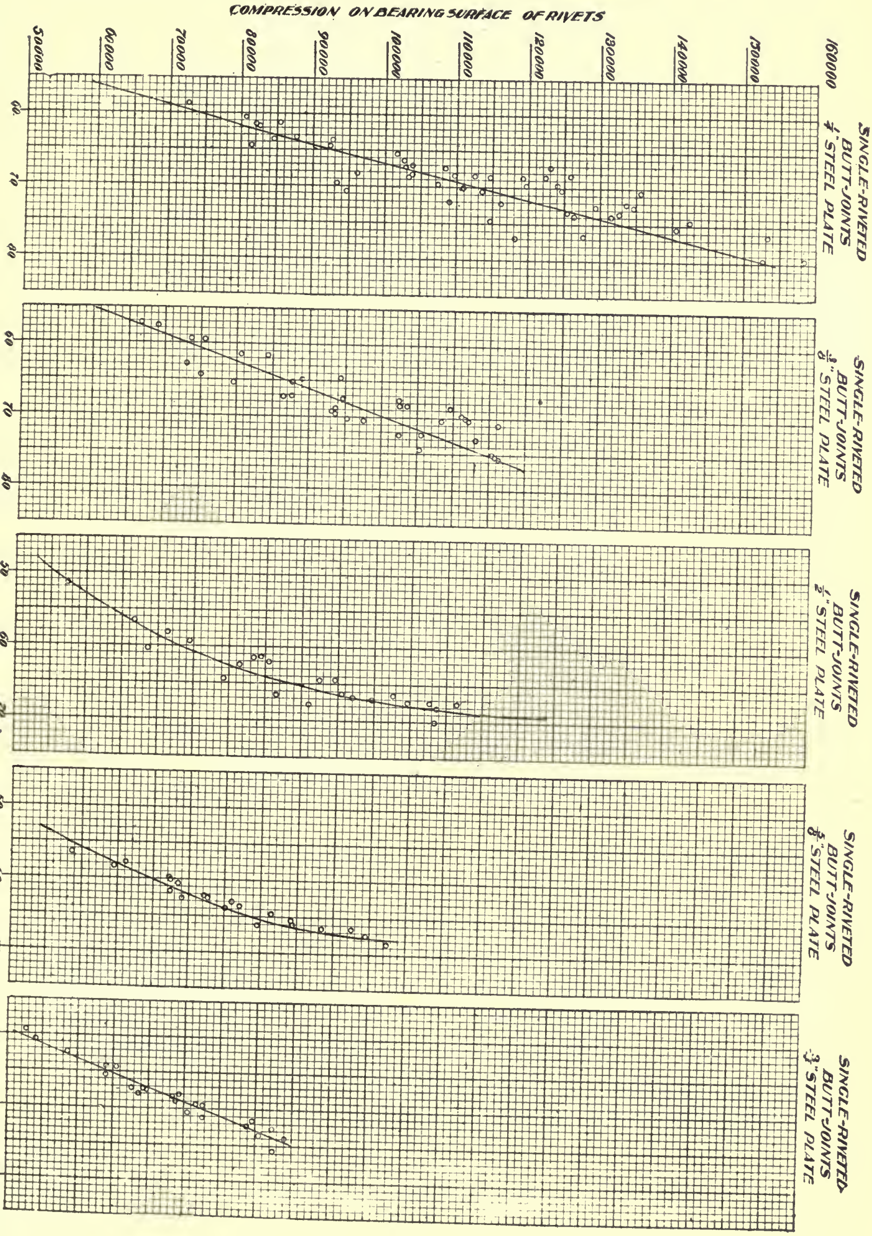
While more experiments are needed, it would seem probable that we might deduce some conclusions, at least, of a general nature, in regard to the ultimate compression by a study of the relations existing between the compression per square inch on the bearing-surface at fracture and the efficiency of the joint as shown by the Government tests.

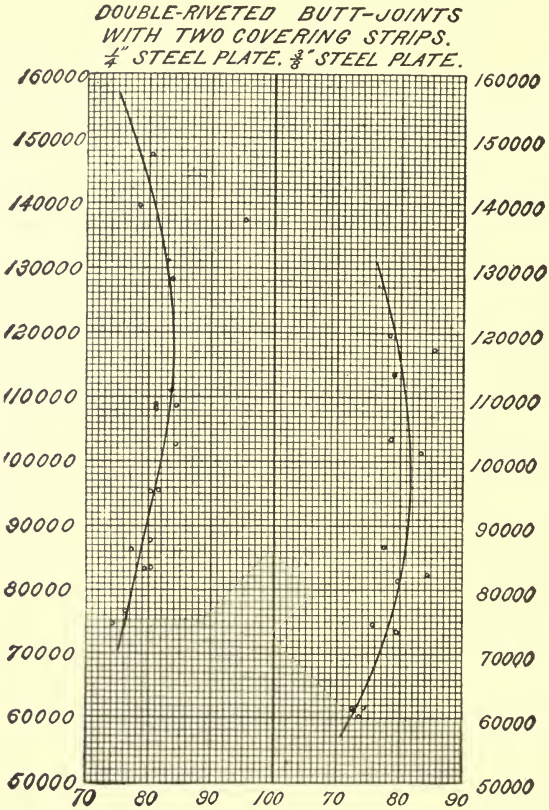
For this purpose the following diagrams (see pages 631 and 632) have been plotted, with the efficiencies as abscissæ and the compression per square inch on the bearing-surface at fracture as ordinates. If similar diagrams were plotted with the efficiencies as abscissæ and the ratio of the compression per square inch on the bearing-surface at fracture to the tensile strength of the plate as ordinates the character of the diagrams would be substantially the same, as the plates used in the tests were all of mild steel of approximately the same quality, and hence the difference in tensile strength of different samples was not great.

A study of these diagrams shows that in the case of the $\frac{1}{4}$-inch plates experiments were made with compressions up to about 158,000 pounds per square inch, but that the highest compression reached with any other thickness of plate was about 120,000 pounds per square inch.

Inasmuch as Kennedy advises the use of 96,000 pounds per square inch, and as this is higher than the values that have been customarily advocated, it would hardly seem wise to adopt a much higher value unless the tests furnish us sufficient evidence for such a procedure. Considering the facts stated above, and also the fact that in the cases of the double-riveted joints some of the highest compressions were accompanied by a decrease in efficiency, it would seem best to limit our estimate of the ultimate compression on the bearing-surface to from 90,000 to 100,000 pounds per square inch until we have further light on the subject derived from experiment; and it is not at all improbable that when we do obtain further light we may find ourselves warranted in using a somewhat higher value.

The reasoning which leads to the above conclusion is, of course, based on evidence which is not conclusive, because of the lack of tests with higher compressions on the bearing sur-


COMPRESSION ON BEARING SURFACE OF RIVETS
50000
60000
70000
80000
90000
100000
110000
120000
130000
140000
150000
160000
SINGLE-RIVETED BUTT-JOINTS 1/4" STEEL PLATE
SINGLE-RIVETED BUTT-JOINTS 3/8" STEEL PLATE
SINGLE-RIVETED BUTT-JOINTS 1/2" STEEL PLATE
SINGLE-RIVETED BUTT-JOINTS 5/8" STEEL PLATE
SINGLE-RIVETED BUTT-JOINTS 3/4" STEEL PLATE



face, with plates thicker than one quarter of an inch. On the other hand, the quarter-inch plates show higher efficiencies with compressions above 100000 pounds than they do with compressions of 100000 pounds or less, and the author knows of tests upon riveted joints in $\frac{7}{16}$-inch plates which tend to show that, with good wrought-iron rivets, it would be perfectly safe to use a considerably larger number for compression on the bearing-surface, in designing riveted joints—at least 110000 pounds per square inch, and probably more.

It will be observed that no reference has been made to the friction, and it is safer to leave this out of account, as the tests show that slipping takes place at all loads, and as there is no friction at the time of fracture.

By far the greater part of the tests at Watertown Arsenal were made with wrought-iron rivets in mild-steel plates, this being, at the time, the most usual practice, although steel rivets were sometimes used. At the present time, notwithstanding the fact that steel long ago superseded wrought-iron for boiler-plate, and that it has, to-day, superseded wrought-iron for structural shapes, as I beams, channel-bars, angles, etc., and that the use of steel rivets has become very extensive, nevertheless a great many still adhere to the use of wrought-iron rivets, and feel more confidence in them than they do in steel rivets. Whereas the use of wrought-iron rivets had been practically universal, the qualifications for a good wrought-iron rivet metal became pretty well known, and while sometimes specifications were drawn up giving the requirements of the rivet metal for tensile strength, ductility, etc., which of course would vary more or less, nevertheless the variations would not be large. A study of the Watertown tests shows that the wrought-iron rivet metal used in those tests had a tensile strength of from about 52000 to about 59000 pounds per square inch, with a percentage contraction of area at fracture of from about 30 to about 45. With this metal the shearing strength per square inch seems to be about $\frac{7}{8}$ of the tensile strength per square inch. Of course other tests are necessary to show whether the metal can be properly worked, and whether it is red-short or not, such as that the metal should bend double, whether cold or hot, without cracks, and that cracks should not develop when the shank is hammered down, cold or hot, to a length considerably less than the diameter.

When steel rivets were first used, the steel employed was not an extremely soft steel, as shown by the few cases of steel rivets included in the Watertown Arsenal tests already quoted, where the shearing-strength per square inch varied from about 50000 pounds per square inch up to as high a figure as 65000 pounds per square inch; and by Kennedy's tests, where he apparently fixes on from about 49000 to about 54000 pounds per square inch as the shearing-strength of steel rivets.

Now it would seem that metal with these shearing-strengths would have a tensile strength per square inch which would not warrant us in classifying it as very soft steel.

On the other hand, it is evident that brittleness should not in any way be tolerated in rivet metal, and hence it would seem that at least soft steel should be used for rivets.

The specifications proposed by the American Society for Testing Materials prescribe for tensile strength per square inch of steel for structural rivets from 50000 to 60000 pounds per square inch, and for boiler-rivets from 45000 to 55000 pounds.

While the number of tests that have been made upon joints constructed with steel rivets is not large, the shearing-strength of such steel rivets as are in use to-day is not very far from 45000 pounds per square inch, as a rule.

The number of tests of joints constructed with steel rivets is not sufficiently large to warrant drawing from them definite conclusions regarding the ultimate compression on the bearing surface in such joints. Meanwhile, it would be advisable to use for it the same values as are suitable in the case of joints made with steel plates and wrought-iron rivets.

The following table contains the joints tested at Watertown Arsenal, which were made with steel plate and wrought-iron rivets, and in which the plate broke out in front of the rivet. It is evident that only four of them, viz., 4915, 4916, 4917, 4918, failed in consequence of excessive compression on the bearing surface, and that the breaking out of the plate in the other cases was due to insufficiency of lap. The calculated $\frac{l}{d}$ was obtained by the method described on page 554, assuming $f_t = 55000$, and $f_s = 38000$, and $f_c = 96000$.

Number of Test.	Kind of Joint.	Rivet Material.	Diameter of Hole. Ins.	Punched = P. Drilled = D.	Thickness of Plate. Ins.	Thickness of Cover-plate. Ins.	Lap in Plate. Ins.	Lap in Cover-plate. Ins.	Compression per Sq. In. on Bearing Surface at Fract. Lbs.	Other Methods of Failure.	$\frac{l}{d}$.	$\frac{l}{d}$ Calculated.
718	Single lap	Iron	$1\frac{1}{16}$	P	$\frac{3}{8}$	....	1.25	....	79510	Tore and sheared	1.18	1.55
719	" "	"	$1\frac{1}{16}$	P	$\frac{3}{8}$	....	1.25	....	80200	Tore	1.18	1.55
4947	" "	"	$1\frac{1}{4}$	D	$\frac{1}{4}$	....	2.00	....	112970	....	1.60	1.75
4948	" "	"	$1\frac{1}{4}$	D	$\frac{1}{4}$	....	2.00	....	114760	Tore	1.60	1.75
4949	" "	"	$1\frac{1}{4}$	D	$\frac{1}{4}$	....	2.00	....	120180	....	1.60	1.77
767	Single butt	"	$\frac{3}{4}$	P	$\frac{3}{8}$	$\frac{3}{8}$	1.25	1.25	95210	Tore	1.67	1.65
1442	" "	"	$1\frac{1}{8}$	....	$\frac{1}{2}$	$\frac{5}{16}$	2.00	2.00	107610	....	1.50	1.69
1443	" "	"	$1\frac{1}{8}$	....	$\frac{1}{2}$	$\frac{5}{16}$	2.00	2.00	108830	....	1.50	1.70
4915	" "	"	$\frac{7}{8}$	D	$\frac{1}{4}$	$\frac{3}{16}$	2.00	2.00	151780	....	2.29	1.93
4916	" "	"	$\frac{7}{8}$	D	$\frac{1}{4}$	$\frac{3}{16}$	2.00	2.00	142570	....	2.29	1.89
4917	" "	"	$\frac{7}{8}$	D	$\frac{1}{4}$	$\frac{3}{16}$	2.00	2.00	158150	....	2.29	1.96
4918	" "	"	$\frac{7}{8}$	D	$\frac{1}{4}$	$\frac{3}{16}$	2.00	2.00	152110	....	2.29	1.93
4985	" "	"	1	D	$\frac{5}{16}$	$\frac{3}{16}$	1.00	1.00	83950	....	1.00	1.57
4987	" "	"	1	D	$\frac{5}{16}$	$\frac{3}{16}$	1.25	1.25	93710	....	1.25	1.63
4991	" "	"	1	D	$\frac{5}{16}$	$\frac{3}{16}$	1.75	1.75	119500	....	1.75	1.77
298	Reinforced lap	"	$\frac{15}{16}$	D	$\frac{3}{8}$	$\frac{1}{4}$	$\left\{\begin{matrix}1.15\\1.00\end{matrix}\right.$	$\left\{1.12\right.$	67300	Sheared rivets	1.19	1.66
299	"	"	$\frac{15}{16}$	D	$\frac{3}{8}$	$\frac{1}{4}$	$\left\{\begin{matrix}1.10\\1.12\end{matrix}\right.$	$\left\{1.12\right.$	68040	"	1.19	1.67
5121	Double lap	"	1	D	$\frac{5}{16}$	....	1.50	....	86180	....	1.50	1.56
5122	" "	"	1	P	$\frac{5}{16}$	....	1.50	....	86450	....	1.50	1.59

§ 234. **Wire and Wire Rope.**—It is well known that the process of making wire by cold drawing greatly increases the strength of the metal. Annealing, on the other hand, decreases the strength, and increases the ductility. It is not the purpose of this article to discuss the various qualities of wire required and used for different purposes. Hence, inasmuch as results of tests of wrought-iron, and of steel wire, have already been given, there will be given here only a few tests of hard-drawn, of semi-hard-drawn, and of soft copper wire.

Wire rope.—Wire rope is used for a great many purposes, as in suspension bridges, in hoisting, in haulage, in the transmission of power, etc.

While flat wire rope is used for some purposes, and while wire rope made of parallel wires is used in large suspension bridges, the greater part is made by twisting a number of wire

HARD-DRAWN COPPER WIRE.

Diameter. Inches.	Tensile Strength per Sq. In. Lbs.	Elastic Limit per Sq. In. Lbs.	Contraction of Area. Per Cent.
0.166	53050	37100	....
0.138	60350	22800	....
0.135	56300	28150	....
0.134	51050	27140	....
0.105	61800	41000	51
0.105	57100	35000	49
0.105	58900	34000	33
0.106	60300	36000	42
0.106	59500	34000	39
0.086	58170	27870	....
0.086	58620	29310	....
0.085	61510	21390	....
0.083	66536	29630	....
0.083	65060	37334	....
0.083	66536	29630	....

SOFT COPPER WIRE.

Diameter. Inches.	Tensile Strength per Sq. In. Lbs.	Elastic Limit per Sq. In. Lbs.	Contraction of Area. Per Cent.
0.163	35730	13740	......
0.162	35770	13760	......
0.162	36640	12990	....
0.083	29500	10500	....
0.083	29500	....	70
0.081	33200	....	45
0.080	33100	....	70
0.080	33080	1[illegible]200	....

SEMI-HARD-DRAWN COPPER WIRE.

0.106	44300	30000	60
0.106	45100	29000	65
0.106	45500	29000	55
0.106	45100	31000	67
0.106	44900	30000	64

strands around a central core, which may be of tarred hemp, or which may be, itself, a wire strand, the wire strands being made of wires twisted together.

In the case of a wire core, the strength of the rope is a little greater, but the resistance to wearing is less.

The most usual number of strands is six, each strand containing seven, eighteen, or nineteen wires, though other numbers of wires are sometimes used.

The strength that can be realized in practice is always less than the strength of the rope, and is determined by the method of holding the ends, as the junction point of the rope and the holder is the weakest point.

The usual methods of holding the ends are as follows: splicing, as in the case of the transmission of power, passing the rope around a pulley, or around a thimble, fastening it in a socket, or in a clamp.

The diameter of the drum or sheave around which a rope passes, should not be so small as to cause too much stress to be exerted upon some of the wires, in consequence of the bending-moment introduced by the curvature.

Inasmuch as it may be a matter of convenience to have here some tables giving the strength of rope as claimed by some makers, there will follow here two tables of the strength of different sizes, as given by the Roebling Company for their rope.

The following explanations are given by the Roebling Company, about the quality of the metal used:

Iron, open-hearth steel, crucible steel, and plough steel possess qualities which cover almost every demand upon the material of a wire rope. Copper, bronze, etc., are, however, used for a few special purposes.

The strength of iron wire ranges from 45000 to 100000 pounds per square inch; open-hearth steel, from 50000 to 130000 pounds

SEVEN-WIRE ROPE.

Composed of 6 Strands and a Hemp Center, 7 Wires to the Strand.

Trade No.	Diameter in Inches.	Approximate Circumference in Inches.	Weight per Foot in Pounds.	Approximate Breaking-strain in Tons of 2000 Lbs.			
				Transmission or Haulage Rope.		Extra Strong Cast Steel.	Plough Steel.
				Swedish Iron.	Cast Steel.		
11	$1\frac{1}{2}$	$4\frac{3}{4}$	3.55	34	68	79	91
12	$1\frac{3}{8}$	$4\frac{1}{4}$	3.00	29	58	68	78
13	$1\frac{1}{4}$	4	2.45	24	48	56	64
14	$1\frac{1}{8}$	$3\frac{1}{2}$	2.00	20	40	46	53
15	1	3	1.58	16	32	37	42
16	$\frac{7}{8}$	$2\frac{3}{4}$	1.20	12	24	28	32
17	$\frac{3}{4}$	$2\frac{1}{4}$	0.89	9.3	18.6	21	24
18	$\frac{11}{16}$	$2\frac{1}{8}$	0.75	7.9	15.8	18.4	21
19	$\frac{5}{8}$	2	0.62	6.6	13.2	15.1	17
20	$\frac{9}{16}$	$1\frac{3}{4}$	0.50	5.3	10.6	12.3	14
21	$\frac{1}{2}$	$1\frac{1}{2}$	0.39	4.2	8.4	9.70	11
22	$\frac{7}{16}$	$1\frac{1}{4}$	0.30	3.3	6.6	7.50	8.55
23	$\frac{3}{8}$	$1\frac{1}{8}$	0.22	2.4	4.8	5.58	6.35
24	$\frac{5}{16}$	1	0.15	1.7	3.4	3.88	4.35
25	$\frac{9}{32}$	$.\frac{7}{8}$	0.125	1.4	2.8	3.22	3.65

NINETEEN-WIRE ROPE.

Composed of 6 Strands and a Hemp Center, 19 Wires to a Strand.

Trade No.	Diameter in Inches.	Approximate Circumference in Inches.	Weight per Foot in Pounds.	Approximate Breaking-strain in Tons of 2000 Lbs.			
				Standard Hoisting Rope.		Extra Strong Cast Steel.	Plough Steel.
				Swedish Iron.	Cast Steel.		
—	$2\frac{3}{4}$	$8\frac{5}{8}$	11.95	—	—	—	305
—	$2\frac{1}{2}$	$7\frac{7}{8}$	9.85	—	—	—	254
1	$2\frac{1}{4}$	$7\frac{1}{8}$	8.00	78	156	182	208
2	2	$6\frac{1}{4}$	6.30	62	124	144	165
3	$1\frac{3}{4}$	$5\frac{1}{2}$	4.85	48	96	112	128
4	$1\frac{5}{8}$	5	4.15	42	84	97	111
5	$1\frac{1}{2}$	$4\frac{3}{4}$	3.55	36	72	84	96
$5\frac{1}{2}$	$1\frac{3}{8}$	$4\frac{1}{4}$	3.00	31	62	72	82
6	$1\frac{1}{4}$	4	2.45	25	50	58	67
7	$1\frac{1}{8}$	$3\frac{1}{2}$	2.00	21	42	49	56
8	1	3	1.58	17	34	39	44
9	$\frac{7}{8}$	$2\frac{3}{4}$	1.20	13	26	30	34
10	$\frac{3}{4}$	$2\frac{1}{4}$	0.89	9.7	19.4	22	25
$10\frac{1}{4}$	$\frac{5}{8}$	2	0.62	6.8	13.6	15.8	18
$10\frac{1}{2}$	$\frac{9}{16}$	$1\frac{3}{4}$	0.50	5.5	11.0	12.7	14.5
$10\frac{3}{4}$	$\frac{1}{2}$	$1\frac{1}{2}$	0.39	4.4	8.8	10.1	11.4
10*a*	$\frac{7}{16}$	$1\frac{1}{4}$	0.30	3.4	6.8	7.8	8.85
10*b*	$\frac{3}{8}$	$1\frac{1}{8}$	0.22	2.5	5.0	5.78	6.55
10*c*	$\frac{5}{16}$	1	0.15	1.7	3.4	4.05	4.50
10*d*	$\frac{1}{4}$	$\frac{3}{4}$	0.10	1.2	2.4	2.70	3.00

per square inch; crucible steel from 130000 to 190000 pounds per square inch; and plough steel from 190000 to 350000 pounds per square inch. Plough steel wire is made from a high grade of crucible cast-steel.

§ 235. **Other Metals and Alloys.**—Copper is, next to iron and steel, the metal most used in construction, sometimes in the pure state, especially in the form of sheets or wire, but more frequently alloyed with tin or zinc; those metals where the tin predominates over the zinc being called bronze, and those where zinc predominates over tin, brass. Copper in the pure state was used not long ago for the fire-box plates of locomotive and other steam-boilers, as it was believed to stand better the great strains due to the changes of temperature that come upon these plates, than iron or steel; but now steel or iron has almost entirely superseded it for this purpose, except in some cases where the feed-water is very impure, and where the impurities are such as corrode iron.

The alloys of copper, tin, and zinc which are used most where strength and toughness are needed, are those where the tin predominates over the zinc; and the composition, mode of manufacture, and resisting properties of these metals, together with the effect of other ingredients, as phosphorus, have been very extensively investigated with reference to their use as a material for making guns, instead of cast-iron.

Accounts of tests made on these alloys will be found as follows:—

Major Wade: Ordnance Report, 1856.
T. J. Rodman: Experiments on Metals for Cannon.
Executive Document No. 23, 46th Congress, 2d session.
Materials of Engineering: Thurston.

No attempt will be made to give a complete account of the results of these tests; but a table will be given on page 639 for convenience of use, showing rough average values of the resisting powers of some metals and alloys other than iron.

§ 236. **Timber.** — However extensively iron and steel may have superseded timber in construction, nevertheless, there are many cases in which iron is entirely unsuitable, and where timber is the only material that will answer the purpose; and in many cases where either can be used, timber is much the cheaper. Hence it follows that the use of timber in construction is even now, and as it seems always will be, a very important item.

Another advantage possessed by timber is, that, on yielding, it gives more warning than iron, thus affording an opportunity to foresee and to prevent accident.

If we make a section across any of the exogenous trees, as

	Specific Gravity.	Tensile Strength per Sq. In.	Modulus of Elasticity.
Brass cast	8.396	18000	9170000
Brass wire	–	49000	14230000
Bronze unwrought:			
84.29 copper + 15.71 tin (gun metal) .	8.561	36060	–
82.81 " + 17.19 " " .	8.462	34048	–
81.10 " + 18.90 " " .	8.459	39648	–
78.97 " + 21.03 " (brasses) . . .	8.728	30464	–
34.92 " + 65.08 " (small bells) .	8.056	3136	–
15.17 " + 84.83 " (speculum metal)	7.447	6944	–
Tin	7.291	5600	–
Zinc	6.861	7500	–
Copper cast	8.712	24138	–
Copper bolts	8.878	33000	–
Copper wire	–	60000	17000000
Gold cast	19.258	20000	–
Silver cast	10.476	40000	–
Platinum wire	22.069	56000	–
Lead cast	11.352	1800	–

the oak, pine, etc., we shall find a series of concentric layers; these layers being called annual rings, because one is generally deposited every year.

Radiating from the heart outwards will be found a series of radial layers, these being known as the medullary rays.

Of the annual rings, the outer ones are softer and lighter in color than the inner ones; the former forming the sap-wood, and the latter the heart-wood. When the log dries, and thus tends to contract, it will be found that scarcely any contraction takes place in the medullary rays; but it must take place along the line of least resistance, viz., along the annual rings, thus causing radiating cracks, and drawing the rays nearer together on the side away from the crack. This action is exhibited in Fig. 241, where a log is shown with two saw-cuts at right angles to each other; when this log becomes dry, the four right angles all becoming acute through the shrinkage of the rings.

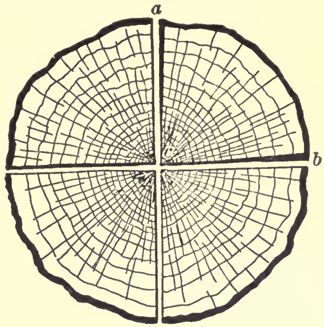


FIG. 241.

If the log be cut into planks by parallel saw-cuts, the planks will, after drying, assume the forms shown in Fig. 242, as is pointed out in Anderson's "Strength of Materials," from which these two cuts are taken.

This internal construction of a plank has an important influence upon the side which should be uppermost when it is used for flooring; for, if the heart side is uppermost, there will be a liability to having layers peel off as the wood dries: indeed, boards for flooring should be so cut as to have the annual rings at right angles to the side of the plank. Before discussing any other considerations which affect the adaptability of timber to use in construction, we will consider the question of its strength.

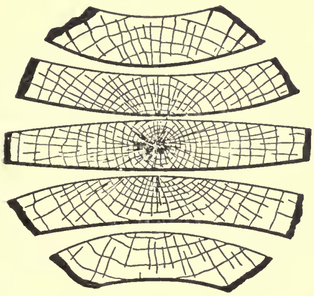


FIG. 242.

§ 237. **Strength of Timber.** — In this regard we must observe, that, whereas the strength and elasticity and other properties of iron and steel vary greatly with its chemical composition and the treatment it has received during its manufacture, the strength, etc., of timber is much more variable, being seriously affected by the soil, climate, and other accidents of its growth, its seasoning, and other circumstances; and that over many of these things we have no control: hence we must not expect to find that all timber that goes by one name has the same strength, and we shall find a much greater variation and irregularity in timber than in iron. The experiments that have been made on strength and elasticity of timber may be divided into the following classes: —

1°. Those of the older experimenters, except those made on full-size columns by P. S. Girard, and published in 1798. A fair representation of the results obtained by them, all of which were deduced from experiments on small pieces, is to be found in the tables given in Professor Rankine's books, "Applied Mechanics," "Civil Engineering," and "Machinery and Millwork."

2°. Tests made by modern experimenters on small pieces. Such tests have been made by —

(*a*) Trautwine: Engineers' Pocket-Book.
(*b*) Hatfield: Transverse Strains.
(*c*) Laslett: Timber and Timber Trees.
(*d*) Thurston: Materials of Construction.
(*e*) A series of tests on small samples of a great variety of American woods, made for the Census Department, and recorded in Executive Document No. 5, 48th Congress, 1st session.

Timber Physics, Division of Forestry, U. S. Department of Agriculture. For a fairly complete bibliography of tests of timber see a paper by G. Lanza, Trans. Am. Soc. C. E., 1905.

3°. Tests made by Capt. T. J. Rodman, U.S.A., the results of which are given in the "Ordnance Manual."

4°. All tests that have been made on full-size pieces.

In regard to tests on small pieces, such as have commonly been used for testing, it is to be observed, that, while a great deal of interesting information may be derived from such tests as to some of the properties of the timber tested, nevertheless, such specimens do not furnish us with results which it is safe to use in practical cases where full-size pieces are used. Inasmuch as these small pieces are necessarily much more perfect (otherwise they would not be considered fit for testing), having less defects, such as knots, shakes, etc., than the full-size pieces they have also a far greater homogeneity. They also season much more quickly and uniformly than full-size pieces. In making this statement, I am only urging the importance of adopting in this experimental work the same principle that the physicist recognizes in all his work; viz., that he must not apply the results to cases where the conditions are essentially different from those he has tested.

Moreover, it will be seen in what follows, that, whenever full-size pieces have been tested, they have fallen far short of the strength that has been attributed to them when the basis in computing their strength has been tests on small pieces; and, moreover, the irregularities do not bear the same proportion in all cases, but need to be taken account of.

The results of the first class of experiments named in the following table are taken from Rankine's "Applied Mechanics;" and, inasmuch as the table contains also the strengths of some other organic fibres, it will be inserted in full. The student may compare these constants with those that will be given later.

Kind of Material.	Tenacity or Resistance to Tearing.	Modulus of Tensile Elasticity.	Resistance to Crushing.	Modulus of Rupture.	Resistance to Shearing along Grain.	Modulus of Shearing Elasticity along the Grain.
Ash	17000	1600000	9000	12000 14000	1400	76000
Bamboo	6300	–	–	–	–	–
Beech	11500	1350000	9360	9000 12000	–	–
Birch	15000	1645000	6400	11700	–	–
Blue gum	–	–	8800	16000 20000	–	–
Box	20000	–	10300	–	–	–
Bullet-tree	–	–	14000	15900 16000	–	–
Cedar of Lebanon . .	11400	486000	5860	7400	–	–

Kind of Material.	Tenacity or Resistance to Tearing.	Modulus of Tensile Elasticity.	Resistance to Crushing.	Modulus of Rupture.	Resistance to Shearing along Grain.	Modulus of Shearing Elasticity along the Grain.
Chestnut	10000 to 13000	1140000	–	10660	–	–
Cowrie	–	–	–	11000	–	–
Ebony	–	–	19000	27000	–	–
Elm	14000	700000 to 1840000	10300	6000 to 9700	1400	76000
Fir, Red pine	12000 to 14000	1460000 to 1900000	5375 to 6200	7100 9540	500 800	62000 116000
" Yellow pine (Am.)	–	–	5400	–	–	–
" Spruce	12400	1400000 to 1800000	–	9900 12300	600	–
" Larch	9000 to 10000	900000 to 1360000	5570	5000 10000	970 1700	–
Hoxen yarn	25000	–	–	–	–	–
Hazel	16000	–	–	–	–	–
Hempen rope	12000 to 16000	–	–	–	–	–
Ox-hide, undressed	6300	–	–	–	–	–
Hornbeam	20000	–	–	–	–	–
Lancewood	23400	–	–	–	–	–
Ox-leather	4200	24300	–	–	–	–
Lignum-vitæ	11800	–	9900	12000	–	–
Locust	16000	–	–	–	–	–
Mahogany	8000 to 21800	1255000	8200	7600 11500	–	–
Maple	10600	–	–	–	–	–
Oak, British	–	–	10000	10000 13600		
" Dantzic	–	–	7700	8700		
" European	10000 to 19800	1200000 1750000	–	–	2300	82000
" American red	10250	2150000	6000	10600		

Kind of Material.	Tenacity or Resistance to Tearing.	Modulus of Tensile Elasticity.	Resistance to Crushing.	Modulus of Rupture.	Resistance to Shearing along Grain.	Modulus of Shearing Elasticity along the Grain.
Silk fibre	52000	1300000	–	–	–	–
Sycamore	13000	1040000	–	9600	–	–
Teak, Indian . . .	15000	2400000	12000	12000 19000	–	–
" African . . .	21000	2300000	–	14980	–	–
Whalebone	7700	–	–	–	–	–
Willow	–	–	–	6600	–	–
Yew	8000	–	–	–	–	–

In regard to the tests of the second class, a few comments are in order: —

1°. These experiments, like those of the first class, were all made upon small pieces; and the results are correspondingly high.

The usual size of the specimens for crushing being one or two square inches in section, and of those for transverse strength being about two inches square in section and four or five feet span, those for tension had even a much smaller section than those for compression; as it is necessary, in order to hold the wood in the machine, to give it very large shoulders.

The only exception to this is the tests of Sir Thomas Laslett, an account of which is given in his "Timber and Timber Trees," and also in D. K. Clark's "Rules and Tables." In these tests he gives very much lower tensile strengths than those given above; and he states that his specimens were three inches square, but does not say how he managed to hold them in such a way as to subject them to a direct tensile stress. His results for crushing and transverse strength are about as great as those given in Rankine's tables, and as were obtained by the other experimenters on small pieces, as his specimens were of about the same dimensions as those used by the others. The figures obtained by these experimenters will only be given incidentally, as —

(*a*) They are very similar to those given in Rankine's table.

(*b*) They are not suitable for practical use on the large scale.

(*c*) While they have been used, it has only been done by employing a very large factor of safety for timber.

The series of tests made for the Census Department, and recorded in Executive Document No. 5, 48th Congress, first session, form a very interesting series of experiments upon small specimens of an exceedingly large number of American woods. In order to have working figures, we should need to test large pieces of the same; as the proportion between the strengths of the different kinds would be liable to be different in the latter case.

The work done by the Division of Forestry of the U. S. Dept. of Agriculture before 1898 was mostly of this class, but little having been done with full-size pieces, and that with imperfect apparatus.

The only record of Rodman's experiments available is a table of results in the "Ordnance Manual." These are lower, as a rule, than those obtained by the experimenters of the first or second class. This is to be accounted for by the fact that, while he did not experiment on full-size pieces, he used much larger pieces than those heretofore employed; his specimens for transverse strength, many of which are still stored at the Watertown Arsenal, being $5\frac{3}{4}$ inches deep, $2\frac{7}{8}$ inches thick, and 5 feet span.

The fourth class of tests are those which furnish reliable data for use in construction; and we will proceed to a consideration of these, taking up (1°) tension, (2°) compression, (3°) transverse strength, and (4°) shearing along the grain.

TENSION.

In all cases where the attempt has been made to experiment upon the tensile strength of timber, a great deal of difficulty has been encountered in regard to the manner of holding the specimens. In all cases it has been found necessary to provide them with shoulders, each shoulder being five or six times as long as the part of the specimen to be tested, and to bring upon these shoulders a powerful lateral pressure, to prevent the specimen from giving way by shearing along the grain, and pulling out from the shoulder, instead of tearing apart.

The specimens tested have generally had a sectional area less than one square inch, and it seems almost impossible to provide the means of holding larger specimens. This being the case, it is plain, that, whenever timber is used as a tie-bar in construction (except in exceedingly rare and out-of-the-way cases), it will give way by some means other than direct tension ; i.e., either by the pulling-out of the bolts or fastenings, and the consequent shearing of the timber, or else by bending if there is a transverse stress upon the piece ; and, this being the case, these other resistances should be computed, instead of the direct tension. Hence, while the direct tensile strength of timber may be an interesting subject of experiment, it can serve hardly any purpose in construction ; and the conclusion follows, that the resistances of timber to breaking we may expect to meet in practice are its crushing, transverse, and shearing strength. Indeed, the use of timber for a tie-bar should be avoided whenever it is possible to do so ; and, when it is used, the calculations for its strength should be based upon the pulling-out of the fastenings, the shearing or splitting of the wood, etc., and not on the tensile resistance of the solid piece.

Moreover, when a wooden tie-bar is subjected not merely to direct tension, but also to a bending-moment, whether the latter is caused by a transverse load, or by an eccentric pull, as it generally is in the case of timber joints, we must compute the greatest tension per square inch at the outside fibre due to the bending, and to that add the direct tension per square inch: and this sum must be less than the modulus of rupture if the piece is not to give way; i.e., the modulus of rupture and not the ultimate tensile strength per square inch must be our criterion of breaking in such a case, the working-strength per square inch being the modulus of rupture divided by a suitable factor of safety.

COMPRESSIVE STRENGTH.

Tests of the compressive strength of full-size wooden columns, with the exception of one set of tests, date from about 1880.

TESTS OF FULL-SIZE COLUMNS.

The following are references to tests of full-size timber columns:

1°. Trautwine, in his "Handbook," speaks of some tests of wooden pillars 20 feet long and 13 inches square, made by David Kirkaldy, which, as he says, gave results agreeing with Mr. C. Shaler Smith's rule.

2°. A series of tests made at the Watertown Arsenal for the Boston Manufacturers' Mutual Fire Insurance Company, under the direction of the author.

3°. Eleven tests of old spruce pillars made at the Watertown Arsenal, for the Jackson Company, under the direction of Mr. J. R. Freeman, and reported in the Journal of the Assoc. Eng. Societies for November, 1889.

4°. The tests that have been made at the Watertown Arsenal on the government testing-machine.

5°. Tests made in the Laboratory of Applied Mechanics of the Massachusetts Institute of Technology.

6°. A series of tests of full-size columns of oak and fir, made by P. S. Girard in 1798.

In regard to the first, no details or results are given: hence nothing will be said about them.

In regard to the second, a summary only will be presented here.

TESTS OF YELLOW-PINE POSTS AND BLOCKS.

TESTED WITH FLAT ENDS.

Set	Distinguishing No.	Weight, in lbs.	Length, in feet and inches. ft.	in.	Diameter of Small End, in inches.	Diameter of Large End, in inches.	Diameter of Core, in inches.	Sectional Area, in square inches.	Ultimate Strength.	Ultimate Strength, in lbs. per Sq. In.	Modulus of Elasticity, in lbs., per Square Inch.	
1st set	Post 1	320	12	0.15	9.31	10.55	1.67	65.90	270000	4097	1885264	Flat ends.
1st set	Post 2	235	12	0.20	8.30	10.07	1.70	51.90	190000	3661	1631035	Flat ends.
1st set	Post 3	211	12	0.15	7.54	8.99	1.70	42.50	200000	4706	2087347	Flat ends.
1st set	Post 4	164	12	0.20	6.40	7.79	1.67	30.00	138000	4600	2204585	Flat ends.
2d set.	Post 1	342	12	0.00	10.45	–	1.60	83.75	390000	4657	–	Tested with round pintles.
2d set.	Post 3	249	11	11.20	8.96	–	1.54	61.20	250000	4085	2169882	Flat ends.
2d set.	Post 4	180	11	11.70	7.70	–	1.53	44.72	205000	4584	2081321	Flat ends.
2d set.	Block 1	62	2	0.13	10.46	–	–	85.90	380000	4424	1657425	Flat ends.
2d set.	Block 2	52	2	0.00	9.98	–	–	78.23	368000	4704	2443411	Flat ends.
2d set.	Block 3	41	2	0.33	8.91	–	–	62.35	270000	4330	1644453	Flat ends.
2d set.	Block 4	29	2	0.00	7.79	–	–	47.66	200000	4511	1900252	Flat ends.
3d set.	Post 2	428	12	10.25	10.05 × 10.13	Rectangular	–	99.80	510000	5220	–	One flat end, one with rectangular pintle.
3d set.	Post 3	213	13	11.90	7.90	–	1.60	47.01	200000	4254	–	Flat ends.
3d set.	Post 4	193	11	11.04	8.00	–	1.60	48.26	225000	4662	–	One flat end, one rounded bearing.
3d set.	Block 1	58	2	0.00	8.98 × 9.02	Rectangular	–	81.00	482000	5350	–	Flat ends.
3d set.	Block 2	62.5	2	0.06	10.20 × 10.07	Rectangular	–	102.70	560000	5452	–	Flat ends.
3d set.	Block 3	26	1	11.95	7.70	–	1.60	44.56	218000	4892	–	Flat ends,
3d set.	Block 4	29.5	1	11.80	7.98	–	1.60	48.00	173000	3604	–	Flat ends.
	Average	. .	. . .	. . .	. . . .	. . . .	. . . .	. . . .	. . . . .	4544	1996351	

TESTS OF WHITE-OAK POSTS AND BLOCKS.

Tested with Flat Ends.

	Distinguishing No.	Weight, in lbs.	Length, in feet and inches. ft.	in.	Diameter of Small End, in inches.	Diameter of Large End, in inches.	Diameter of Core, in inches.	Sectional Area, in square inches.	Ultimate Strength. Lbs.	Ultimate Strength per Square Inch. Lbs.	Modulus of Elasticity, in lbs., per Square Inch.	
1st set.	Post 1	395	12	0.17	9.15	10.15	1.67	63.2	190000	3006	1222222	Flat ends.
	Post 2	325	12	0.20	8.37	10.23	1.67	52.8	200000	3788	1633987	Flat ends.
	Post 3	289	12	0.20	7.55	9.05	1.67	42.6	160000	3756	1504389	Flat ends.
	Post 4	215	12	0.20	6.60	8.06	1.67	32.0	110000	3438	1748817	Flat ends.
2d set.	Post 2	352	12	0.00	10.00	–	1.53	76.7	210000	2738	–	Tested with round pintles.
	Post 4	200	12	0.00	7.74	–	1.60	45.04	145000	3219	1508906	Flat ends.
2d set.	Block 1	72	2	0.00	10.91	–	–	93.48	416000	4450	–	Flat ends.
	Block 2	59	2	0.06	9.98	–	–	78.23	245000	3132	1165382	Flat ends.
	Block 3	41	2	0.00	8.18	–	–	52.55	165000	3139	1104938	Flat ends.
	Block 4	33	1	11.88	7.73	–	–	46.93	155000	3303	1302623	Flat ends.
	Average . .	. .	. .	. .	. . . .	. . . .	. . . .	. . . .	. . . . .	3470		

TESTS OF OLD AND SEASONED WHITE-OAK POSTS.

Distinguishing Mark.	Weight, in lbs.	Length, in feet and inches. ft.	in.	Diameter of the Small End, in inches.	Diameter of the Large End, in inches.	Diameter of Core, in inches.	Sectional Area, in square inches.	Ultimate Strength.	Ultimate Strength per Square Inch.	Modulus of Elasticity, in lbs., per Square Inch.	
1914	113	12	0.92	5.85	6.84	1.95	23.90	110000	4602	1843725	Ends brought to even bearing.
1915	116	12	0.92	5.85	6.85	1.90	24.00	145000	6032	2138804	Ends brought to even bearing.
1916	118	12	1.40	5.87	6.70	2.00	23.90	112000	4682	1657824	Ends not brought to even bearing.
1917	113	12	0.60	6.02	6.75	1.95	25.50	75000	2943	2052454	Tested with maple cap and oak base.
1918	108	12	0.80	5.75	6.88	1.95	22.98	75000	3263	–	Tested with maple cap and oak base.
1931	118	12	0.10	5.97	6.74	1.92	25.10	106000	4223	–	Tested with maple cap and oak base.
1932	118	11	8.65	6.10	6.83	2.00	26.08	90000	3450	–	Tested with maple cap and oak base.
A	398	13	10.70	10.54	–	1.90	84.40	360000	4265	–	Ends not brought to even bearing.
B	360	13	9.40	10.50	–	1.90	83.76	325000	3880	–	Ends not brought to even bearing.
C	383	14	0.00	10.54	–	1.95	87.26	420000	4084	2053024	Ends not brought to even bearing.
D	388	13	10.50	10.56	–	1.90	84.70	390000	4604	1448964	Ends brought to even bearing.
E	310	13	11.04	10.20	–	1.80	79.17	370000	4662	1574555	Ends not brought to even bearing.
F	378	13	11.63	10.40	–	1.70	82.68	400000	4838	1790964	Iron cap and pintle at one end, iron base at the other.
G	301	13	8.20	9.30	–	1.95	64.94	317000	4881	–	Ends not brought to even bearing.
H	293	13	8.08	9.25	–	1.90	64.37	221000	3433	–	Ends not brought to even bearing.
I	267	13	7.80	9.40	–	1.90	66.56	265000	3981	–	Ends not brought to even bearing.
J	310	13	7.80	9.35	–	1.90	65.83	215000	3266	–	Tested with rectangular pintles.
K	121	11	6.20	5.98	–	1.62	26.03	160000	6147	–	
Average .		. . .		. . .	. . .	. .	. . .	. . .	3957		

In all the experiments enumerated in the tables given above, the columns gave way by direct crushing, and hence the strength of columns of these ratios of length to diameter can properly be found by multiplying the crushing-strength per square inch of the wood by the area of the section in square inches.

This conclusion is deduced from the fact that the deflections were measured in every case, and found to be so small as not to exert any appreciable effect.

In regard to other tests of this same set, there were eight tests made, in addition to those already enumerated; and in five the loads were off centre. A summary of the results is appended, together with a comparison of their actual strength with that which would be computed on the basis of 4400 per square inch for yellow pine, and 3000 for oak. The first three tests were made on yellow-pine columns, and the last two on oak.

	Weight, in lbs.	Length, in feet and inches.		Diameter of Column.	Diameter of Core.	Sectional Area, in square inches.	Eccentricity, in inches.	Ultimate Strength.	Computed Ultimate Strength.
		ft.	in.						
2, 2d series	320	11	11.27	9.92	1.53	75.45	2.33	265000	331980
5, 3d series	298	12	6.8	8.30 × 7.60	–	63.1	2.07	240000	277640
1, 3d series	386	12	9.3	8.75 × 8.92	–	76.04	2.25	280000	334576
1, 2d series	451	11	11.4	10.95	1.80	92.16	2.75	170000	276480
3, 2d series	236	11	11.2	8.2	1.55	50.92	1.91	100000	152760

These results exhibit a great falling-off of strength due to the eccentricity of the load; and emphasizes the importance of taking into account eccentric loading in our calculations in a manner similar to that already mentioned on pages 370, 371, and 448.

The remaining experiments were: (1°) Two tests of white-wood columns, average strength 3000 pounds per sq. in., and very brittle. (2°) One yellow-pine square column (sectional area 68.8 sq. in., length 12′ 6″.85) with one end resting against a thick yellow-pine bolster.

The maximum load was 120,000 lbs. = 1744 lbs. per sq. in., the post beginning to split due to eccentricity of bearing caused by uneven yielding of the bolster. The bolster was then removed, the post cut off $1\frac{1}{2}$ in. at the end and tested without the bolster. Ultimate strength 375,000 lbs. = 5451 lbs. per sq. in.

The table of results of the tests on old and seasoned oak columns were made upon columns that had been in use for a number of years in different mills, from which they were removed, and replaced by new ones. Ten of them had been in use about twenty-five years, and the remainder for shorter periods. An inspection of this table will, I think, convince the reader that it would not be safe to calculate upon a higher breaking-strength per square inch in these than in the green ones.

TESTS FOR THE JACKSON COMPANY.

Eleven tests of old spruce pillars, which had been in use in a cotton-mill of the Jackson Company, were tested on the government machine at Watertown, under the direction of Mr. J. R. Freeman. The manner of making them was as follows:

In the first two the ends were brought to an even bearing.

In the third the ends came to an even bearing under a load of 60000 pounds.

In the fourth, fifth, ninth, tenth, and eleventh, the cap, and also the base-plate, were planed off on the back to a slope of 1 in 24, and placed with their inclinations opposite.

In the eighth they had their inclinations the same way one as the other.

In the sixth and seventh the base-plate was not used, the larger end of the post having a full bearing on the platform of the machine.

The results are given in the following table:

	Length in Feet and Inches.	Diameter at Small End, Inches.	Diameter at Large End, Inches.	Area at Small End, Sq. In.	Ultimate Strength, Lbs.	Ultimate Strength per Sq. In., Lbs.
	ft. in.					
1	10 4.75	5.82	7.78	31.87	142000	4088
2	10 4.	5.85	7.49	27.15	192800	6225
3	10 5.75	5.85	7.74	32.17	166100	4900
4	10 5.5	5.70	7.77	31.67	108200	
5	10 5.1	5.70	7.70	30.78	105000	
6	10 5.2	5.80	7.61	30.39	168000	
7	9 7	5.74	7.81	32.88	194100	
8	10 5.4	5.85	7.90	34.21	155000	
9	10 4.9	5.82	7.77	40.72	96100	
10	10 5.13	5.73	7.78		125000	
11	10 4.38	5.74	7.81		60000	

All but the first three of the tests were made with inclined bearings of one kind or another, hence the ultimate strength per square inch is only given here for the first three; which, as Mr. Freeman says, were of "well-seasoned spruce, of excellent quality." Hence the average crushing-strength of spruce is doubtless considerably lower than the average of these three.

TESTS MADE ON THE GOVERNMENT MACHINE.

In Executive Document 12, 47th Congress, first session, will be found a series of tests of white and yellow pine posts made at the Watertown Arsenal; and these tests probably furnish us the best information that we possess in regard to the strength of wooden columns.

The summary of results is appended: —

COMPRESSION OF WHITE-PINE POSTS.

No. of Test.	Weight.	Average Rate of Growth.	Size of Post.				Sectional Area.	Ultimate Strength.		Manner of Failure.
			Length.		Width.	Depth.		Actual.	Lbs. per Sq. In.	
	lbs.	rings per in.	ft.	in.	in.	in.	sq. in.	lbs.		
511	7.0	–	0	15.00	5.50	5.50	30.25	108000	3570	Fibres crushed.
491	26.5	–	4	11.95	5.48	5.48	30.00	111000	3700	Failed at knot 14 in. from end.
492	31.0	–	5	0.05	5.48	5.48	30.00	99300	3100	" " " 15 " " "
493	43.5	–	7	6.10	5.50	5.52	30.40	67400	2217	" " knots 28 " " "
494	37.5	–	7	6.10	5.47	5.47	29.90	70800	2368	" " " 16 " " "
495	45.0	–	7	6.10	5.50	5.45	29.80	74100	2487	" " " 20 and 25 in. from end.
512	62.5	–	10	0.10	5.48	5.49	30.10	82000	2724	" " " 40 in. from end.
519	49.5	–	10	0.10	5.48	5.42	29.70	75400	2539	" " " near middle.
520	59.5	–	10	0.10	5.45	5.45	29.70	61700	2077	" " " 30 in. from end.
516	60.0	–	12	6.15	5.46	5.46	29.80	68000	2282	" " " 64 " " "
517	70.0	–	12	6.16	5.50	5.50	30.25	70800	2340	" " " 58 " " "
518	67.5	–	12	6.10	5.48	5.52	30.25	100000	3306	" " " 31 " " "
513	84.5	–	15	0.27	5.47	5.47	29.90	54000	1806	" " " 66 " " "
514	90.0	–	15	0.31	5.48	5.48	30.00	93500	3117	Defl. diagonally, and failed at knots.
515	91.0	–	15	0.28	5.33	5.33	28.40	94000	3310	" " " " " "
521	96.5	–	17	6.24	5.35	5.35	28.60	53000	1853	" " " " " "
522	101.0	–	17	6.25	5.40	5.38	29.00	40500	1396	" downward, wind-shake on concave side.
523	95.0	–	17	6.22	5.33	5.35	28.50	64800	2274	" horizontally.
524	109.5	–	20	0.25	5.28	5.26	27.80	45000	1619	" diagonally.
525	106.5	–	20	0.27	5.29	5.27	27.90	39500	1416	" "

COMPRESSION OF WHITE-PINE POSTS.—*Continued.*

No. of Test.	Weight.	Average Rate of Growth.	Size of Post.				Sectional Area.	Ultimate Strength.		Manner of Failure.
			Length.		Width.	Depth.		Actual.	Lbs. per Sq. in.	
	lbs.	rings per in.	ft.	in.	in.	in.	sq. in.	lbs.		
526	95.0	–	20	0.28	5.25	5.29	27.80	37000	1331	Defl. horizontally.
505	110.0	–	22	6.26	5.16	5.18	26.70	37000	1386	" "
506	113.0	–	22	6.23	5.20	5.22	27.10	37200	1373	" "
507	118.0	–	22	6.26	5.18	5.18	26.80	46700	1743	" "
502	127.0	–	25	0.31	5.27	5.25	27.70	30900	1116	" diagonally.
503	125.0	–	25	0.34	5.25	5.25	27.60	25800	935	" sidewise.
504	127.0	–	25	0.37	5.25	5.25	27.60	22200	804	" diagonally.
496	169.0	–	27	6.35	5.32	5.34	28.40	32500	1144	" "
497	173.0	–	27	6.40	5.35	5.35	28.60	32000	1119	" "
498	157.0	–	27	6.30	5.35	5.35	28.60	28000	979	" "
569	94.0	–	6	8.07	7.77	9.64	74.90	204000	2724	Failed at knots 10 in. from middle.
570	93.0	7	6	8.10	7.70	9.70	74.70	177000	2369	" " " near middle.
571	120.5	12	6	8.13	7.73	9.63	74.40	185000	2487	" " " 9 in. from end.
575	166.0	15	10	0.13	7.75	9.62	74.60	186000	2493	" " " 21 " " "
576	166.0	11	10	0.16	7.72	9.76	75.30	204600	2717	" " " 13 and 27 in. from end.
577	160.0	10	10	0.13	7.73	9.74	75.30	135000	1793	" " " 8 " 11 " " "
581	221.0	7	13	4.12	7.75	9.75	75.60	168000	2222	" " " 14 " 44 " " "
582	182.0	3	13	4.14	7.75	9.75	75.60	127500	1687	" " " 13 " 18 " " "
583	196.5	13	13	4.14	7.48	9.23	69.00	208500	3022	" " " 50 in. from end.
590	242.0	7	16	8.20	7.72	9.63	74.30	161000	2167	" " " 20 " " "

COMPRESSION OF WHITE-PINE POSTS. — *Concluded.*

No. of Test.	Weight.	Average Rate of Growth.	Size of Post.				Sectional Area.	Ultimate Strength.		Manner of Failure.
			Length.		Width.	Depth.		Actual.	Lbs. per Sq. In.	
	lbs.	rings per in.	ft.	in.	in.	in.	sq. in.	lbs.		
591	210.5	9	16	8.22	7.78	9.58	74.50	163000	2188	Failed at knots near middle.
592	274.0	8	16	8.21	7.75	9.75	75.60	175300	2319	" " " 75 in. from end.
533	278.0	–	20	0.24	7.59	9.62	73.00	140000	1918	Deflected horizontally.
534	285.0	–	20	0.26	7.38	9.29	68.60	166000	2420	Failed at knot near middle.
535	295.0	–	20	0.25	7.39	9.27	68.50	206000	3007	" " " 24 in. from middle.
539	352.0	–	23	4.20	7.72	9.75	75.40	140000	1857	Defl. horizontally.
540	357.0	–	23	4.20	7.72	9.61	74.20	170000	2291	" "
541	340.0	–	23	4.18	7.63	9.50	72.50	150000	2069	" diagonally.
536	427.0	–	26	8.20	7.46	9.36	69.80	169000	2421	" horizontally.
537	397.0	–	26	8.39	7.49	9.34	70.00	140000	2000	" "
538	384.0	–	26	8.27	7.46	9.37	69.90	134000	1917	" " knots on concave.
702	300.0	10	15	0.02	5.60	15.58	87.20	159000	1823	Failed at knots 23 in. from end.
703	267.0	5	15	0.08	5.61	15.60	87.50	168000	1926	Defl. horizontally.
704	267.0	12	15	0.00	5.65	15.60	88.10	165000	1873	" "
699	356.0	7	15	0.00	6.60	15.64	103.20	240000	2326	Failed at knots 40 in. from end.
700	395.0	10	14	11.94	6.60	15.64	103.20	239000	2316	" " " 10 and 60 in. from end.
701	376.0	7	15	0.00	6.60	15.60	103.00	203000	1971	" " " 15 in. from middle.
705	657.0	6	15	0.03	8.47	16.43	139.20	279400	2007	" " " at middle.
706	437.0	5	15	0.00	8.48	16.46	139.60	292000	2092	" " " " "
707	447.0	5	14	11.93	8.48	16.48	139.80	359000	2568	Defl. horizontally, fibres crushed near end.

COMPRESSION OF WHITE PINE. — Single Sticks and Built Posts.

In the multiple ones, dimensions of each stick are given.

No. of Test.	Weight.	Average Rate of Growth. Rings per Inch.	Dimensions of Post. Length.	Width.	Depth.	Sectional Area.	Compression in 150 In. Load = 500 lbs. per Sq. In.	Ultimate Strength. Actual.	Lbs. per Sq. In.
	lbs.		in.	in.	in.	sq. in.	in.	lbs.	
664	153	11	177.50	4.48	11.65	52.2	0.0545	110000	2107
665	143	10	180.00	4.48	11.64	52.1	0.1010	81500	1564
666	163	5	179.97	4.47	11.63	52.0	0.0895	70000	1346
667	228	13	180.00	5.40	11.30	61.0	0.0505	160000	2623
668	193	5	179.93	5.61	11.73	65.8	0.0622	156300	2375
669	253	5	180.00	5.64	11.76	66.3	0.0608	152300	2297
638	181 134 315	6 8	180.00 180.00	4.50 4.50	11.60 11.59	52.2 52.2 104.4	0.0670 0.0750	200000	1916
639	182 167 349	7 5	180.00 180.00	4.52 4.49	11.66 11.62	52.7 52.2 104.9	0.0645 0.0670	212000	2021
640	131 158 289	7 5	180.00 180.00	4.53 4.52	11.59 11.59	52.5 52.5 105.0	0.1060 0.0955	149000	1419
642	192 306 498	6 7	179.98 179.98	5.57 5.58	11.61 11.61	64.7 64.7 129.4	0.0770 0.0390	215000	1661
643	203 189 392	17 5	179.92 179.92	5.65 5.61	11.61 11.62	65.6 65.2 130.8	0.0440 0.0600	261000	1995
644	227 236 463	7 7	179.96 179.96	5.60 5.60	11.63 11.62	65.1 65.1 130.2	0.0596 0.0690	257800	1980
648	206 203 409	12 10	180.00 180.00	5.60 5.61	11.72 11.72	65.6 65.6 131.2	0.0590 0.0700	268000	2042
649	187 184 371	9 4	180.00 180.00	5.60 5.61	11.71 11.74	65.6 65.9 131.5	0.0600 0.0705	277000	2107
650	194 190 384	12 9	180.00 180.00	5.61 5.61	11.75 11.71	65.9 65.7 131.6	0.0530 0.0885	240000	1824
645	215 266 481	5 7	179.97 179.97	5.59 5.60	11.59 11.60	64.8 65.0 129.8	0.0560 0.0540	263200	2028
646	206 204 410	6 12	180.00 180.00	5.59 5.60	11.61 11.62	64.9 65.1 130.0	0.0493 0.0620	249000	1915
647	192 192 384	11 13	180.00 180.00	5.62 5.62	11.62 11.62	65.3 65.3 130.6	0.0630 0.0700	248000	1899
678	251 215 466	16 8	179.94 179.94	5.58 5.57	11.47 11.45	64.0 63.8 127.8	0.0529 0.0642	245500	1921
679	209 190 399	12 7	180.00 180.00	5.62 5.62	11.76 11.72	66.1 65.9 132.0	0.0664 0.0705	249000	1886
680	268 278 546	8 8	180.00 180.00	5.60 5.61	11.72 11.73	65.6 65.8 131.4	0.0650 0.0495	278000	2116
663	230 208 438	14 6	180.00 180.00	5.60 5.63	11.75 11.75	65.8 66.2 132.0	0.0621 0.0657	300000	2273
676	201 184 385	12 6	179.94 179.94	5.60 5.61	11.71 11.73	65.6 65.8 131.4	0.0530 0.0593	274500	2089
677	185 193 378	6 6	180.00 180.00	5.61 5.68	11.72 11.72	65.7 65.4 131.1	0.0551 0.0625	255000	1945

COMPRESSION OF WHITE PINE.— *Concluded.*

Single Sticks and Built Posts.

No. of Test.	Weight.		Average Rate of Growth. Rings per Inch.	Dimensions of Post. Length.	Width.	Depth.	Sectional Area.		Compression in 150 In. Load = 500 lbs. per Sq. In.	Ultimate Strength. Actual.	Lbs. per Sq. In.
	lbs.			in.	in.	in.	sq. in.		in.	lbs.	
690	175 226 199	600	18 11 18	180.00 180.00 180.00	4.52 5.56 4.46	11.62 11.70 11.62	52.5 65.0 51.8	169.3	0.0460 0.0580 0.0480	310000	1831
691	164 197 159	520	9 14 12	179.98 179.98 179.98	4.48 5.56 4.45	11.60 11.60 11.61	52.0 64.5 51.7	168.2	0.0526 0.0430 0.0390	372500	2215
692	248 153 213	614	13 12 5	177.25 177.25 177.25	5.62 4.50 5.60	11.60 11.60 11.57	65.2 52.2 64.8	182.2	0.0580 0.0641 0.0768	363000	1992
687	151 218 167	536	11 8 9	180.00 180.00 180.00	4.50 5.58 4.52	11.60 11.62 11.59	52.2 64.8 52.4	169.4	0.0460 0.0587 0.0533	325500	1919
688	176 236 163	575	10 10 11	180.00 180.00 180.00	4.50 4.62 4.50	11.60 11.60 11.60	52.2 65.2 52.2	169.6	0.0565 0.0645 0.0703	306000	1804
689	188 203 159	550	7 11 9	180.05 180.05 180.05	4.48 5.60 4.46	11.60 11.62 11.57	52.0 65.1 51.6	168.7	0.0510 0.0660 0.0789	340000	2015
681	193 192 146 157	688	5 5 9 9	179.95 179.95 179.95 179.95	4.47 4.52 4.48 4.51	11.60 11.65 11.65 11.65	51.9 52.7 52.2 52.5	209.3	0.0700 0.0714 0.0531 0.0762	362000	1734
682	161 156 163 151	631	7 14 7 9	180.00 180.00 180.00 180.00	4.50 4.50 4.49 4.50	11.63 11.64 11.60 11.63	52.3 52.4 52.1 52.3	209.1	0.0612 0.0546 0.0542 0.0916	414000	1980
683	169 219 218 181	787	10 11 9 7	180.03 180.03 180.03 180.03	4.46 5.62 5.60 4 46	11.63 11.69 11.70 11.61	51.9 65.7 65.5 51.8	234.9	0.0570 0.0530 0.0494 0.0590	501000	2133
684	146 210 234 166	756	10 12 7 10	180.00 180.00 180.00 180.00	4.50 5.64 5.60 4.48	11.64 11.59 11.58 11.61	52.4 65.4 64.8 52.0	234.6	0.0664 0.0610 0.0548 0.0514	529000	2255
685	145 220 209 166	740	10 10 12 9	180.00 180.00 180.00 180.00	4.50 5.63 5.61 4.48	11.60 11.62 11.62 11.62	52.2 65.4 65.2 52.0	234.8	0.0645 0.0650 0.0506 0.0546	430000	1831
686	145 183 159 150	637	11 10 12 8	180.00 180.00 180.00 180.00	4.48 4.50 4.50 4.52	11.61 11.56 11.36 11.61	52.0 52.0 51.1 52.5	207.6	0.0500 0.0315 0.0543 0.0680	395000	1903

COMPRESSION OF YELLOW-PINE POSTS.

No. of Test.	Weight.	Average Rate of Growth.	Size of Post. Length.		Width.	Depth.	Sectional Area.	Ultimate Strength. Actual.	Lbs. per Sq. In.	Manner of Failure.
	lbs.	rings per in.	ft.	in.	in.	in.	sq. in.	lbs.		
565	47.0	20	5	0.06	5.50	5.50	30.3	172000	5677	Fibres crushed 12 in. from end.
566	44.0	12	5	0.10	5.50	5.49	30.2	141000	4669	Failed at knots 15 " " "
567	45.0	25	5	0.10	5.48	5.54	30.4	168000	5526	" " " 9 and 17 in. from end.
568	47.5	20	5	0.10	5.46	5.50	30.0	108000	3600	" " " 8 in. from end.
562	72.5	–	7	5.98	5.45	5.52	30.1	140000	4651	" " season cracks, crushed.
563	59.5	–	7	6.14	5.46	5.46	29.8	124000	4161	" " knots 30 in. from end.
564	57.0	–	7	6.15	5.47	5.47	29.9	143500	4799	" " " 27 " " "
559	86.0	–	10	0.17	5.49	5.48	30.1	146200	4857	Defl. upward.
560	88.5	–	10	0.16	5.50	5.47	30.1	139000	4618	" downward.
561	110.0	–	10	0.15	5.51	5.50	30.3	143600	4739	Failed at knots in middle.
556	126.0	–	12	6.12	5.54	5.51	30.5	156000	5114	Defl. horizontally.
557	128.0	–	12	6.17	5.57	5.52	30.2	156000	5166	" diagonally.
558	141.5	–	12	6.20	5.50	5.50	30.3	150000	4950	" "
553	140.0	–	15	0.22	5.50	5.48	30.1	99200	3296	" downward.
554	148.0	–	15	0.24	5.49	5.47	30.0	130000	4333	" horizontally.
555	148.0	–	15	0.25	5.50	5.50	30.3	129000	4257	" diagonally.
550	172.0	–	17	6.24	5.48	5.47	30.0	90000	3000	" horizontally.
551	180.5	–	17	6.25	5.46	5.46	29.8	109000	3658	" "
552	167.5	–	17	6.26	5.50	5.50	30.3	93000	3069	" "
548	185.0	–	20	0.30	5.40	5.45	29.4	82000	2789	" "

COMPRESSION OF YELLOW-PINE POSTS. — *Continued.*

No. of Test.	Weight.	Average Rate of Growth.	Size of Post. Length.		Width.	Depth.	Sectional Area.	Ultimate Strength. Actual.	Lbs. per Sq. In.	Manner of Failure.
	lbs.	rings per in.	ft.	in.	in.	in.	sq. in.	lbs.		
549	185.0	–	20	0.28	5.45	5.46	29.8	87000	2946	Defl. horizontally.
545	212.0	–	22	6.28	5.64	5.59	31.5	73000	2340	" "
546	189.0	–	22	6.37	5.42	5.49	29.8	56000	1879	" upward.
547	266.0	–	22	6.30	5.65	5.61	31.7	62500	1972	" diagonally.
508	210.0	–	25	0.36	5.46	5.48	29.9	52000	1739	" horizontally.
509	235.0	–	25	0.30	5.46	5.49	30.0	59000	1967	" "
510	210.0	–	25	0.33	5.45	5.46	29.8	55500	1862	" "
499	258.0	–	27	6.44	5.31	5.29	28.1	60800	2163	" "
500	236.0	–	27	6.43	5.32	5.30	28.2	39000	1383	" diagonally.
501	256.0	–	27	6.40	5.31	5.30	28.1	44400	1580	" horizontally.
572	124.0	6	6	8.10	7.78	9.75	75.8	260000	3430	Failed at knot 30 in. from end.
573	131.0	14	6	8.08	7.76	9.77	75.8	286500	3780	" " " 26 " " "
574	161.5	21	6	8.10	7.74	9.81	75.9	383000	5046	" " knots 8 " " "
578	233.5	23	10	0.13	7.74	9.73	75.3	304000	4037	" " " 18 " " "
579	240.0	19	10	0.14	7.78	9.75	75.9	399500	5264	" 36 in. from end, grain wavy.
580	240.0	28	10	0.14	7.75	9.75	75.6	350000	4629	" at knot 17 in. from end.
584	266.0	18	13	4.18	7.68	9.72	74.6	307500	4122	" " " 15 and 29 in. from end.
585	258.0	6	13	4.15	7.75	9.74	75.5	284000	3762	" " " 45 in. from end.
586	336.0	23	13	4.20	7.75	9.75	75.6	296000	3921	" " " near middle.
587	318.0	24	16	8.17	7.70	9.74	75.0	274500	3660	" " " 50 in. from end.

COMPRESSION OF YELLOW-PINE POSTS.—*Continued.*

No. of Test.	Weight.	Average Rate of Growth.	Size of Post. Length.		Width.	Depth.	Sectional Area.	Ultimate Strength. Actual.	Lbs. per Sq. In.	Manner of Failure.
	lbs.	rings per in.	ft.	in.	in.	in.	sq. in.	lbs.		
588	381.0	12	16	8.24	7.74	9.73	75.3	382000	5073	Defl. horizontally.
589	339.5	21	16	8.26	7.25	9.49	68.8	304000	4419	" "
530	461.0	–	20	0.25	7.71	9.80	75.6	287000	3796	" "
531	410.0	–	20	0.28	7.76	9.58	74.3	275000	3701	Failed at knots 70 in. from end.
532	391.0	–	20	0.25	7.61	9.68	73.7	220000	2985	Defl. horizontally.
542	501.0	–	23	4.20	7.69	9.77	75.1	225000	2996	" "
543	501.0	–	23	4.20	7.69	9.73	74.8	220500	2948	" "
544	560.0	–	23	4.22	7.70	9.76	75.1	298000	3968	" "
527	531.0	–	26	8.30	7.40	9.47	70.0	210000	3000	" "
528	627.0	–	26	8.29	7.45	9.45	70.4	205000	2912	" "
529	535.0	–	26	8.30	7.46	8.94	66.7	184500	2766	" diagonally.
711	375.0	12	15	0.08	5.60	15.60	87.4	349000	3993	" horizontally.
712	397.0	21	15	0.05	5.63	15.57	87.7	339000	3865	" "
713	382.0	15	15	0.00	5.65	15.50	87.6	272500	3111	" "
484	432.0	–	15	0.25	6.90	15.85	109.4	270000	2468	" " knot on concave side.
708	464.0	17	15	0.00	6.61	15.51	102.5	340000	3317	Failed at knots at middle.
709	557.0	18	15	0.00	6.60	15.64	103.2	455000	4400	Defl. horizontally.
710	491.0	18	14	11.92	6.61	15.61	103.2	432000	4186	" "
641	561.0	5	15	0.10	8.25	16.25	134.0	428000	3194	Fibres crushed at middle, in vicinity of knots.
671	585.0	12	15	0.07	8.25	16.20	133.6	425000	3181	Defl. horizontally.

COMPRESSION OF YELLOW-PINE POSTS.—*Concluded.*

No. of Test.	Weight.	Average Rate of Growth.	Size of Post.			Sectional Area.	Ultimate Strength.		Manner of Failure.	
			Length.	Width.	Depth.		Actual.	Lbs. per Sq. In.		
	lbs.	rings per in.	ft. in.	in.	in	sq. in.	lbs.			
672	578.0	13	14 11.93	8.20	16.20	132.8	526000	3961	Defl. horizontally.	
485	434.0	–	15 0.27	6.75	15.95	107.7	175000	1625	" " knots on concave side.	
487	450.0	–	15 0.25	6.80	15.72	106.9	310000	2900	Defl. horizontally, initial bend in post.	Second test of these posts.
486	705.0	–	15 0.24	8.63	16.83	145.2	380000	2617	Opened shakes and seasoned cracks.	

COMPRESSION OF YELLOW PINE.

Single Sticks and Built Posts.

No. of Test.	Weight, in lbs.	Average Rate of Growth. Rings per Inch.	Dimensions of Post. Length.	Width.	Depth.	Sectional Area.	Compression in 150 In. Load = 500 lbs. per Sq. Inch.	Ultimate Strength. Actual.	Lbs. per Sq. In.
			in.	in.	in.	sq. in.			
673 673*a* 673*b*	260 13 7¼	11	180.05 20.00 20.00	4.09 4.01 3.95	11.35 4.01 3.97	46.04 16.08 15.68	0.0370	142200 94000 99600	3065 5846 6352
674	233	17	180.00	4.51	11.60	52.30	0.0492	131500	2515
675	194	22	180.00	4.34	11.60	50.30	0.0490	121200	2410
490	269	–	180.00	5.05	12.10	61.10	–	230000	3764
670	309	12	180.00	5.65	11.74	66.30	0.0455	205900	3106
489	287	–	180.08	5.85	12.05	70.50	–	250000	3546
654	326 290 } 616	11 6	180.00 180.00	5.63 5.62	11.71 11.71	65.9 65.9 } 131.8	0.0418 0.0540	470000	3566
655	292 329 } 621	34 24	179.93 179.93	5.64 5.63	11.72 11.72	66.1 66.1 } 132.2	0.0292 0.0315	580000	4387
656	282 287 } 569	13 11	180.00 180.00	5.61 5.61	11.71 11.71	65.7 65.7 } 131.4	0.0368 0.0466	480000	3653
651	262 275 } 537	11 7	180.00 180.00	5.58 5.58	11.71 11.71	65.3 65.3 } 130.6	0.0620 0.0514	360000	2756
652	310 370 } 680	16 16	180.00 180.00	5.58 5.58	11.70 11.71	65.3 65.3 } 130.6	0.0395 0.0360	588500	4506
653	310 311 } 621	16 12	180.00 180.00	5.63 5.59	11.71 11.68	65.9 65.3 } 131.2	0.0559 0.0550	436600	3328
657	340 319 } 659	9 13	179.96 179.96	5.63 5.64	11.72 11.71	66.0 66.0 } 132.0	0.0375 0.0305	580000	4394
658	319 332 } 651	12 9	179.98 179.98	5.59 5.59	11.71 11.72	65.5 65.5 } 131.0	0.0320 0.0436	448000	3420
659	320 293 } 613	12 8	180.00 180.00	5.61 5.61	11.73 11.73	65.8 65.8 } 131.6	0.0312 0.0372	600000	4559
660	283 294 } 577	25 18	180.03 180.03	5.61 5.63	11.22 11.24	62.9 63.3 } 126.2	0.0325 0.0400	510000	4041
661	274 310 } 584	12 9	180.00 180.00	5.66 5.60	11.70 11.72	66.2 65.6 } 131.8	0.0410 0.0365	410000	3111
662	292 280 } 572	14 8	180.00 180.00	5.61 5.61	11.75 11.75	65.9 65.9 } 131.8	0.0540 0.0500	388000	2944
693	242 276 256 } 774	11 15 23	179.97 179.97 179.97	4.50 5.50 4.49	11.61 11.56 11.62	52.2 63.6 52.2 } 168.0	0.0320 0.0325 0.0148	564000	3357
694	193 290 208 } 691	17 24 10	180.00 180.00 180.00	4.50 5.59 4.46	11.35 11.36 11.35	51.1 63.5 50.6 } 165.2	0.0500 0.0650 0.0610	500000	3027
695	207 290 246 } 743	8 16 16	180.00 180.00 180.00	4.49 5.20 4.50	11.35 11.34 11.35	51.0 59.0 51.1 } 161.1	0.0429 0.0290 0.0410	474000	2942

COMPRESSION OF YELLOW PINE.—*Concluded.*

SINGLE STICKS AND BUILT POSTS.

No. of Test.	Weight, in lbs.		Average Rate of Growth. Rings per Inch.	Dimensions of Post.			Sectional Area.		Compression in 150 In. Load = 500 lbs. per Sq. Inch.	Ultimate Strength.	
				Length.	Width.	Depth.				Actual.	Lbs. per Sq. In.
				in.	in.	in.	sq. in.				
696	217 253 190	660	25 11 15	180.00 180.00 180.00	4.51 5.50 4.48	11.24 11.23 11.23	50.7 61.8 50.3	162.8	0.0337 0.0567 0.0715	480000	2948
697	242 249 215	706	15 12 15	180.00 180.00 180.00	4.52 5.43 4.47	11.60 11.60 11.58	52.4 63.0 51.8	167.2	0.0240 0.0330 0.0336	540000	3230
698	224 255 296	775	15 8 15	179.94 179.94 179.94	4.46 5.53 4.50	11.60 11.70 11.59	51.7 64.7 52.2	168.6	0.0290 0.0385 0.0341	544000	3227
488	911		–	180.20	6.88 6.72	15.75 15.75	108.36 105.84	214.2	–	700000	3268

On page 668 will be found a series of tests of spruce columns made in the Laboratory of Applied Mechanics of the Massachusetts Institute of Technology, these columns having their ends resting against the platforms of the testing-machine. On the same page will be found also a series of tests made at the same place on timber columns with one end resting against a timber bolster.

A perusal of this table will show a great decrease in strength due to the presence of the bolster.

The four diagrams on the left of page 669 represent all the tests, with central load and no bolsters, of the four woods named, the abscissæ being ratios of length to least diameter and the ordinates breaking-strengths per square inch.

By whatever curve we may attempt to represent the values to be used in practice, it should pass nearly through the lowest results, as the timber was all of at least fair quality. It is also evident that up to a certain ratio of length to diameter, not far from 25, the strength is not affected by the length, and

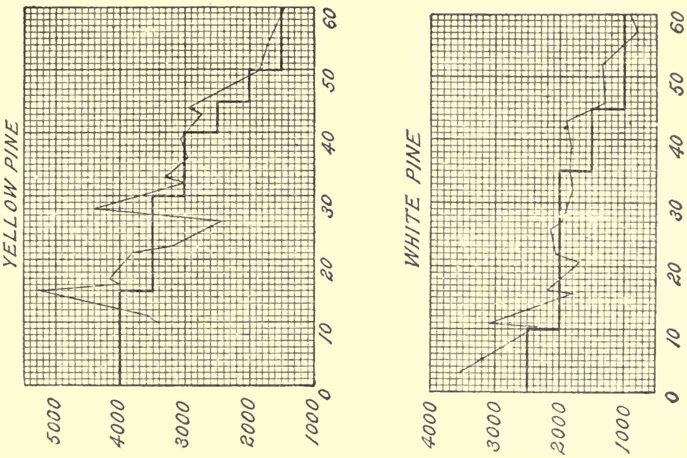
COMPRESSION OF SPRUCE COLUMNS.

Date.	Weight.	Gauge Length.	Size of Column. Length.		Width.	Depth.	Sectional Area.	Ratio of Length to Least Side.	Ultimate Strength. Actual.	Lbs. per Sq. In.	Ratio of Stress to Strain, commonly called Modulus of Elasticity.	Remarks.
1894	lbs.	in.	ft.	in.	in.	in.	sq. in.					
Mar. 19	226	100	17	0	8.00	8.00	64.00	25.00	155,000	2,422	1,297,800	Deflected diagonally.
" 20	276	"	18	0	8.25	8.06	66.52	26.79	140,000	2,105	1,385,600	" "
" 21	357	"	16	0	10.00	10.00	100.00	23.00	274,000	2,745	1,656,300	" " Crushed 5 feet from platform.
" 23	337	"	15	0	10.00	10.00	100 00	18.00	261,300	2,613	1,552,800	" downward. " at centre.
Apr. 3	162	"	11	1.00	8.13	8.13	66.02	16.37	180,000	2,727	1,405,200	" diagonally. " 4½ feet from platform.
" 4	188	"	9	10.75	8.00	12.13	97.00	14.84	191,000	1,969	1,009,200	" horizontally. " 1′ 3″ from respective ends.
" 6	182	"	9	7.88	7.88	12.00	94.50	14.71	188,500	1,995	982,540	" downward. " at knot 18″ from end.
" 26	126	50	9	0.13	8.13	8.13	66.02	13.31	191,500	2,901	1,439,900	Crushed 1′ from platform.
" "	100	"	7	6.00	8.13	8.13	66.02	11.07	170,800	2,507	1,442,700	" 6″ " "
" "	133	"	9	6.38	8.13	8.13	66.02	14.45	175,700	2,662	1,357,300	" 3′ " "
" "	66	"	7	8.38	7.75	7.75	60.06	8.82	191,900	3,195	1,222,400	" 1.5′ " "
" "	66.5	"	5	9.75	7.88	9.88	61.03	9.00	156,000	2,556	834,270	" " " "
" "	104	"	7	11.00	8.25	8.25	67.03	11.69	175,400	2,617	857,380	" " " "
Dec. 28	200	100	9	10.25	10.13	10.13	102 62	11.67	182,800	1,784	884,000	Deflected horizontally. Crushed at end. Part of column tested December 8, 1894.
1895 Jan. 3	....	100	12	00.00	10.00	10.00	100.00	14.40	190,000	1,900	1,180,000	Deflected diagonally. Crushed at centre.
Feb.	......		12	00.00	10.00	10.00	100.00	14.40	192,000	1,920	....	Crushed near one end.

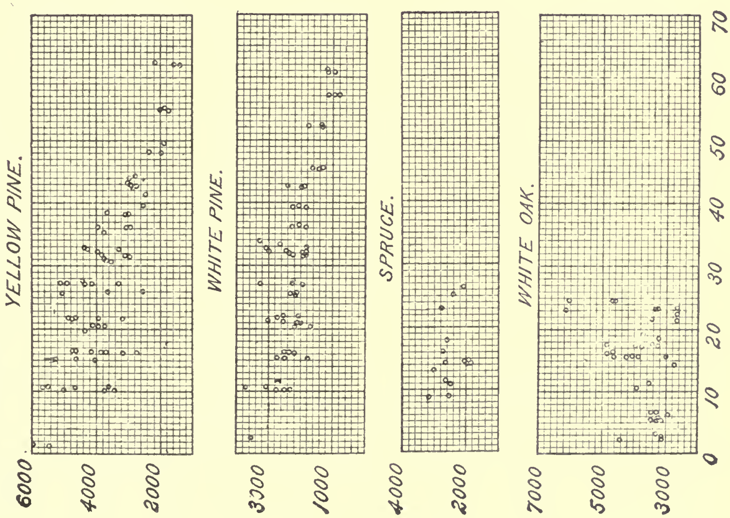
COLUMNS TESTED WITH A BOLSTER.

Date.	Weight.	Gauge Length.	Length (ft.)	Length (in.)	Width.	Depth.	Sectional Area.	Ratio of Length to Least Side.	Actual.	Lbs. per Sq. In.	Modulus of Elasticity.	Material of Column.	Material of Bolster.	Manner of Failure.
1894 Oct. 24	240	50	8	4.00	8.00	10.38	83.04	12.50	159,400	1,921	994,000	Oak	Oak	Split'g and crush'g at end next to bolster.
" 25	130	50	4	6.25	8 00	10.25	82.00	6.78	156,000	1,902	1,575.000	"	"	Splitting at end next to bolster.
" 29	85	50	5	0.50	9.50	9.88	93.86	6 37	166,000	1.770	1,082,000	Spruce	"	" " " " " "
Nov. 10	125	50	7	2.25	7.50	12.00	90.co	11.50	169,600	1,900	1,812,000	"	"	" " " " " "
" 12	88	50	5	5.69	9.50	9.88	93.86	6.91	198,000	2,110	1,316,000	"	Maple	" " " " " "
" 14	88	50	5	6.00	9.75	9.75	95.06	6.77	250,000	2,630	1,154,000	"	Oak	" " " " " "
" 26	151	50	4	6.75	4.75	7.75	36.81	11.53	170,300	1,870	884,000	Oak	"	Crush'g and split'g at end next to bolster.
" 27	183	100	11	0.00	9.50	9.63	91.49	13.89	143,000	1,560	1,443.000	Spruce	"	Splitting at end next to bolster.
Dec. 6	183	100	9	9.50	9 50	9.50	90.25	12.37	153,600	1.700	1,526,000	"	"	" " " " " "
" 7	275	100	11	11.50	10.00	10.00	100.00	14.35	138,000	1,380	889.700	"	"	" " " " " "
" 8	260	100	11	5.75	10.13	10.13	102.62	13.60	145.000	1.410	1.001.000	"	yel. pine	Split'g and crush'g at end next to bolster.

MR. E. F. ELY'S CURVES OF LOWEST RESULTS AND OF PROPOSED VALUES OF CRUSHING STRENGTH OF TIMBER COLUMNS TO BE USED IN PRACTICE.



DIAGRAMS OF RESULTS OF TESTS OF TIMBER COLUMNS



hence this part of the curve should be a horizontal line, its ordinate being about 3500 for yellow pine, 2000 for white pine or spruce, and 3000 for white oak.

For larger ratios than 25 the ordinate decreases, and might well be represented by some curve similar in character to those on page 425.

The right-hand diagrams are due to Mr. E. F. Ely, and represent the lowest results of the Government tests on yellow pine and white pine, and also the values he proposes for use in practice. Mr. Ely's diagrams are represented by the following rules: Let $A =$ area of section in square inches, $f =$ constant given in the tables following, $\frac{l}{r} =$ ratio of length to least side of rectangle.

Then breaking strength $= fA$.

White Pine.		Yellow Pine.	
$\frac{l}{r}$	f	$\frac{l}{r}$	f
0 to 10	2500	0 to 15	4000
10 to 35	2000	15 to 30	3500
35 to 45	1500	30 to 40	3000
45 to 60	1000	40 to 45	2500
		45 to 50	2000
		50 to 60	1500

In the case of spruce and white oak, if it is desired to apply the results to greater ratios of length to diameter than those tested, a similar reduction can be made in the value of f to that which takes place here in the case of white and yellow pine.

§ 238. **Factor of Safety.**—Whereas, in the case of iron bridge-work, it is very common to use a factor of safety 4, the apparent factors of safety that have been used and recommended for timber have varied very greatly, and naturally so, because the values assumed for breaking-strength have been so very variable, and while some have advised the use of apparent factors of safety greater than 4, nevertheless most of the building laws only require an apparent factor of safety 3, while making use of values of breaking-strength deduced from tests of small pieces.

In view of the above facts, it is true that the values of working-strength used in many cases have been very near the actual breaking-strength; and, indeed, it would be impossible to recommend any suitable factor of safety to be used with results derived from tests of small pieces. But if the true values of the breaking-strength as derived from tests of full-size pieces be used, it would seem to the writer that a factor of safety 4 will be sufficient for most ordinary timber constructions; i.e., that we should use for working-strength per square inch one-fourth of the breaking-strength per square inch. In the case of mill-work, and in other cases where there is the jarring of moving machinery, it is advisable to use a somewhat larger factor. This same reasoning will also apply to the case of beams bearing a transverse load, where they are designed with reference to their breaking-weight.

§ 239. **Transverse Strength of Timber.**—The table of Rankine, already given, represents the values of modulus of rupture as obtained from small specimens. Other values, not differing essentially from these, are given by Hatfield, Laslett, Thurston, Trautwine, and others, all based upon tests of small pieces. Confining ourselves to tests of full-size pieces, we find an account of a set of tests attributed by D. K. Clark, in his "Rules and Tables," to Edwin Clark and C. Graham Smith. The results are given below, and it will be seen that they are very much below those given by experimenters on small pieces.

Kind of Timber.	Breadth and Depth.	Span.	How Loaded.	Breaking-Weight.	Modulus of Rupture.
	in.	ft.			
American red pine	12.0 × 12.0	15.00	Centre	33497	5238
" " "	12.0 × 12.0	15.00	"	29908	4680
" " "	6.0 × 6.0	7.50	"	7370	4608
Memel fir . . .	13.5 × 13.5	10.50	Distributed	68560	5274
" " . . .	13.5 × 13.5	10.50	"	68560	5274
Baltic fir	6.0 × 12.0	12.25	Centre	19145	4878
" "	6.0 × 12.0	12.25	"	23625	6020
Pitch pine . . .	6.0 × 12.0	12.25	"	23030	5868
" " . . .	6.0 × 12.0	12.25	"	23700	6048
" " . . .	14.0 × 15.0	10.50	"	134400	8064
" " . . .	14.0 × 15.0	10.50	"	132610	7956
Red pine	6.0 × 12.0	12.25	"	16800	4284
" "	6.0 × 12.0	12.25	"	19040	4860
Quebec yellow pine	14.0 × 15.0	10.50	Distributed	68600	4122
" " "	14.0 × 15.0	10.50	"	68600	4122
" " "	14.0 × 15.0	10.50	Centre	85792	5148
" " "	14.0 × 15.0	10.50	"	76160	4572

Two tests by R. Baker are also mentioned by D. K. Clark.

Bauschinger also made quite an extensive series of tests of German woods, an account of which will be given later on.

A great many tests of the strength and stiffness of full-size beams of spruce, yellow pine, oak, and white pine, both under centre loads and distributed loads, have been carried on in the Laboratory of Applied Mechanics of the Massachusetts Institute of Technology. Tests have also been made upon the effect of time on the stiffness of such beams, also on the strength of built-up beams, and of floors and framing-joints, all full size. A summary of the results obtained will be given, and conclusions drawn as to the proper values of the modulus of rupture and modulus of elasticity, etc., to be used in practice.

SUMMARY OF THE TESTS.

The tests recorded may be divided into six classes: —

1°. Spruce beams.
2°. Yellow-pine beams.
3°. Time tests.
4°. Oak beams.
5°. White-pine beams.
6°. Framing-joints.

1°. *Spruce Beams.* — Before giving a summary of the tests made in this laboratory, I will insert some of the moduli of rupture and moduli of elasticity given by different authorities.

Moduli of rupture are given as follows: —

	Maximum.	Minimum.	Mean.
Hatfield	12996	7506	9900
Rankine	12300	9900	11100
Laslett	9707	7506	9045
Trautwine	–	–	8100
Rodman	–	–	6168

Hatfield's, Laslett's, Trautwine's, and Rodman's figures are from their own experiments. Trautwine advises, for practical use, to deduct one-third on account of knots and defects, hence to use 5400. The tables show the values obtained in these tests, and I will add a recommendation as to the values of modulus of rupture and modulus of elasticity suitable to use in practice.

As a result of the tests thus far made in my laboratory, it seems to me safe to say, if our Boston lumber-yards are to be taken as a fair sample of the lumber-yards in the case of spruce, — if such lumber is ordered from a dealer of good repute, no selection being made except to discard that which is rotten or has holes in it, — that 3000 lbs. per square inch is all that could with any safety be used for a modulus of rupture, and even this might err in some cases in being too large; (2°) that, if the lumber is carefully selected at any one lumber-yard, so as to take only the best of their stock, it would not be safe to use for modulus of rupture a number greater than 4000; and if we required a lot of spruce which should have a modulus of rupture of 5000, it would be necessary to select a very few pieces from each lumber-yard in the city. With a factor of safety four, we should have for greatest allowable outside fibre stress in the three cases respectively, 750, 1000, and 1250.

The modulus of elasticity (i.e., that determined from the immediate deflections) was: maximum, 1588548; minimum, 897961; mean, 1332500.

As will be explained when the results of the time tests are given, if by means of the ordinary deflection formulæ we wish to compute the deflection which a spruce beam will acquire under a given load after it has been applied for a long time, we should use for modulus of elasticity in the formulæ not more than one-half of the values given above, or about 666300.

SPRUCE BEAMS.

No of Test.	Width and Depth.	Distance between Supports.		Manner of Loading.	Breaking Load.	Modulus of Rupture in Lbs. per Sq. In.	Modulus of Elasticity in Lbs. per Sq. In.	Calculated Intensity of Shear in Lbs. per Sq. In.	Manner of Breaking.
	inches.	ft.	in.		lbs.				
3	2 × 12	15	0	Centre	5,894	5.526	1,237,200	181	Crushing and tension.
4	2 × 9	6	7 1/2	"	7,322	5,389	1,067,900	301	" " "
5	2 × 12	15	0	"	5,586	5,237	....	174	Tension.
5a	2 × 12	7	0	"	8,982	..	....	... .	"
6	2 3/4 × 9	6	8	"	7,586	4.082	938,500	230	"
7	3 × 9	4	0	"	11,086	3,285	....	308	"
8	3 × 9	10	0	"	6,086	4,508	.....	170	"
9	3 × 9	15	0	"	8,086	5.651	....	141	Tension and crushing.
10	3 7/8 × 12	20	0	"	6,586	4.253	....	106	Tension.
11	2 1/2 × 13 1/2	10	0	"	9.585	3,787	.. ..	208	"
12	13 3/4 × 12	16	0	4 1/2 ft. from end	7,585	3,271	....	126	"
14	7 × 2	7	0	Centre	1,944	8,748	....	105	Crushing.
15	1 3/4 × 6 3/4	7	0	"	4,785	7.562	....	304	Tension.
16	3 × 9	6	8	"	9,985	4,931	....	277	"
17	3 × 9	6	8	4 points 16″ apart	16,744	4,961	....	465	"
18	3.9 × 1	16	0	4 1/2 ft. from end	12.585	5.218	....	202	"
19	2 × 12	14	0	Centre	4,404	3.854	1,482,600	138	"
20	2 × 12	14	0	"	5,108	4.469	1,588,500	160	"
21	3 15/16 × 12	14	0	"	8,627	3,834	1,187,100	137	"
22	3 7/8 × 12	14	0	"	12,545	5.666	1,332,700	......	Tension and shearing.
23	3 7/8 × 12 1/4	14	0	"	6,917	2,995	898,000	108	Tension.
24	3 × 11 7/8	14	0	"	8,927	5.442	1,572,500	......	Shearing.
25	2 × 9 3/4	14	0	"	3,198	4,139	1,460,600	123	Crushing and tension.
26	2 3/4 × 12	14	0	"	6,819	4.339	1,396,700	155	Tension.
27	1 15/16 × 10	14	0	"	4,306	5,601	1,355,900	167	"
28	4 1/5 × 12	18	0	"	8,829	4,816	1,397,100	134	"
29	4 × 12 1/4	18	0	"	8,324	4,586	1,259,200	129	"
31	3 1/8 × 12	18	0	"	7,721	5,559	1,231,500	......	Crushing and shearing.
35	6 × 12	18	0	"	11,188	4,196	1,347,900	......	Shearing.
36	2 × 11 7/8	7	2	"	7,870	3,599	....	......	Shearing and crushing.
37	4 × 11 3/4	12	0	"	10,572	4,135	....	169	Tension.
45	3 1/8 × 11 3/4	16	4	"	8,072	4,436	... .	133	"
46	3 11/16 × 12	10	2	Centre	13,772	4,746	. ...	..	Shearing.
49	3 3/4 × 11 3/4	14	0	"	12,076	5,878	....	205	Tension and crushing.
60	4 × 12	17	4	12 points	26,000	7.448	1,461,000	406	" " "
66	4 × 12	15	8	Centre	14.576	7,211	1.336.200	230	Tension.
70	4 1/4 × 12 1/2	17	4	12 points	14,633	3,748	1,551,800	205	"
72	4 × 12 3/16	17	4	"	11,333	3,252	1.228,600	177	"
90	6 × 12	17	4	"	26,100	5,102	1.587,600	.....	Shearing.
196	3 13/16 × 11 15/16	19	10	Centre	9.187	6,037	1,513,400	153	Crushing.
197	3 23/32 × 11	18	4	"	6,486	4,757	1,282.000	120	Tension.
198	3 7/8 × 11 3/4	17	6	"	10,178	5,993	1.529,000	170	"
199	3 4/5 × 11 15/16	19	6	"	9,784	6,649	1,510.900	171	"
200	4 × 12 5/8	19	10	"	6,597	3,845	1,100,200	101	"
201	3 11/16 × 11 3/4	18	4	"	10.958	5,941	1,490,200	191	"
202	3 31/32 × 12 1/32	18	10	"	10,077	5.946	1,475.100	160	"
203	3 13/16 × 11 23/32	19	10	"	11,184	7,626	1,850.600	189	Crushing.
204	3 23/32 × 11 27/32	18	8	"	10,970	7,066	1.470,400	188	"
205	3 31/32 × 12 1/32	18	8	"	8,082	4,727	1,208,000	123	Tension.
297	4 × 12	15	5	"	6,025	2,923	1,010,000	97	"
298	3 11/16 × 12	9	8	"	15,335	4,700	1,215,900	......	
299	3 13/16 × 12	9	8	"	14,134	4,484	1,126,400	232	"

SPRUCE BEAMS.

No. of Test.	Width and Depth.	Distance between Supports.	Manner of Loading.	Breaking Load.	Mod. of Rupture in Lbs. per Sq. In.	Mod. of Elasticity in Lbs. per Sq. In.	Cal. Intensity of Shear in Lbs. per Sq. In.	Manner of Breaking.
	inches.	ft. in.		lbs.				
300	4×12	9 6	Centre	19,040	5,652	1,173,000	298	Tension.
301	$4 \times 12\frac{1}{8}$	9 6	"	19,041	5,537	1,210,000	295	"
304	$4\frac{1}{8} \times 12$	15 0	"	8,475	3,852	1,466,800	.....	Shearing.
305	$4\frac{1}{16} \times 12$	15 0	"	9,779	4,515	1,345,600	152	Tension.
307	$4\frac{7}{16} \times 12$	16 0	"	10,233	4,594	1,238,100	156	"
308	$4\frac{1}{4} \times 12$	16 0	"	12,186	5,735	1,466,700	179	
309	$3\frac{15}{16} \times 12\frac{1}{8}$	16 0	"	9,283	4,618	1,067,300	141	Tension.
311	$4\frac{1}{4} \times 12$	16 0	"	8,891	4,184	1,156,100	129	"
314	$3\frac{7}{8} \times 12$	16 0	"	6,670	3,442	978,700	102	"
316	$4 \times 11\frac{7}{8}$	16 0	"	11,885	6,068	1,290,300	184	"
317	$4\frac{5}{16} \times 12$	16 0	"	12,189	5,638	1,479,400	175	"
318	$4 \times 11\frac{7}{8}$	16 0	"	11,875	6,113	1,414,800	191	"
319	$3\frac{7}{8} \times 12$	16 0	"	12,386	6,393	1,470,900	201	Crushing.
320	$3\frac{15}{16} \times 11\frac{13}{16}$	16 0	"	5,386	2,828	1,092,600	85	
321	$3\frac{3}{4} \times 11\frac{1}{8}$	16 0	"	13,086	8,120	1,899,800	.....	Tension and shearing.
322	$4 \times 12\frac{1}{16}$	16 0	"	9,571	4,639	1,081,400	145	Tension.
323	$3\frac{3}{4} \times 12$	15 0	"	9,170	4,585	1,196,600	154	"
325	$4 \times 12\frac{1}{8}$	16 0	"	8,175	4,003	979,300	128	"
327	4×12	15 0	"	4,665	2,187	890,700	74	....
328	$3\frac{7}{8} \times 11\frac{7}{8}$	16 0	"	12,375	6,522	1,701,500	203	Shearing.
330	$3\frac{3}{4} \times 12$	15 0	"	6,075	3,037	1,089,500	103	
333	$4\frac{1}{8} \times 11\frac{13}{16}$	15 0	"	5,353	2,512	849,500	83	
336	$4 \times 12\frac{1}{8}$	15 0	"	7,961	3,655	1,002,400	124	Tension.
344	$3\frac{13}{16} \times 12\frac{1}{8}$	15 0	"	10,761	5,184	1,369,100	175	"
348	$3\frac{7}{8} \times 11\frac{3}{4}$	15 0	"	13,162	6,652	1,689,600	217	"
349	$3\frac{7}{8} \times 11\frac{15}{16}$	15 0	"	7,760	3,801	1,116,300	126	Tension at knot.
351	$3\frac{13}{16} \times 11\frac{3}{4}$	15 0	"	10,464	5,385	1,382,000	176	Crushing.
353	$3\frac{3}{4} \times 12$	15 0	"	11,978	5,989	1,253,900	200	"
354	$4 \times 12\frac{1}{16}$	16 0	"	8,184	4,049	1,169,000	127	Tension.
355	$3\frac{15}{16} \times 11\frac{11}{16}$	15 0	"	7,760	3,893	1,233,400	126	"
356	$3\frac{7}{8} \times 11\frac{7}{8}$	15 0	"	9,963	4,774	1,464,700	157	"
359	$3 \times 9\frac{7}{8}$	17 0	"	4,740	4,958	1,147,500	120	Compression and tension
363	$4\frac{1}{16} \times 12\frac{1}{8}$	15 0	"	10,170	4,598	1,285,100	157	" " "
365	$3\frac{7}{8} \times 12\frac{1}{16}$	17 0	"	9,176	4,980	1,129,400	147	....
366	$3 \times 9\frac{7}{8}$	17 0	"	5,540	5,793	1,411,200	141	Compression and tension
367	$2\frac{3}{4} \times 9\frac{7}{8}$	17 0	"	1,630	1,869	1,211,000	47	Tension.
369	$3\frac{15}{16} \times 12\frac{1}{8}$	17 0	"	9,876	5,217	1,373,000	157	Tension and compression
370	4×12	17 0	"	10,789	5,732	1,408,700	170	....
372	$3\frac{1}{16} \times 9\frac{7}{8}$	12 0	2 points 2′ from centre	3,722	1,794	854,100	94	Tension.
373	$3 \times 9\frac{7}{8}$	12 0	2 points 2′ from centre	9,595	4,733	1,126,500	244	Tension and compression
374	$2\frac{7}{8} \times 9\frac{3}{4}$	12 0	2 points 2′ from centre	10,767	5,673	1,593,000	289	Shearing.
377	$2\frac{13}{16} \times 9\frac{7}{8}$	12 0	$\frac{1}{2}$ at centre $\frac{1}{4}$—3′ each from centre	4,626	2,722	1,072,200	128	Tension.
378	$3\frac{7}{8} \times 12\frac{1}{8}$	15 6	2 points 2′ from centre	17,259	6,252	1,720,000	277	Tension.
380	$1\frac{13}{16} \times 9\frac{15}{16}$	12 0	Centre	4,216	5,086	1,361,000	176	Tension.
383	$3\frac{7}{8} \times 12\frac{1}{8}$	16 0	"	9,474	4,840	1,205,000	153	
384	$2\frac{7}{8} \times 10$	11 6	"	9,360	6,739	1,537,000	244	Compression.
385	$2\frac{7}{8} \times 10$	11 6	"	5,005	3,604	1,082,000	131	Tension.
387	$4\frac{1}{8} \times 12\frac{1}{8}$	16 0	"	9,869	4,735	1,666,600	149	Tension and compression
387a	$4 \times 11\frac{3}{4}$	16 0	"	11,956	6,232	1,442,800	191	Shearing, crushing & ten.
389	$2\frac{3}{4} \times 9\frac{7}{8}$	11 6	"	7,702	5,945	1,445,000	213	Tension and compression
392	$3\frac{1}{2} \times 12$	16 0	"	15,813	9,036	2,132,400	284	Tension.
395	$3\frac{11}{16} \times 11\frac{15}{16}$	12 0	"	8,132	3,339	1,055,500	138	Shearing.
397	4×12	17 0	"	9,574	5,084	1,102,000	151	Tension.

SPRUCE BEAMS.

No. of Test.	Width and Depth.	Distance between Supports.		Manner of Loading.	Breaking Load.	Modulus of Rupture in Lbs. per Square Inch.	Modulus of Elasticity in Lbs. per Square Inch.	Calculated Intensity of Shear in Lbs. per Square Inch.	Manner of Breaking.
	inches.	ft.	in.		lbs.				
398	$3\frac{7}{8} \times 12$	17	0	Centre	10,677	5,854	1,602,000	174	Tension.
400	$2\frac{13}{16} \times 10\frac{1}{8}$	15	0	"	5,725	5,361	1,433,400	152	"
401	$1\frac{1}{8} \times 9\frac{3}{4}$	15	0	"	4,107	6,221	1,609,600	170	"
404	3×10	15	6	"	6,828	6,350	1,563,900	173	"
405	4×12	15	0	"	7,160	3,356	1,244,300	113	Shearing.
407	$4\frac{1}{8} \times 11\frac{7}{8}$	16	0	"	8,660	4,287	1,099,300	134	Tension.
408	$4 \times 12\frac{3}{16}$	16	0	"	8,762	4,248	1,280,300	136	"
409	$3\frac{13}{16} \times 12$	15	6	"	6,152	3,127	986,000	102	....
411	4×12	17	6	12 points	9,157	2,645	1,117,000	143	Tension.
414	$2\frac{15}{16} \times 10$	15	0	Centre	5,200	4,902	1,270,500	138	Crushing at knot.
415	$3 \times 9\frac{13}{16}$	16	0	"	3,700	3,723	1,105,700	98	Tension.
416	$2\frac{7}{8} \times 9\frac{13}{16}$	15	6	"	7,900	8,087	1,548,400	215	Longitudinal shear and tension.
417	$4\frac{1}{4} \times 11\frac{3}{4}$	14	0	"	12,100	5,139	1,622,500	182	Tension.
421	$3\frac{7}{8} \times 11\frac{7}{8}$	15	0	"	12,000	6,010	1,318,000	199	Tension and longitudinal shear.
422	$3\frac{7}{8} \times 11\frac{3}{4}$	14	0	"	13,200	6,182	1,304,700	217	Tension.
423	4×12	14	0	"	13,260	5,762	1,213,100	207	Tension, crushing, and longitudinal shear.
424	$3\frac{1}{8} \times 11\frac{7}{8}$	16	0	"	12,000	7,105	2,100,000	221	Tension.
425	4×12	14	0	"	9,710	4,206	1,067,800	152	"
426	$4\frac{1}{8} \times 12$	14	0	"	6,160	2,562	966,880	93	"
427	$4 \times 12\frac{1}{4}$	14	0	"	8,110	3,663	1,249,900	135	"
428									
429	4×12	14	0	"	8,665	3,753	1,148,300	135	"
430	$2\frac{5}{8} \times 10$	14	0	"	4,440	4,217	1,377,600	127	"
431	$2\frac{3}{4} \times 10$	14	0	"	6,500	5,897	1,464,200	177	"
432	$2\frac{3}{4} \times 10$	14	0	"	5,500	4,991	1,156,400	150	"
433	$2\frac{13}{16} \times 10\frac{1}{16}$	14	0	"	2,925	2,542	896,800	78	"
434	$2\frac{11}{16} \times 10$	14	0	"	7,020	6,525	1,819,900	196	"
435	$2\frac{3}{4} \times 10$	14	0	"	4,990	4,518	1,180,300	136	"
436	$2\frac{3}{4} \times 9\frac{13}{16}$	14	0	"	5,120	4,830	1,242,800	139	"
438	$2\frac{3}{4} \times 9\frac{7}{8}$	14	0	"	6,400	5,969	1,358,100	177	"
439	$2\frac{3}{4} \times 10$	14	0	"	6,000	5,467	1,263,700	164	"
441	$2\frac{3}{4} \times 10$	14	0	"	8,080	7,347	1,732,800	220	"
442	$1\frac{15}{16} \times 9\frac{9}{16}$	11	0	"	4,130	4,519	1,368,000	166	"
443	4×12	14	0	"	10,760	4,780	1,129,800	172	"
444	4×12	14	0	"	10,000	4,495	1,238,200	163	"
445	$3\frac{7}{8} \times 12$	14	0	"	7,875	3,673	1,327,600	133	Longitudinal shear and tension.
446	4×12	14	0	"	10,400	4,633	1,286,800	167	Tension.
448	$4\frac{1}{8} \times 12\frac{1}{16}$	14	0	"	10,800	4,639	1 236,400	167	"
449	$3\frac{7}{8} \times 12$	14	0	"	8,500	3,929	1,213,000	142	"
450	$4\frac{1}{16} \times 11\frac{7}{8}$	14	0	"	10,400	4,658	1,160,900	166	Longitudinal shear and tension.
456	$7\frac{13}{16} \times 12\frac{3}{8}$	19	0	"	14,000	4,098	1,267,000	113	Tension.
457	$7\frac{7}{8} \times 12$	19	0	"	14,750	4,456	1,176,700	121	"
458	$7\frac{7}{8} \times 12$	19	0	"	16,950	5,216	1,440,000	139	"

SPRUCE BEAMS.

No. of Test.	Width and Depth.	Distance between Supports.		Manner of Loading.	Breaking Load.	Modulus of Rupture in Lbs. per Square Inch.	Modulus of Elasticity in Lbs. per Square Inch.	Calculated Intensity of Shear in Lbs. per Square Inch.	Manner of Breaking.
	inches.	ft.	in.						
459	6 × 12	14	0	Centre	11,600	3,345	915,000	123	Tension.
460	5¾ × 11¾	14	0	"	16,350	7,580	1,640,800	184	Crushing, tension, and shear
461	6 × 12	14	0	"	15,600	4,590	1,369,000	166	Shear.
481	5½ × 12	15	0	"	21,000	6,120	1,845,000	231	"
488	6¼ × 12	19	0	"	12,950	4,900	1,271,000	130	Tension.
489	6 × 11¾	22	0	"	11,700	5,800	1,549,000	129	Crushing and tension.
491	5¾ × 12	14	0	"	13,500	4,070	1,117,000	147	Tension.
495	5¾ × 12	14	0	"	15,500	4,660	1,195,000	168	"
496	6 × 12	10	0	"	22,500	4,700	1,186,000	236	Shear.
497	6 × 12	10	0	"	24,600	5,150	....	258	"
498	6 × 12	15	0	"	5,550	1,850	923,000	60	Tension.
499	6 × 12	15	0	"	16,700	5,230	1,274,000	176	"
501	6 × 12	16	0	"	10,800	3,640	948,000	115	"
Average values......................						4,521	1,310,584		

MAPLE BEAMS.

No. of Test.	Width and Depth.	Distance between Supports.		Manner of Loading.	Breaking Load.	Modulus of Rupture in Lbs. per Square Inch.	Modulus of Elasticity in Lbs. per Square Inch.	Intensity of Shear in Lbs. per Square Inch.	Manner of Breaking.
	inches.	ft.	in.						
463	4 × 12	15	6	Centre	9,650	4,732	1,396,000	155	Tension.
467	4 × 12	16	0	"	12,300	6,200	1,627,000	196	"
469	4 × 12	13	0	"	17,800	7,280	1,448,000	282	"
470	4 × 11¾	14	0	"	9,200	4,260	1,262,000	150	"
473	3½ × 12	15	0	"	14,600	7,900	1,587,000	265	"
475	3⅞ × 11⅞	14	0	"	18,450	8,570	1,597,000	305	"
477	2⅝ × 11⅝	14	0	"	15,500	11,080	1,702,000	385	"
Average values...........................						7,146	1,517,000		

Yellow-Pine Beams. — The moduli of rupture in common use are given as follows by different authorities; viz., —

	Maximum.	Minimum.	Mean.
Hatfield	21168	9000	15300
Laslett	14162	10044	12254
Trautwine	-	Yellow pine	9000
		Pitch pine	9900
Rodman	9876	8796	9293

A summary of the figures obtained from these tests will be given in a table at the end of these remarks.

It will be observed that we have for

	Maximum.	Minimum.	Mean.
Modulus of rupture . .	11360	3963	7486
Modulus of elasticity .	2386096	1162467	1757900

If by means of the ordinary deflection formulæ we wish to compute the deflection which a yellow-pine beam will acquire under a given load after it has been applied for a long time, we should use for modulus of elasticity in the formulæ about one-half of the values above, or about 878950 (see report of time tests).

For the modulus of rupture of yellow pine of fair quality, in the light of the tables on pages 683 and 684, I should not feel justified in using a number greater than 5000 pounds per square inch. With a factor of safety four we should have about 1200 as our greatest allowable fibre-stress.

YELLOW-PINE BEAMS.

No. of Test.	Width and Depth.	Distance between Supports.		Manner of Loading.	Breaking Load.	Modulus of Rupture in Lbs. per Square Inch.	Modulus of Elasticity in Lbs. per Square Inch.	Calculated Intensity of Shear in Lbs. per Square Inch.	Manner of Breaking.
	inches.	ft.	in.		lbs.				
30	3 × 13⅞	14	0	Centre	15,158	6,614	1,937,000	......	Shearing. [shearing.
32	4 1/16 × 12 3/16	18	0	"	13,751	7,383	1,734,000	....	Tension, crushing, and
33	3 15/16 × 12¼	18	0	"	9,832	5,386	1,794,000	......	Shearing.
47	3 × 13¾	14	0	"	19,574	8,696	2,386,000	359	Crushing.
50	4 × 14 1/16	21	0	"	12,875	5,914	1,256,300	....	Shearing. [shearing.
53	3⅛ × 14	24	6	"	10,076	7,206	1,784,400	179	Tension, crushing, and
54	3 × 12⅛	24	0	"	9,576	9,380	2,116,800	203	Crushing.
56	3⅛ × 14	15	4	"	10,572	4,764	1,490,400	185	Tension.
57	2 15/16 × 12	19	2	"	8,472	6,950	1,444,500	182	"
59	9 × 13¾	24	0	"	21,083	5,352	1,417,800	133	Tension and crushing.
62	4⅛ × 12⅛	19	10	"	15,461	9,102	2,038,000	237	Crushing.
63	4 3/16 × 12 3/16	20	0	"	14,073	8,145	1,599,000	211	Tension.
64	4⅛ × 12¼	19	10	"	10,573	6,098	1,918,000	161	"
65	4 × 12¼	19	8	"	11,573	6,782	1,966,700	183	Crushing.
67	4¼ × 12	18	6	"	13,374	7,277	1,787,600	196	Tension.
68	4 × 12⅛	19	9	"	17,676	10,872	2,381,700	278	"
69	3 3/16 × 14	20	0	"	6,675	3,963	1,169,300	115	Crushing.
71	4¼ × 12	18	2	"	16,074	8,248	1,512,200	227	Tension.
74	4 × 12	20	0	"	11,071	7,004	1,628,100	175	"
75	4 × 11¾	19	9	"	13,771	9,391	1,850,700	233	"
76	4⅛ × 12 7/16	17	4	12 points	15,825	4,207	1,344,100	233	"
77	4¼ × 12	17	4	" "	37,325	10,286	2,123,200	551	"
78	4 × 12½	22	10	Centre	7,172	4,845	1,455,300	109	....
79	4 × 12	19	8	"	....	......	2,087,600	.....	....
81	4⅛ × 12¼	17	4	12 points	16,025	4,349	1,162,500	238	Tension.
82	4 × 12	19	8	Centre	15,571	9,671	1,607,300	246	"
84	4¼ × 12¼	21	4	"	11,374	6,985	1,501,900	167	....
85	4 × 11¾	20	6	"	16,874	11,360	2,246,200	271	Tension.
87	4 × 12¼	21	4	"	11,272	7,335	1,535,600	175	"
88	6 × 12¼	20	4	"	15,283	6,112	1,613,000	161	"
91	4 × 12	19	10	"	18,074	11,303	2,223,800	282	Tension and shearing.
92	6 × 12	6	5	"	38,090	5,092	......	397	....
144	4¼ × 12	21	0	"	9,433	6,012	1,628,100	145	Tension.
145	4⅛ × 12⅛	19	3	"	11,201	6,400	1,472,000	191	"
146	4 × 12	19	4	"	13,341	8,060	1,839,700	206	"
147	4 × 12	19	2	"	16,748	10,032	2,286,000	.....	Shearing.
216	4⅛ × 12¼	15	8	"	15,453	7,040	1,557,300	......	Crushing and shearing.
218	4¼ × 12⅛	15	6	"	15,613	7,484	2,010,300	244	Crushing and tension.
219	4¼ × 12⅛	15	6	"	16,632	7,427	1,623,800	244	Tension.
220	4⅛ × 12¼	15	6	"	17,710	7,982	1,775,100	265	"
221	4⅛ × 12 1/16	16	0	"	16,330	7,836	1,931,200	248	"
222	4⅛ × 12¼	16	0	"	18,515	8,884	1,786,000	285	"
223	4 × 12	17	0	"	13,492	7,167	1,638,500	212	Crushing.
224	3 15/16 × 11 15/16	17	0	"	16,426	8,958	1,938,900	264	Tension.
225	4⅛ × 12	15	6	"	18,705	8,786	1,729,900	285	"
226	4⅛ × 12	15	6	"	16,594	7,794	1,611,200	253	"
252	3 15/16 × 11 15/16	17	0	"	11,006	6,002	1,271,000	177	"
254	4 × 12	16	0	"	15,226	7,613	1,617,700	240	"
255	4 7/16 × 12⅝	15	6	"	19,425	7,975	2,214,400	267	Crushing.
257	4¼ × 12⅛	16	0	"	17,424	8,031	1,789,800	256	"
258	4 7/16 × 12	16	0	"	18,319	8,256	1,830,000	260	Tension.
261	4 3/16 × 12	15	0	"	17,818	7,978	1,989,700	268	"
324	3¾ × 12¼	15	0	"	12,510	6,002	1,719,500	206	Shearing.
326	4 × 12⅛	16	0	"	14,011	6,862	1,980,600	219	Tension.
329	4¼ × 12¼	15	0	"	23,324	9,971	1,845,600	340	"

YELLOW-PINE BEAMS.

No. of Test.	Width and Depth.	Distance between Supports.		Manner of Loading.	Breaking Load.	Modulus of Rupture in Lbs. per Square Inch.	Modulus of Elasticity in Lbs. per Square Inch.	Calculated Intensity of Shear in Lbs. per Square Inch.	Manner of Breaking.
	inches.	ft.	in.		lbs.				
331	$3\frac{7}{8} \times 11\frac{7}{8}$	15	0	Centre	13,709	6.774	1,619,800	225	Tension.
332	$4 \times 13\frac{1}{4}$	15	0	"	21,733	8.356	2.109,600	310	Shearing.
334	$3\frac{7}{8} \times 13$	15	0	"	17.400	7.105	1,586,600	259	Tension.
335	4×12	15	0	"	16,228	7,686	1,911.400	257	Shearing.
339	$4 \times 11\frac{13}{16}$	16	6	"	15,737	8,373	2,128,800	250	Tension.
342	$4\frac{3}{16} \times 12\frac{1}{8}$	16	0	"	9,616	4.502	1,380,500	142	"
346	$4 \times 11\frac{7}{8}$	16	0	"	16,227	8.284	1,857,800	256	"
347	4×12	17	0	"	13.536	7,191	2,041,000	211	"
350	4×12	15	0	"	19,212	8,912	1.790.500	299	"
352	$4\frac{1}{8} \times 12\frac{1}{8}$	17	0	"	18,238	9,087	2,059,000	271	Shearing.
357	$4\frac{1}{8} \times 12\frac{3}{4}$	16	0	"	22.546	10.278	2.595.600	331	Tension & compression.
358	$4\frac{1}{8} \times 12\frac{1}{4}$	17	0	"	20.825	10.371	2.137.340	312	" " "
360	3×10	15	0	"	12,0,1	10.864	2,122,000	302	Tension.
361	3×10	15	0	"	7 264	6,485	1.632.700	179	"
362	$3 \times 9\frac{7}{8}$	15	0	"	8.467	7,815	1,833.700	214	Shearing.
364	$4 \times 12\frac{1}{4}$	15	0	"	15,208	6,379	1,667,500	237	Tension.
368	$4 \times 12\frac{1}{8}$	15	0	"	17,714	8,133	1,899,100	276	"
376	$3\frac{3}{4} \times 11\frac{7}{8}$	15	0	"	14,712	7,388	1,800,300	246	"
381	$3\frac{3}{4} \times 11\frac{7}{8}$	15	0	"	15,201	7.761	1,566.400	258	Tension and shearing.
382	$4 \times 12\frac{1}{4}$	15	0	"	17,525	7,803	1,882,700	269	Compression.
386	4×12	16	0	"	12.110	6,055	1,310,000	191	Tension.
388	$4 \times 12\frac{1}{4}$	16	0	"	10.534	5,059	1,658,800	162	Shearing and tension.
390	$4 \times 12\frac{1}{8}$	15	0	"	22,100	10.147	2,234,800	342	Shearing.
391	$4\frac{1}{8} \times 12$	17	0	"	18,391	9.474	2,068,200	279	Tension.
394	$4\frac{1}{4} \times 12\frac{1}{4}$	16	0	"	19,713	8.902	1.996,400	286	Shearing.
396	$3\frac{1}{2} \times 12$	20	0	"	10,296	7.354	1,889,000	186	Tension.
396*a*	$3\frac{1}{2} \times 12\frac{1}{4}$	8	6	"	25,827	7,392	2,217,000	445	Shearing.
399	$4\frac{1}{4} \times 12\frac{1}{4}$	16	0	"	8,602	3,828	1,306,000	124	Tension.
402	$4\frac{5}{8} \times 12\frac{3}{16}$	15	0	"	15.192	6.303	1,542,000	215	"
403	$4\frac{1}{4} \times 12\frac{1}{8}$	15	0	"	10,886	4.636	1,058,200	158	"
410	$3\frac{3}{4} \times 12\frac{5}{8}$	15	0	2 points 2′ from cen.	18,590	6.304	1.733.400	297	"
412	$4\frac{1}{8} \times 12\frac{1}{4}$	14	0	do. do.	15,865	4,627	1,774,100	237	Shearing.
413	$3\frac{7}{8} \times 12\frac{1}{4}$	15	0	do. do.	14.852	5,069	1.778,900	236	Tension.
418	4×12	14	0	Centre	20,500	9.064	2,278,500	326	"
419	4×12	16	0	"	12,850	6,541	1,664,800	207	Crushing at top & shearing above neutral axis.
420	4×12	14	6	"	14,000	6.442	1,684,400	225	Tension.
429	4×12	14	0	"	15,600	6,759	1,498,500	244	Tension, compression, & longitudinal shear.
437	$6 \times 16\frac{1}{2}$	17	6	"	38,000	7.407	1,777,700	289	Tension and shear.
440	4×12	16	0	"	19,500	9.908	1,778,400	312	Tension.
447	$3\frac{7}{8} \times 12\frac{1}{4}$	18	0	"	13,900	7,908	1,672,900	227	"
451	$4 \times 12\frac{1}{8}$	18	6	"	9.350	5.446	2,013.200	151	Longitudinal shear and tension.
452	$4\frac{1}{8} \times 12\frac{3}{16}$	14	0	"	15,750	6,570	1,704,900	240	Tension and longitudinal shear.
453	4×12	14	0	"	13,000	5.800	1,804,000	209	Tension
454	$4\frac{1}{4} \times 12$	19	0	"	12,000	6.848	1,603.800	182	Tension & compression.
455	$4\frac{1}{8} \times 12$	16	0	"	16,750	8.200	1.656,000	259	Tension.
Average values.						7,442	1.783,000		

WHITE-OAK BEAMS.

No. of Test.	Width and Depth.	Span.		Description.	Breaking-Weight, in lbs.	Modulus of Rupture.	Modulus of Elasticity.	Remarks.
	inches.	ft.	in.					
48	6 × 12	19	6	Load at middle	13776	5596	1766839	
51	$4\frac{1}{4}$ × $14\frac{3}{8}$	15	6	"	19076	6060	1240728	
55	3 × $13\frac{1}{4}$	13	8	"	10671	4984	853098	
80	4 × 12	18	0	"	13371	7659	1307180	
109	4 × $10\frac{1}{4}$	15	4	"	8421	5530	823000	Nos. 109–112.—Lumber cut in Ohio winter 1884; tested spring of 1885.
110	6 × $12\frac{1}{4}$	15	8	"	17834	5586	1227666	
111	3 × 9	15	6	"	2821	3240	1134498	
112	4 × 12	19	0	"	8783	5214	1057378	
113	$6\frac{3}{8}$ × $12\frac{1}{4}$	19	4	"	15104	5494	744774	Nos. 113–125.—Lumber was cut in Pennsylvania in the winter of 1884–5; tested in spring 1885.
115	4 × 12	19	6	"	7885	4805	1607910	
116	6 × 12	20	0	"	20098	8374	1666844	
117	4 × 12	17	0	"	10546	5603	973213	
118	4 × 12	19	6	"	8080	4924	979930	
119	4 × 12	16	0	"	12360	6180	1343768	
122	4 × 12	19	0	"	5953	3535	672724	
123	4 × 12	19	0	"	10082	5986	1130217	
125	4 × 12	19	9	"	10269	6338	1115578	
149	$2\frac{7}{8}$ × $11\frac{1}{2}$	15	10	"	6287	4713	878900	Ohio White Oak.
150	3 × $11\frac{3}{4}$	17	6	"	7000	5324	889000	

WHITE-OAK BEAMS.—*Concluded.*

No. of Test.	Width and Depth.	Span.	Description.	Breaking-Weight, in lbs.	Modulus of Rupture.	Modulus of Elasticity.	Remarks.
	inches.	ft. in.					
151	4 × $12\frac{3}{16}$	16 8	Load at middle	9231	4661	800870	Ohio White Oak.
152	4 × $10\frac{3}{4}$	17 9	"	9432	6519	1544600	
153	$3\frac{1}{8}$ × $12\frac{1}{4}$	18 0	"	9122	6302	1143900	
168	$3\frac{7}{8}$ × $11\frac{15}{16}$	20 0	"	11474	7482	1464500	Tests Nos. 168–176, inclusive, were made on White Oak from Ohio, sawed latter part of August '86; received at Institute Oct. 11, '86; tested Oct. 15, '86–Nov. 30, '86.
169	4 × $11\frac{7}{8}$	19 8	"	9275	5821	1141900	
171	$3\frac{27}{32}$ × $12\frac{7}{16}$	19 4	"	9372	5485	972200	
172	$3\frac{7}{8}$ × $11\frac{13}{16}$	17 8	"	7501	4411	1059600	
173	4 × 12	19 8	"	8971	5513	1777500	
174	4 × $11\frac{7}{8}$	20 0	"	12274	7834	1465800	
175	$4\frac{1}{16}$ × $11\frac{13}{16}$	16 0	"	11231	5706	946470	
176	4 × $11\frac{5}{8}$	17 8	"	15043	8850	1735800	
259	5 × 12	15 6	"	16169	6266	990300	Nos. 259–269 are selected Oak beams from Ohio, straight-grained, and nearly free from knots.
260	$5\frac{1}{8}$ × $12\frac{3}{16}$	13 6	"	15138	4832	813400	
262	$5\frac{1}{16}$ × $12\frac{1}{4}$	17 0	"	14994	6040	1134000	
264	$5\frac{1}{16}$ × $10\frac{3}{16}$	13 6	"	14720	6808	1089000	
265	5 × $12\frac{1}{8}$	9 10	"	26088	6281	861200	
269	$4\frac{7}{8}$ × $12\frac{1}{8}$	9 6	"	29788	7107	953800	Green Lumber.
			Mean . . .	. . . . .	5863	1131100	

WHITE-PINE BEAMS.

No. of Test.	Width and Depth.	Span.	Breaking Centre Load, in lbs.	Modulus of Rupture.	Modulus of Elasticity.	Remarks.
	inches.	ft. in.				
94	$3 \times 11\frac{1}{2}$	15 8	5088	3613	924252	
95	3×13	14 0	12588	7251	1280832	Pattern stock. Clear piece. Seasoned 3 yrs.
96	3×13	16 6	9088	5324	1072889	
97	3×11	15 8	6088	4729	978256	
98	$2\frac{7}{8} \times 9\frac{3}{4}$	16 0	6088	6415	1234880	
99	$2\frac{7}{8} \times 13$	15 6	5988	3438	1020390	
100	$3 \times 9\frac{3}{4}$	16 0	4288	4330	1165937	
102	$3 \times 10\frac{3}{4}$	15 6	4790	3855	999190	
103	3×11	16 6	6588	5390	1242649	
104	$3 \times 11\frac{1}{4}$	15 6	5088	3739	931760	
128	6×12	19 10	12922	5340	1380660	
129	6×12	19 08	15060	6170	1565000	
130	$6 \times 12\frac{1}{8}$	19 08	12340	4954	1222100	
131	6×12	19 10	13023	5380	1307900	
132	6×12	19 10	6231	2575	1103500	
133	6×12	19 08	12912	5290	1297000	
134	6×12	20 00	11254	4689	1345700	
137	$6\frac{1}{16} \times 12\frac{1}{16}$	19 08	13650	5478	1367700	
138	6×12	19 09	14010	5765	1247100	
140	6×12	19 09	9761	4016	1105600	
244	$4\frac{3}{32} \times 12\frac{29}{32}$	16 00	10179	4300	948600	
245	$4\frac{1}{8} \times 12\frac{1}{2}$	15 6	12984	5620	1271000	
247	$4\frac{7}{16} \times 10\frac{7}{8}$	15 8	6770	3638	1111900	
248	$4\frac{1}{8} \times 12\frac{1}{4}$	15 8	7790	3549	1057300	
279	$4 \times 12\frac{1}{2}$	15 6	9085	4222	1084000	
280	$4 \times 12\frac{1}{4}$	15 6	7575	3521	854300	
281	$3\frac{7}{8} \times 12\frac{5}{16}$	15 0	12070	5547	1281000	
282	$3\frac{15}{16} \times 12\frac{8}{16}$	15 0	8660	3998	1053000	
283	$3\frac{15}{16} \times 12\frac{1}{4}$	15 0	7182	3285	970300	
284	$4 \times 12\frac{1}{2}$	15 0	5685	2456	873300	
285	$3\frac{15}{16} \times 12\frac{5}{16}$	15 6	6385	3031	901500	
286	$3\frac{7}{8} \times 12\frac{7}{16}$	15 6	11791	5449	1258700	
287	$4 \times 12\frac{5}{16}$	15 6	8265	3802	1049500	
288	$3\frac{7}{8} \times 12\frac{5}{16}$	15 6	11272	5353	1295382	
289	$3\frac{7}{8} \times 12\frac{1}{16}$	15 6	5671	2806	825300	
296	$4 \times 12\frac{5}{16}$	15 6	5571	2563	727200	
315	$3\frac{3}{4} \times 12$	16 0	7165	3820	1158411	
	Average values..................			4451	1122000	

It would seem to the writer that about the same modulus of rupture should be used for white pine as for spruce.

KILN-DRIED WESTERN WHITE PINE.

No. of Test.	Depth and Width.	Span.	Manner of Loading.	Breaking Load (lbs).	Modulus of Rupture (lbs. per Sq. In.).	Modulus of Elasticity (lbs. per Sq. In.).	Remarks.
	inches.	ft. in.					
206	$2\frac{1}{16} \times 13\frac{3}{8}$	15 05	Centre	9325	7014	1505000	
207	$2 \times 12\frac{7}{8}$	13 00	"	8814	6432	1193000	
208	$2\frac{1}{16} \times 12\frac{9}{16}$	15 00	"	3420	2836	752300	
210	$2\frac{1}{16} \times 11\frac{3}{4}$	15 00	"	4518	4284	1241400	
212	$2\frac{1}{16} \times 13$	15 06	"	6120	4898	1099300	
213	$2\frac{1}{16} \times 13\frac{15}{16}$	15 06	"	....	....	1276300	
214	$2 \times 12\frac{1}{16}$	15 06	"	7420	6430	1166000	
237	$2\frac{3}{32} \times 12\frac{15}{16}$	15 04	"	8225	6478	1231000	
				Mean =	5482	1183037	

HEMLOCK.

No. of Test.	Depth and Width.	Span.	Manner of Loading.	Breaking Load (lbs.).	Modulus of Rupture (lbs. per Sq. In.).	Modulus of Elasticity (lbs. per Sq. In).	Remarks.
	inches.	ft. in.					
154	$3\frac{1}{4} \times 11\frac{5}{8}$	15 00	Centre	6449	3965	870960	Nos. 154–160 are Eastern hemlock seasoned about 1 year. The remainder of the hemlock tests are from hemlock cut in Vermont June, 1886; first growth grown on high ground. Sawed Sept. 25, 1886. Received at Institute Nov. 15, 1886. Tested Dec. 2, 1886, to March 9, 1887.
155	$3\frac{1}{16} \times 10$	14 08	"	4648	4007	971710	
156	$2\frac{7}{8} \times 9\frac{3}{4}$	12 09	"	4425	3716	750400	
157	$3\frac{3}{16} \times 10\frac{1}{8}$	11 10	"	3223	2381	770900	
158	$3\frac{3}{16} \times 10\frac{3}{16}$	14 00	"	4137	3151	735800	
159	$3 \times 9\frac{5}{8}$	13 06	"	2939	2570	833000	
160	$2\frac{7}{8} \times 11\frac{1}{4}$	12 08	"	9433	5911	1086600	
177	$4\frac{1}{8} \times 12$	15 08	"	9605	4560	1081100	
178	4×12	17 00	"	5502	2923	821990	
179	$4\frac{1}{8} \times 11\frac{15}{16}$	15 08	"	3192	1531	688960	
180	$4 \times 12\frac{1}{8}$	15 06	"	4584	2175	926560	
181	$4 \times 11\frac{15}{16}$	15 08	"	5133	2539	758390	
182	4×12	15 06	"	11073	5363	1296600	
183	4×12	15 06	"	13274	6499	1269800	
184	$3\frac{7}{8} \times 11\frac{7}{8}$	15 08	"	9964	5142	1075600	
185	$3\frac{7}{8} \times 11\frac{15}{16}$	17 02	"	3679	2059	412670	
186	$3\frac{25}{32} \times 11\frac{15}{16}$	15 08	"	12488	6535	1327200	
				Mean =	3825	922250	
114	$7\frac{3}{4} \times 10\frac{1}{2}$	20 04	Centre	19244	8243	2011188	Yellow birch, N.H.
120	$7\frac{1}{4} \times 10\frac{1}{8}$	19 06	"	16150	7627	1583201	" " "
127	3×10	13 06	"	14965	12122	....	N. H. ash, seasoned 2 yrs.

TIME TESTS.

The following is a record of the time tests made at the Institute, and at the close will be found a statement in regard to the proper value of modulus of elasticity for use in computing deflections.

TIME TEST NO. 1.

Spruce from Maine, received at Institute October 30, 1885. All the beams when received were green lumber, except F, which was seasoned on the wharf about six months. Beams A, B, C, D, E, and F were seasoned under a centre load in the laboratory in steam heat from November 10, 1885, to May 8, 1886. Beams G, H, I were seasoned in the same room, without load; span = 20 feet for all the beams under load.

Beam.	A (164)	B (163)	C (162)	D (161)	E (169)	F (168)
Description of lumber,	clear	knotty	knotty	clear	clear	clear
Dimensions, original,	4″ × 12″	4″ × 12″	4″ × 12″	4″ × 12″	6″ × 12″	6″ × 12″
Dimensions, final,	3¾″ × 11¾″	4″ × 11¾″	3¾″ × 11¾″	3¾″ × 11¾″	5¾″ × 11⅝″	5⅞″ × 11¾″
Max. fibre stress, lbs. per sq. in., with wt. of beam,	1078	1076	1074	1070	1133	1136
Max. fibre stress, lbs. per sq. in., without wt. of beam,	1003	1003	1003	1003	1057	1051
Deflection, load first applied,	0″.5488	0″.6355	0″.7675	0″.6108	0″.7188	0″.5454
Deflection at end of test,	1″.0164	1″.4707	1″.5154	1.3734	1.2667	0.3151
E (immediate),	1462200	1262900	1045700	1314000	1175000	1540000
E (final),	789000	546000	529000	584000	667000	2666000
Modulus of rupture, lbs. per sq. in., with wt. of beam,	6574	7260	5007	6066	6779	8574
Modulus of rupture, lbs. per sq. in., without wt. of beam,	6500	7187	4937	6000	6708	8500
Weight per cu. ft. at beginning, lbs.,	35.5	34.9	33.6	31.9	34.1	35.4
Weight per cu. ft. at end, lbs.,	27.1	28.1	25.5	25.7	28.2	29.6
Date of testing,	May 11, '86	May 11, '86	May 10, '86	May 10, '86	May 14, '86	May 13, '86
E (after seasoning), lbs. per sq. in. of final section,	1866900	1509500	1367300	1625500	1737400	1718300

Average E (immediate) = 1300000.
Average E (final) = 963500.
Average modulus of rupture beams under load = 6710.
All quantities in the above table except the last were calculated by using the origina section.

Beam.	G (165)	H (166)	I (167)
Description of lumber, ...	clear	clear	knotty
Dimensions { original, ...	4″ × 12″	4″ × 12″	4″ × 12″
Dimensions { final, ...	3¾″ × 11⅝″	3¾″ × 12″	4″ × 12″
Modulus of rupture, lbs. per sq. in. { with wt. of beam, . . .	....	....	.. .
Modulus of rupture, lbs. per sq. in. { without wt. of beam, . .	6525	7187	5812
Weight per cu. ft. at beginning, lbs., ...	....	....	...
Weight per cu. ft. at end, lbs., ...	27.3	27	27.2
Date of testing, ...	May 11, '86	May 12, '86	May 12, '86
E (after seasoning), lbs. per sq. in. of final section,	1603700	1748900	1457000

Average E (final section), beams without load, ... 1603200
Average modulus of rupture (original section), ... 6508

TIME TEST FOR SHORT PERIODS OF TIME.

No.	Depth and Width.	Span.	Original Modulus of Elasticity.	Total increased Deflection.	Modulus of Rupture.	Description of Test.
	inches.	ft. in.		inches.	lbs.	
108	6 × 12	17 04	1269670	.1241	5066	Spruce-beam load equally distributed at 12 points. The beam was subjected to a load of 5031 lbs. for 898 hrs.
148	$4\frac{3}{16} \times 11\frac{15}{16}$	17 04	1211800	.3765	4668	Yellow-pine beam, load distributed equally over 12 points. The beam was subjected to a load of 6355 lbs. for 29 days.

TIME TEST NO. 2.

Spruce beams cut in Maine in the spring of 1886. Received at Institute September 13, 1886. Beams A, B, C, D, E, and F were seasoned under a centre load, in the laboratory, in steam heat, from September 15, 1886, to April 2, 1887 (200 days). Beams G, H, I were seasoned in the same room, but were not loaded. All the beams were without load between

April 2 and date of testing. Span = 18′ 00″ for all beams under load.

Beam.	A (194)	B (193)	C (191)	D (190)	E (195)	F (192)
Description of lumber, ...	clear	clear	cross-grained	knotty	straight-grained, some knots	straight-grained, some knots
Dimensions, original, .	$4'' \times 12''$	$4'' \times 12\frac{1}{16}$	$3\frac{1}{4}'' \times 11\frac{1}{4}''$	$4\frac{3}{16}'' \times 11\frac{7}{8}''$	$6'' \times 12\frac{1}{12}''$	$6\frac{1}{8}'' \times 12''$
Dimensions, final, . .	$3\frac{13}{16}'' \times 11\frac{11}{16}''$	$3\frac{13}{16}'' \times 11\frac{13}{16}''$	$3\frac{5}{8}'' \times 11\frac{5}{16}''$	$4\frac{1}{8}'' \times 11\frac{19}{32}''$	$5\frac{7}{8}'' \times 11\frac{7}{8}''$	$5\frac{31}{32}'' \times 11\frac{13}{16}''$
Max. fibre stress, lbs. per sq. in., with wt. of beam,	1020	1011	1136	994	1095	1093
Max. fibre stress, lbs. per sq. in., without wt. of beam,	966	956	1075	943	1037	1041
Deflection, load first applied, ...	0″.3707	0″.4253	0″.5335	0″.4794	0″.5024	0″.5747
Deflection at end of test, ...	0″.9181	0″.8447	1″.2336	1″.0453	1″.0170	1″.5050
E (immediate), . .	1689000	1449000	1333000	1288000	1327000	1173000
E (final), ...	682000	730000	577000	591000	655000	448000
Modulus of rupture, lbs. per sq. in., with wt. of beam,	10283	7838	3243	5038	7448	5116
Modulus of rupture, lbs. per sq. in., without wt. of beam,	10229	7783	3182	4987	7392	5064
Weight per cu. ft. at beginning, lbs.,	31.8	32.8	35.4	30.0	34.1	30.5
Weight per cu. ft. at end, lbs., . . .	28.6	26.1	29.1	27.0	27.7	26.3
Date of testing, .	Apr. 25, '87	Apr. 19, '87	Apr. 5, '87	Apr. 4, '87	Apr. 27, '87	Apr. 8, '87
E (after seasoning), lbs. per sq. in. of final section, . .	2125500	1852300	1731600	1517800	1662100	1294200

Average E (immediate) = 1376500.
Average E (final) = 614000.
Average modulus of rupture beams under load = 6494.

Beam.	G (188)	H (187)	I (189)
Description of lumber,	knotty	clear	knotty
Dimensions { original,	$4'' \times 12''$	$4'' \times 12''$	$4'' \times 12''$
Dimensions { final	$3\frac{3}{4}'' \times 11\frac{7}{8}''$	$3\frac{15}{16}'' \times 11\frac{5}{8}''$	$3\frac{3}{4}'' \times 11\frac{11}{16}''$
Modulus of rupture, lbs. per sq. in. { with wt. of beam,	5218	8667	4796
Modulus of rupture, lbs. per sq. in. { without wt. of beam,	5167	8598	4751
Weight per cu. ft. at beginning, lbs.,	29.5	35.8	27.3
Wt. per cu. ft. at end, lbs.,	24.9	29.3	24.8
Date of testing,	Mch. 21, '87	Apr. 1, '87	Mch. 16, '87
E (after seasoning), lbs. per sq. in. of final section,	1355300	1914500	1573600

Average E (final section), beams without load = 1614500.
Average modulus of rupture (original section) = 6227.

TIME TEST NO. 3.

Yellow-pine beams from Georgia, cut in season of 1886. Received at Institute September 13, 1887. The lumber was purchased in sticks of double length, and cut in two for testing. The numbers indicate the stick, and the letter "B" the butt, and "T" the top end of the same. Beams 1 T, 2 B, 3 B, 4 T, 5 B, 5 T were seasoned under load, the remainder being seasoned without load. Span = 18′.

Beam.	1 T (238)	2 B (239)	3 B (240)	4 T (241)	5 B (242)	5 T (243)
Description of lumber, . .	clear, cross-grained	clear, some dry rot	clear, straight-grained	clear, cross-grained	clear, straight-grained	clear, straight-grained
Dimensions { original	$4\frac{3}{16}'' \times 12\frac{7}{32}''$	$4\frac{11}{32}'' \times 12\frac{7}{16}''$	$4\frac{1}{8}'' \times 12\frac{7}{32}''$	$3\frac{15}{16}'' \times 12\frac{7}{32}''$	$6\frac{7}{32}'' \times 11\frac{23}{32}''$	$6\frac{9}{32}'' \times 12\frac{1}{8}''$
Dimensions { final, .	$4'' \times 11\frac{13}{16}''$	$4\frac{3}{16}'' \times 12\frac{1}{16}''$	$3\frac{15}{16}'' \times 11\frac{27}{32}''$	$3\frac{13}{16}'' \times 11\frac{29}{32}''$	$6'' \times 11\frac{7}{16}''$	$6\frac{1}{8}'' \times 11\frac{27}{32}''$
Max. fibre stress, lbs. per sq. in. { with wt. of beam,	1358	1259	1381	1435	1626	1509
Max. fibre stress, lbs. per sq. in. { without wt. of beam,	1283	1193	1302	1364	1531	1426
Deflection, load first applied, .	0''.5310	0''.5518	0''.4700	0''.5618	0''.4431	0''.4836
Deflection at end of test, . . .	0''.9817	1''.2387	0''.9424	1''.4566	0''.8364	0''.8449
E (Immediate), .	1536000	1351000	1763000	1545000	2290000	1889000
E (final), . . .	832000	602000	879000	596000	1213000	1081000
Modulus of rupture, lbs. per sq. in. { with wt. of beam,	6290	6234	9545	6570	10630	10000
Modulus of rupture, lbs. per sq. in. { without wt. of beam,	6213	6163	9461	6495	10540	9924
Weight per cu. ft. at beginning, lbs., ...	45.6	40.6	48.2	43.1	50.9	45.8
Weight per cu. ft. at end, lbs. . .	38.4	36.8	43.3	39.0	45.8	41.4
Date of testing, .	May 22, '88	May 28, '88	May 28, '88	May 28, '88	May 31, '88	June 1, '88
E (after seasoning), lbs. per sq. in. of final section, . . .	1898000	1651000	2240000	1803700	2501400	2103400

Average E (immediate) = 1729000.
Average E (final) = 867000.
Average modulus of rupture beams under load = 8211.

Beam.	1 B (229)	2 T (232)	3 T (233)	4 B ([illegible])
Description of lumber, ...	clear, straight-grained	dry rot	unsound	clear, straight-grained
Dimensions, original, ...	$4\frac{1}{4}'' \times 12\frac{5}{32}''$	$4\frac{9}{32}'' \times 12\frac{3}{8}''$	$4\frac{3}{32}'' \times 12''$	$3\frac{31}{32}'' \times 12\frac{7}{32}''$
Dimensions, final, ...	$4\frac{1}{8}'' \times 11\frac{1}{4}''$	$4\frac{9}{32}'' \times 12\frac{3}{16}''$	$4'' \times 11\frac{13}{16}''$	$3\frac{13}{16}'' \times 12''$
Modulus of rupture, lbs. per sq. in., with wt. of beam, . . .	9417	3069	5838	9960
Modulus of rupture, lbs. per sq. in., without wt. of beam, . . .	9330	3007	5754	9892
Weight per cu. ft. at beginning, lbs., ...	46.1	...	....	....
Weight per cu. ft. at end, lbs., .	41.4	....	....	...
Date of testing, ...	March 28, '88	April 6, '88	April 9, '88	March 26, '88
E (after seasoning), lbs. per sq. in. of final section, ...	2141400	1293000	1951000	1895600

Average E (final section), beams without load = 1820200.
Average modulus of rupture (original section) = 7071.

TIME TEST NO. 4.

Spruce from Maine. Green lumber. Received at Institute September 11, 1888. Beams A, B, C, D, J, K were seasoned under load in the laboratory in steam heat from September 13, 1888, to March 25, 1889 (194 days). Beams E, F, G, and H were seasoned in the same room, but not loaded, All the beams were without load from March 25, 1889, to the date of testing. Span = 18′ for the beams while under load.

Beam.	A (273)	B (274)	C (276)	D (278)	J (277)	K (275)
Description of lumber, ...	ave. stock	ave. stock		knotty, straight-grained	clear, straight-grained	clear, straight-grained
Dimensions original, .	$4\frac{1}{8}'' \times 11\frac{15}{16}''$	$4\frac{1}{16}'' \times 11\frac{15}{16}''$	$4\frac{3}{16}'' \times 11\frac{3}{4}''$	$4\frac{1}{8}'' \times 11\frac{7}{8}''$	$5\frac{11}{16}'' \times 11\frac{15}{16}''$	$5\frac{3}{4}'' \times 12''$
Dimensions final, . .	$4'' \times 11\frac{1}{2}''$	$3\frac{15}{16}'' \times 11\frac{9}{16}''$	$4'' \times 11\frac{5}{8}''$	$3\frac{15}{16}'' \times 11\frac{9}{16}''$	$5\frac{7}{16}'' \times 11\frac{5}{8}''$	$5\frac{5}{8}'' \times 11\frac{9}{16}''$
Max. fibre stress, lbs. per sq. in., with wt. of beam,	1422	1442	1444	1436	1750	1705
Max. fibre stress, lbs. per sq. in., without wt. of beam,	1364	1385	1387	1379	1698	1645
Deflection, load first applied, ...	0''.6699	0''.7716	0''.6618	0''.7379	0''.8446	0''.7439
Deflection at end of test, ...	1''.3023	1''.8375	1''.5369	1''.6980	1''.7359	1''.4284
E (immediate), .	1326000	1169000	1387000	1224000	1308000	1431000
E (final), ...	682000	491000	597000	532000	637000	745000
Modulus of rupture, lbs. per sq. in., with wt. of beam,	7545	7211	5597	6622	6484	7099
Modulus of rupture, lbs. per sq. in., without wt. of beam,	7487	7155	5540	6565	6431	7038
Weight per cu. ft. at beginning, lbs., .	34.3	33.5	33.0	33.6	31.5	36.4
Weight per cu. ft. at end, lbs., ...	27.3	25.4	26.8	26.9	27.1	28.6
Date of testing, . .	Mch. 26, '89	Apr. 3, '89	Apr. 9, '89	Apr. 11, '89	Apr. 10, '89	Apr. 8, '89
E (after seasoning), lbs. per sq. in. of final section, . .	1631700	1405500	1747900	1503200	1576000	1722600

Average E (immediate) = 1307500.
Average E (final), = 614000.
Average modulus of rupture beams under load = 5760.

Beam.	E (270)	F (271)	G (272)	H (251)
Description of lumber,	nearly clear, straight-grained			knotty
Dimensions original,	$3\frac{7}{8}'' \times 12''$	$4'' \times 11\frac{15}{16}''$	$3\frac{31}{32}'' \times 12''$	$4\frac{7}{32}'' \times 11\frac{7}{8}''$
Dimensions final,	$3\frac{5}{8}'' \times 11\frac{5}{8}''$	$3\frac{13}{16}'' \times 11\frac{5}{8}''$	$3\frac{13}{16}'' \times 11\frac{9}{16}''$	$4\frac{1}{16}'' \times 11\frac{3}{4}''$
Modulus of rupture, lbs. per sq. in. with wt. of beam,	8293	7615	8558	4732
Modulus of rupture, lbs. per sq. in. without wt. of beam,	8237	7552	8496	4676
Weight per cu. ft. at beginning, lbs.,	33.5	37.2	36.8	33.0
Weight per cu. ft. at end, lbs.,	27.3	27.6	26.3	27.7
Date of testing,	Mch. 13, '89	Mch. 19, '89	Mch. 22, '89	Nov. 6, '88
E (after seasoning), lbs. per sq. in. of final section,	1632500	1771000	1770000	1266500

Average E (final section), beams without load = 1610000.
Average modulus of rupture (original section) = 7300.

DEFLECTIONS WITH TIME.

From the above it is plain that the deflection of a timber beam under a long-continued application of the load may be 2 or more times that assumed when the load was first applied; and in order to compute it by means of the ordinary deflection formulæ, we should use for E not more than $\frac{1}{2}$ the value derived from quick tests.

LONGITUDINAL SHEARING.

Below are given tables showing the greatest intensity of the shear at the neutral axis of each beam at fracture as calculated from the formula on page 675.

For breaking shearing-strength per square inch, in the case of each wood, it seems to the author that it would be proper to use a value somewhere near the lowest of those given in the table of beams which gave way by longitudinal shearing.

It will also be observed that these shearing-forces are less than those obtained from the experiments on direct shearing along the grain, made at the Watertown Arsenal; and this is naturally to be expected, for the shearing in their case took place along a section that was perfectly sound, while in these cases it took place at the weakest point.

TIME TEST NO. 5.

Yellow-pine beams received at the Institute in Feb., 1899. The beams were subjected to center load from Feb. 12, 1899, until Dec. 11, 1902. The load was then removed, and the beams were subsequently tested in the ordinary way.

Beam.	A	B	C	D	E	F	G	H
Description of Lumber...	Clear, straight-grained.	Clear, straight-grained.	Clear, straight-grained, about 1″ of sap on top.	Clear, straight-grained.	Clear, straight-grained, about 1″ of sap on bottom.	Clear, straight-grained.	Clear, straight-grained.	Clear, straight-grained.
Dimensions { Original......	$6\frac{1}{8}''\times12\frac{3}{16}''$	$6\frac{3}{16}''\times12\frac{5}{16}''$	$6\frac{3}{16}''\times12\frac{3}{16}''$	$6\frac{3}{8}''\times12\frac{3}{16}''$	$6\frac{5}{16}''\times12\frac{1}{4}''$	$6\frac{1}{16}''\times11\frac{7}{8}''$	$6\frac{1}{16}''\times12''$	$6\frac{1}{16}''\times12\frac{1}{16}''$
Dimensions { Final........	$5\frac{15}{16}''\times11\frac{3}{4}''$	....	$6''\times11\frac{3}{4}''$	$6\frac{3}{16}''\times11\frac{3}{4}''$	$6\frac{1}{8}''\times11\frac{13}{16}''$	$5\frac{7}{8}''\times11\frac{3}{8}''$	$5\frac{7}{8}''\times11\frac{3}{4}''$	$5\frac{7}{8}''\times11\frac{3}{4}''$
Max. fibre stress, lbs. per sq. in., including wt. of beam.	1980	1920	1970	1900	1920	2130	2080	2070
Deflection, load first applied	.578″	.429″	.457″	.471″	.495″	.533″	.462″	.516″
Deflection at end of test....	1.086″	.912″	.811″	.936″	1.522″	1.549″	1.073″	1.342″
E (immediate)...........	1476000	1910000	1840000	1741000	1647000	1745000	1949000	1731000
E (final).................	786000	898000	1042000	876000	536000	601000	839000	666000
Modulus of Rupture, lbs. per sq. in., including weight of beam...............	5000	....	11570	11240	11720	7880	11280	10950
Weight per cu. ft. at beginning lbs..............	45.6	48.5	52.8	47.0	48.0	52.2	53.6	46.4
Weight per cu. ft. at end, lbs.	39.9	....	48.3	43.6	43.9	48.7	50.0	42.4
Date of testing..........	April, 1903	....	April, 1903	April, 1903	Jan., 1903	Jan., 1903	Jan., 1903	Jan., 1903
E (after seasoning) lbs. per sq. in. of final section....	1240000	....	2110000	1960000	2110000	2320000	2300000	1970000

Average E (immediate) = 1755000
Average E (final) = 781000
Average modulus of rupture = 9950

TABLE OF BEAMS WHICH GAVE WAY BY LONGITUDINAL SHEARING.

Spruce.		Yellow Pine.		Oak.		White Pine.	
No.	Intensity of Shear, Lbs. per Sq. In.	No.	Intensity of Shear, Lbs. per Sq. In.	No.	Intensity of Shear, Lbs. per Sq. In.	No.	Intensity of Shear, Lbs. per Sq. In.
22	202	30	273	109	152	129	155
24	190	32	242	269	379	134	119
31	154	33	153	....	....	233	180
35	117	50	177	....	...	....	....
36	248	147	264	...	....	....	....
46	233	419	207	....	....	....	....
90	273	429	244	....	....	....	...
304	130	437	289	...	....	....	....
321	233	451	151	...	....	....	....
416	215	452	240	....	....	....	....
421	199	....	....	....	....	...	....
423	207	....	....	....	....	..	....
445	133	....	....	...	....	...	....
450	166	....	....	....	....	....	....
Average, 200.		Average, 224.		Average, 266.		Average, 151.	

COMPRESSION OF TIMBER AT RIGHT ANGLES TO THE GRAIN.

On page 698 will be found a table giving the averages of a series of tests of compression of timber at right angles to the grain.

The pieces of timber tested were all about 13″ long in the direction of the grain, the other two dimensions being as given in the table, the pressure being applied in the direction of the longer of these two dimensions.

In the cases given in the tables a maximum load was found. Evidently, however, as the ratio of length in the direction of the pressure to least diameter decreases, we reach a point where no maximum load is found, but where continuous crushing goes on, the pressure continuously increasing. Thus, in the case of spruce specimens 10″ × 12″, 4″ × 6″, 4″ × 8″, and 6″ × 8″, while a maximum load was sometimes found, at other times it was not.

COMPRESSION OF TIMBER AT RIGHT ANGLES TO THE GRAIN.

Kind of Wood.	Approximate Dimensions.	Number of Tests.	Average Compressive Strength per Square Inch.	Kind of Wood.	Approximate Dimensions.	Number of Tests.	Average Compressive Strength per Square Inch.
	lbs.		lbs.		lbs.		lbs.
Spruce ...	$2\frac{3}{4} \times 10$	60	396	Yellow pine..	4×12	60	535
	2×12	21	252		6×12	16	402
	3×12	23	350		8×12	18	471
	4×12	59	397		8×8	5	696
	6×12	26	319		4×6	17	566
	8×12	31	350	Oak..........	4×12	60	898*
Maple...... ..	4×12	21	1808		4×12	51	980*
Hemlock.......	4×12	20	330		6×12	17	835
					8×12	21	913

* Two different lots.

6°. *Framing-Joints.* — Another matter intimately connected with the strength of timber beams is the strength of the beam after it has been cut in some of the various ways commonly employed in framing. We are often told that a notch cut on top of a beam, or at the middle of its depth, or near the support, does but little injury; but the tests made, show the injury to be very large, amounting to a reduction of the strength of the beam to one-fourth or one-fifth of its original strength, with some of the most approved framing.

The fact is that, in the case of timber, the shearing-strength along the grain is small, and that, in almost any case of notching timber, as in a notched beam, or in a header, there is developed, in consequence of the cutting, a tendency to tear the timber across the grain, and the resistance of timber to this kind of stress is very small. Moreover, the injury due to notching, so far from being a small quantity, is very large. Hence it follows that almost any cutting does a great deal of injury; and it is much better to avoid framing whenever it is possible, and use stirrup-irons instead. In these tests, only two of the most approved framing-joints have been tested; viz., the joint known as the "tusk-and-tenon," shown in Fig. 243, and used for framing the tail-beams of a floor into the headers, and the "double tenon and joint bolt," shown in Fig. 244, and used for framing the headers into the trimmers.

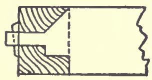


FIG. 243.

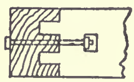


FIG. 244.

The arrangement is shown in plan in Fig. 245, where 1 and 2 are the trimmers, 3 is the header, and 4, 5, and 6 are the tail-beams; the latter being supported at one end on the header, and at the other on the wall, the header being supported by the trimmers, and the trimmers being supported on the walls at both ends.

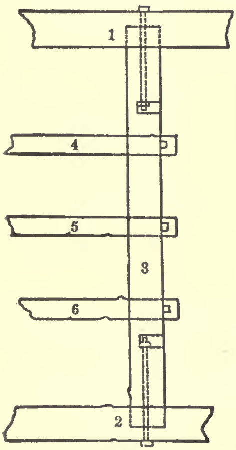


FIG 245.

It is sometimes the practice to hang the header in stirrup irons, and this is an improvement; but it is very seldom that the tail-beams are hung in stirrup irons, and these tests have shown the weakening already referred to, from the mortises cut in the header to admit the tail-beams.

A spruce floor was first built and tested, the following being a partial account of the test: —

No. 52. — Section of a floor between the trimmers. Spruce: three tail-beams, 2 inches by 12 inches each, framed into a $3\frac{3}{4}$-inch by $11\frac{3}{4}$-inch header; header in turn framed into sections of the trimmers by double tenon and joint-bolt, cross-bridged in two places; tail-beams framed by tusk-and-tenon joint, pinned, floored over and furred below; load at centre, distributed between the three tail-beams by bridging.

Span = 16 feet; weight of joist, flooring, etc., = 331 lbs.

11238 lbs. = breaking-load.

Joist on east side broke by splitting off at the tenon, bore 7988 lbs. after. The load was then increased. Centre tail-beam broke by tension at 9988 lbs., on account of cross-grain in the lower fibres. A split also started at the lower tenon of the header, which at the time of breaking was rapidly increasing.

Average modulus of rupture of the tail-beams, including their own weight, etc., = 3801 lbs. per square inch.

Average modulus of elasticity of tail-beams = 1399141 lbs. per square inch.

It is to be noticed, that the header already began to crack when the tail-beams broke, and hence that the floor could have borne but little more, even if the load had been uniformly distributed: hence that, in this case, the breaking-strength of the floor would be determined by calculating the loads at the centre of the tail-beams, instead of accounting it as distributed; in other words, the breaking-weight would be about one-half what we should get by considering the load as distributed on the tail-beams.

YELLOW-PINE HEADERS.

A number of tests of the strength of yellow-pine headers, and also of spruce headers, have been made in the Laboratory of Applied Mechanics of the Massachusetts Institute of Technology, and the results will be given here.

It will be seen from these tests that the first of these headers had for its breaking-weight 13163 lbs., and the second 11631, or in each case one-half the load on the floor. To institute a comparison, we may observe that, if a 6-inch by 12-inch yellow-pine header 6 feet 8 inches long, with four tail-beams 18 feet long, were to support a floor, the floor surface would be 96 square feet, giving 48 square feet to be supported by the header. This, if the floor were loaded with 100 lbs. per square foot, would bring upon the header 4800 lbs., or about one-half the breaking-weight of a header only 5 feet 4 inches long; whereas, it would commonly be supposed, that, with such a construction for 100 lbs. per square foot of floor, we should have provided an unnecessarily large margin of safety.

The special source of weakness in the header is, of course, the joint by which the tail-beams are attached to it, while the framing of the header into the trimmer causes great loss of strength in the trimmer. Nevertheless, even for the sake of the header, hanging it in stirrup-irons on the trimmer is better than framing.

The fact, also, that a 6-inch by 12-inch yellow-pine beam 5 feet 4 inches long bore 48000 lbs. centre load, equivalent to 96000 distributed, without breaking, while the header broke at 10916, shows what an enormous weakening is caused by cutting mortises, and how much strength would be gained by avoiding all framing, and using stirrup-irons to support the tail-beams in all cases where they cannot be supported on top of the header bearing the latter.

TESTS OF YELLOW-PINE HEADERS.

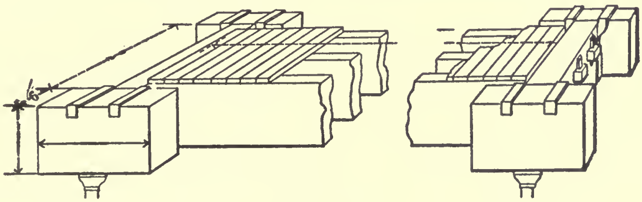


FIG. 246.

The yellow-pine headers were all 6 inches wide by 12 inches deep, and, in every case, the tail-beams were placed 16 inches apart centre to centre, so that the length of the header between the trimmers, in any case, can be found by multiplying sixteen inches by one more than the number of tail-beams. The headers were either framed into the trimmers by a double tenon and joint-bolt, or else they were hung from them by stirrup-irons, the trimmers being supported upon jack-screws. The headers were mortised at the proper places (sixteen inches on centres) for twelve-inch yellow-pine tail-beams sometimes ten feet length between headers, sometimes six, but more often three feet between the headers.

The load was applied at the centre of the tail-beams, and divided equally among them by iron bars and knife edges.

The longer tail-beams were cross-bridged by $2'' \times 3''$ spruce bridging, but the shorter ones were not bridged.

The mortises in the headers were, in every case, those suitable for a tusk and tenon joint, the tail-beams being in most cases three inches wide.

The table giving the results of the tests of the yellow-pine headers will be found on page 703.

In order to form an adequate conception of the amount of the loss of strength of one of these yellow-pine headers carrying three tail-beams, assume the same beam with no notches, but with the load equally divided at three points sixteen inches on centres, compute the breaking strength of such a beam, and compare with it the strength of any one of the headers tested which carried three tail-beams. Use for modulus of rupture 5000 pounds per square inch.

Performing the calculation, we easily obtain for the breaking strength of the six-inch by twelve-inch yellow-pine beam without the notches 67500 pounds. The average of the breaking strength of the four headers carrying three tail-beams is 13127 pounds, which is only 0.194 of 67500 pounds; hence the notches for the tusk and tenon joints where the tail-beams were attached to the headers caused a loss of about eighty per cent in the strength of the latter.

The following table shows the results of the tests of the yellow-pine headers.

YELLOW-PINE HEADERS.

Number of Test.	Width (inches).	Depth (inches).	Length between Trimmers (inches).	Number of Tail Beams.	Breaking Load of each Header. Pounds.
89	6	12	64	3	13163
105	6	12	64	3	11631
106	6	12	64	3	15131
107	6	12	64	3	12581
482	6	12	80	4	15190
483	6	12	80	4	20190
484	6	12	96	5	19870
485	6	12	96	5	16045
486	6	12	96	5	16925
487	6	12	112	6	18610
490	6	12	128	7	13905
492	6	12	80	4	12795
493	6	12	112	6	13010
500	6	12	96	5	16370

DETAILS OF THE TESTS.

No. 89.—The headers were framed into the trimmers by double tenons and joint-bolts.

The tail-beams were each 3 inches by 12 inches, and at first were 10 feet long, the result being that, at a total load of 24866 pounds, i.e., 12433 pounds on each header, one of the tail-beams broke under the tenon by splitting, while the headers were left intact. These tail-beams were then removed and new ones were supplied each 3 inches by 12 inches as before, but only 80 inches long, and the test was repeated, resulting in the breakage of one of the headers by splitting through the middle, following the line of mortises.

No. 105.—The headers were framed into the trimmers by double tenons and joint-bolts. The tail-beams were each 3 inches by 12 inches, and 72 inches long.

One of the headers failed through the line of mortises.

No. 106—The headers were supported on the trimmers by means of stirrup-irons.

The tail-beams were 3 inches by 12 inches, and 72 inches long.

At a total load of 26,662 pounds, i.e., a load on each header of 13331 pounds, one of the tail-beams split below the line of mortises. The total load was then increased to 30,262 pounds, i.e., 15131 pounds on each header, when one of the stirrup-irons broke, but simultaneously with this the header failed.

No. 107.—The unbroken headers of Nos. 105 and 106 were used for this test, supported on the trimmers in stirrup-irons. At the maximum load one of the headers failed suddenly.

Of the remaining headers the last three were supported on the trimmers in stirrup-irons, the other seven being framed into the trimmers.

No. 482.—One of the headers split, starting at the lower tenon at one of the trimmers.

No. 483.—One of these headers was the unbroken one of No. 482. This header broke in the same way as in the case of No. 482.

No. 484.—One of the headers split.

No. 485.—One of the headers split.

No. 486.—One of the headers was the broken one of No. 483, and this one failed by splitting.

No. 487.—One of the headers split.

No. 490.—The splitting of one of the headers was followed almost immediately by that of the other.

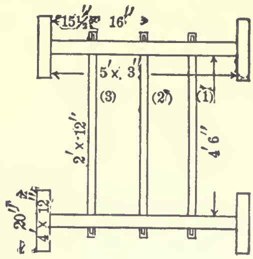
No. 492.—The failure of one header was soon followed by that of the other.

No. 493.—One of the headers failed, splitting very much.

No. 500.—One of the headers split.

TESTS OF SPRUCE HEADERS.

The general dimensions of the floors tested are shown in the figure. In all these tests the tail-beams are joined to the headers by tusk and tenon joints. The load was distributed equally at the centres of the three tail-beams.



The following table shows the results of these tests.

SPRUCE HEADERS (Figure, page 705).

Number of Test.	Width (inches).	Depth (inches).	Length between Trimmers (inches).	Number of Tail Beams.	Breaking Load of each Header. Pounds.
141	4	12	64	3	9020
142	4	12	64	3	9920
143	4	12	64	3	9415
170 (*a*)	$3\frac{3}{4}$	12	64	3	6917
170 (*b*)	$3\frac{3}{4}$	12	64	3	7417
170 (*c*)	$3\frac{3}{4}$	12	64	3	9417
SPRUCE HEADERS (Figure, page 706).					
217 (*A*)	4	12	64	3	10000
217 (*B*)	4	12	64	3	15850
217 (*C*)	4	12	64	3	10450

DETAILS OF THE TESTS.

No. 141.—Headers were framed into trimmers by double tenons and joint-bolts. Tail-beams were made of spruce. One of the headers broke by tension.

No. 142.—Headers were supported on trimmers by stirrup-irons. Spruce tail-beams were used at first, but they broke, leaving the headers intact. Then yellow-pine tail-beams were used. One of the headers failed by splitting along the middle.

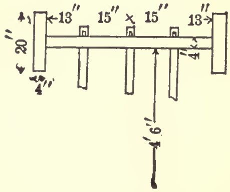
No. 143.—These headers were the unbroken ones of Nos. 141 and 142; the first one framed into the trimmer, the other hung from the trimmer by stirrup-irons. The header framed into the trimmer failed by splitting.

No. 170 (*a*).—Headers joined to trimmers by double tenons and joint-bolts. One of the headers split.

No. 170 (*b*).—Unbroken header of previous test used for one of the headers. One header split.

No. 170 (*c*).—The unbroken header of the previous test used for one of the headers. One header split.

No. 217. Dimensions of floor altered as indicated in the figure.



217A.—Headers framed into trimmers. One header split.

217B.—Headers supported upon trimmers by stirrup-irons. One header split.

217C.—Unbroken header of 217A used for one of the headers. One header split.

GENERAL REMARKS.

The stresses brought into play by the load on a header are not only bending and shearing, but also a tension across the grain, and any method of figuring the load a header would bear without taking account of all three, and especially the latter, would not furnish correct results.

Moreover the character of the results of tests of yellow-pine headers of different lengths, given on page 703, confirm this statement, for the strength with different lengths is not by any means universally proportional to its length, and, on the other hand, the breaking strengths do not increase with the length, as they would do if tension across the grain were the only stress and bending did not take place.

BAUSCHINGER'S TESTS.

In the ninth Heft of the Mittheilungen are given the results of an experimental study which Bauschinger made of the strength of certain pine and spruce woods, in connection with their other properties, as specific-gravity, age, time of felling, etc.; but special attention is given to the variation of strength and specific gravity, with the percentage of moisture which they contain, i.e., their condition of dryness. While he did a considerable amount of work upon the variation of the tensile strength with the percentage of moisture, the results are rather variable, and none of this work will be quoted here for the reasons given on page 646, under Tension of Timber. He himself came to the conclusion that more satisfactory results could be reached by experimenting upon compressive strength.

The following tables give summaries of the tests which he made and reported in this ninth Heft upon compressive and transverse strength:

COMPRESSIVE TESTS.

TEST-PIECES ABOUT $3\frac{1}{2} \times 3\frac{1}{2}$ INCHES AND 6 INCHES LONG.

Timber.	Place.	Summer Felled.		Winter Felled.	
		Percentage of Moisture.	Compressive Strength. Mean Values. Lbs. per Sq. In.	Percentage of Moisture.	Compressive Strength. Mean Values. Lbs. per Sq. In.
Pine . .	Lichtenhof .	9	3997	26	4537
Spruce .	Frankenhofen	20	3449	17	4452
Spruce .	Regenhütte .	27	3328	20	3997
Spruce .	Schliersee .	20	2304	19	3200

TRANSVERSE TESTS.

Timber.	Place.	Breadth and Depth. Inches.	Percentage of Moisture.	Modulus of Elasticity. Lbs. per Sq. In.	Modulus of Rupture. Lbs. per Sq. In.
I. Summer Felled. Span, 8 ft. 2.43 ins.					
Pine . .	Lichtenhof . .	6.70× 6.81	24	1422300	6002
" . .	" . .	7.19× 7.17	23	1536084	6685
" . .	" . .	6.66× 6.72	19	1536084	6742
" . .	" . .	7.06× 7.17	25	1664091	7453
Spruce .	Frankenhofen .	6.72× 6.76	29	1706760	6372
" .	" .	7.32× 7.25	35	1649868	6344
" .	" .	7.77× 7.80	25	1464969	5703
" .	" .	8.10× 8.10	26	1436523	5405
" .	Regenhütte . .	7.52× 7.64	39	1578753	6016
" .	" . .	8.01× 8.07	34	1592976	5476
" .	" . .	7.88× 7.83	30	1635645	6173
" .	" . .	8.36× 8.26	32	1720983	6016
" .	Schliersee . .	10.17×10.24	25	1016945	4025
" .	" . .	10.25×10.34	24	960052	3840
" .	" . .	10.47×10.47	21	1109394	4281
" .	" . .	10.73×10.67	24	1080948	4281
II. Winter Felled. Span, 8 ft. 2.43 ins.					
Pine . .	Lichtenhof . .	6.58× 6.71	33	1422300	6813
" . .	" . .	6.23× 6.30	33	1664091	7609
" . .	" . .	7.19× 7.34	31	1308516	5348
" . .	" . .	8.30× 8.17	34	1479192	5903
Spruce .	Frankenhofen .	7.07× 6.94	31	1479192	5959
"	" .	7.19× 7.29	28	1436523	5903
"	" .	6.65× 6.69	24	1863213	6969
" .	" .	6.64× 6.67	24	1820544	6813
" .	Regenhütte . .	6.38× 6.39	27	1479192	6230
" .	" . .	7.02× 7.08	29	1536084	6329
" .	" . .	6.66× 6.79	30	1635645	6443
" .	" . .	7.65× 7.66	38	1635645	6358
" .	Schliersee . .	10.98×12.66	26	1024056	3641
" .	" . .	10.83×13.18	25	981387	3755
" .	" . .	11.19×11.23	28	967164	3670
" .	" . .	11.11×11.10	25	981387	3570

In Heft 16 of the Mittheilungen, Bauschinger gives an account of a series of tests of the crushing and transverse strength of the more important coniferous woods from the different districts of Bavaria.

In the case of the transverse tests the percentage of moisture was determined by experiment, and is recorded in the tables. Then sections were cut from the same pieces from which the beams were taken, and tested for crushing, one of them being rather wet; one had somewhere near 15 per cent of moisture, which Bauschinger considers to be about the average dryness of the air, and one was somewhat drier; and in each case the percentage of moisture is determined and recorded.

The results of the crushing tests are then plotted, and a curve drawn, from which he determines the crushing-strength with 15 per cent of moisture. A similar proceeding is adopted in regard to the specific-gravity.

The 45 sections were each cut into five specimens about 3 or 4 inches square, one of them containing the heart.

These (which contain the heart) he omits from his curves and calculations, and plots only the results of the others.

His results are given in the following table, a perusal of which will show that the moduli of rupture, and also the crushing-strengths, run somewhat higher than they do for woods of the same name in the tests made at the Massachusetts Institute of Technology. This, of course, may be due to the woods that Bauschinger tested being stronger than American woods of the same name, but it is more probably due to the facts that, 1°, the specimens he used were rather smaller, and, 2°, they were, on the whole, drier than the American woods tested at the Massachusetts Institute of Technology.

No. of Test.	Name.	Place of Growth.	Specific Gravity reduced to 15% of Moisture.	Mean Compressive Strength of Pieces not containing Heart, reduced to a 15% Moisture. Lbs. per Sq. In.	b Same per cent Moisture as in Transverse Test. Lbs. per Sq. In.	Transverse Test. Span, 8 ft. 2.42 ins. Breadth. Inches.	Depth. Inches.	Percentage of Moisture.	Modulus of Elasticity. Lbs. per Sq. In.	Modulus of Rupture. Lbs. per Sq. In.
1	Larch . .	St. Zeno . .	0.62	6827	6756	5.78	6.97	15.5	2076560	10382
2	Larch . .	" . .	0.67	7353	6784	5.85	6.99	17.2	1692540	10596
3	Pine . .	" . .	0.53	6045	6400	5.78	5.74	13.5	1834770	9742
4	Spruce . .	" . .	0.43	4978	4551	7.78	7.61	16.6	1479190	7183
5	Pine . .	" . .	0.52	5263	5120	7.75	7.70	15.8	1649888	7325
6	Spruce . .	" . .	0.45	5405	5334	7.69	9.21	15.4	1635650	6756
7	" . .	" . .	0.48	5547	5831	5.80	6.85	14.3	1592980	7965
8	" . .	" . .	0.51	5974	5689	6.87	6.82	16.1	1578750	7751
9	Larch . .	" . .	0.59	6685	4978	6.10	6.91	16.0	1706760	9245
10	" . .	" . .	0.58	6116	5974	5.85	6.91	15.5	1521860	9743
11	Zürbe . .	" . .	0.41	3200	3271	5.86	7.00	14.7	846269	5191
12	Larch . .	Karlstein . .	0.61	7965	7752	5.85	7.16	15.8	2097898	9529
13	Spruce . .	" . .	0.54	5974	5334	6.79	7.78	17.4	1592980	7965
14	Pine . .	" . .	0.54	5974	6258	5.65	6.93	14.4	1905880	10027
15	Spruce . .	" . .	0.46	5405	5191	6.78	7.80	16.3	1777880	8107
16	Larch . .	" . .	0.68	7965	7325	7.61	9.34	17.9	2039670	8605
17	Spruce . .	" . .	0.54	6969	6898	5.79	6.95	15.1	2026780	9387
18	" . .	" . .	0.49	6045	5476	5.86	7.80	18.3	1578750	6542
19	" . .	" . .	0.52	6187	6045	5.83	6.87	15.6	1692540	8036
20	Pine . .	" . .	0.52	5831	5903	5.77	7.71	14.9	1550307	7823
21	Larch . .	" . .	0.61	7040	6258	5.81	6.93	17.6	1905880	10241
22	Spruce . .	" . .	0.51	6471	6187	5.76	7.77	15.8	1564530	8107
23	" . .	" . .	0.39	4267	4054	7.72	9.31	15.9	1208960	5334
24	Larch . .	Freising . .	0.61	7823	7538	5.75	6.94	15.9	2062340	9814
25	" . .	" . .	0.70	8249	7538	5.84	6.91	16.9	2176120	9814
26	Spruce . .	" . .	0.59	6969	5831	6.87	7.72	18.4	1977000	8249
27	" . .	" . .	0.44	5405	5191	6.96	7.62	15.6	1479190	6685
28	Pine . .	" . .	0.52	6187	5476	6.88	7.75	18.4	1550310	6258
29	" . .	" . .	0.62	7894	7467	6.86	7.64	16.9	2062340	9814
30	Spruce . .	" . .	0.48	5974	5689	6.91	7.74	16.7	1607200	7609
31	" . .	" . .	0.47	5618	4480	6.93	7.80	18.6	1763650	6898
32	Pine . .	" . .	0.53	6116	5689	5.83	6.86	16.8	1848990	8818
33	" . .	" . .	0.49	4907	4551	5.83	6.91	17.0	1223180	5974
34	Spruce . .	" . .	0.59	6827	6400	6.82	7.74	16.7	1948550	9103
35	" . .	" . .	0.48	5618	5334	6.96	7.70	16.4	1678310	6471
36	" . .	" . .	0.44	5334	5191	6.96	7.71	15.8	1550310	7325
37	" . .	" . .	0.49	6258	5974	7.00	7.82	16.4	1806340	7467
38	Larch . .	" . .	0.58	7325	7112	5.77	6.91	15.5	1742320	9387
39	" . .	" . .	0.65	5903	5263	7.00	6.96	18.2	1550310	7752
40	Pine . .	" . .	0.49	4764	4267	6.96	7.81	17.7	1038280	3485
41	Spruce . .	" . .	0.46	6116	5547	6.88	7.80	17.0	1280070	5618
42	" . .	Unterliezheim	0.42	4120	4125	6.85	7.71	20.7	1464970	6144
43	" . .	"	0.39	4907	4551	6.95	7.80	18.8	1251620	5191
44	White Pine	"	0.33	3414	3058	6.99	7.83	18.1	810731	4125
45	" "	"	0.32	3200	2631	7.00	7.84	21.4	725370	3556

AVERAGE COMPRESSIVE STRENGTH OF WHOLE SECTION OF LOG.

POUNDS PER SQUARE INCH.

Time of Felling.	Pine from Lichtenhof.		Spruce from Frankenhofen.		Spruce from Regenhütte.		Spruce from Schliersee.	
	1	2	1	2	1	2	1	2
Summer	7183	5244	6416	4807	6287	5319	4570	3143
Winter	6343	6784	6756	5618	6343	5348	4779	4238

1. Tested 5 years after felling. 2. Tested 3 months after felling.

OTHER FULL-SIZE TESTS.

References to other full-size tests of timber are:

1°. Tests of Pine Stringers, and Floor Beams, by Onward Bates, Trans. Am. Soc. C.E., 1890.

2°. Tests of White Pine of Large Scantling, by Prof. H. T. Bovey, Trans. Canadian Soc. C.E., 1893.

3°. The Strength of Canadian Douglas Firs, Red Pine, White Pine and Spruce (mostly full size). Trans. Canadian Soc. C.E., 1895.

4°. The Proc. Fifth Annual Convention of the Assoc. R. R. Supts. contains, among others, the following references: (*a*) Tests of Strength of State of Washington Timbers, Talbot Hart, and S. K. Smith; (*b*) Tests of the Northwest and Pacific Coast Timbers, S. K. Smith and Thurston; (*c*) Tests of California Redwoods, Soule; (*d*) Old and new White Pine Stringers, Finley; (*e*) Beams of Douglas Fir, Wing, 1895.

5°. U. S. Dept. of Agriculture; preliminary circular issued by the Bureau of Forestry, giving outline of future work, 1903.

6°. U. S. Dept. of Agriculture; Progress Report on the Strength of Structural Timbers, Bureau of Forestry, 1904.

§ 240. **Shearing of Timber along the Grain.**—The shearing of timber almost takes place along, and not across, the grain; for it can be shown, that, wherever we have a tendency to shear on a certain plane, there is an equal tendency to shear on a plane at right angles to it. Hence if there is, at any point in a piece of wood, a tendency to shear across the grain, there must necessarily accompany it an equal tendency to shear it along the grain; and wherever (as is almost always the case) the resistance to the latter is less than the resistance to the former, the timber will give way in this manner, instead of across the grain. As to the shearing-strength per square inch, some values have been given in Rankine's table; and the following table contains results obtained at the Watertown Arsenal, and recorded in Tests of Metals for 1881.

Kind of Wood.	Arsenal No.	Shearing-Strength per Square Inch.	Kind of Wood.	Arsenal No.	Shearing-Strength per Square Inch.
Ash	620	600	Oak (white) . .	631	752
	621	592	Pine (white) . .	752	324
	622	458		753	267
	623	700		754	352
Birch (yellow) .	623	563		755	366
	633	815	Pine (yellow) . .	607	399
	634	672		608	317
	635	612		614	409
Maple (white) . .	636	647		615	415
	637	537		616	409
	638	367		617	364
	639	431		618	286
Oak (red) . . .	624	775		619	330
	625	743	Spruce	748	253
	626	999		749	374
	627	726		750	347
Oak (white) . .	628	966		751	316
	629	803	Whitewood . .	609	406
	630	846		610	382

§ 241. **General Remarks.** — A perusal of the tests on columns and on beams will show that one of the principal sources of weakness in timber is the presence of knots, and it will be noticed that the position of the fracture is in most cases determined by the knots.

Sap-wood, season cracks, and decay are doubtless other sources of weakness. The tests, however, do not present such striking evidence of the deleterious effects of the first two as is the case with knots. In general, it may be said, however, that timber used in construction should be free, or nearly free, from sap-wood; as an excessive amount of sap-wood renders it weak.

It will often be found to be a common opinion among lumber-dealers, that a piece of timber which contains the heart is not as good as one which is cut from the wood on one side of the heart. This is very often true; as the timber which is sold in the market is very liable to have cracks at the heart, and also, if the tree has passed maturity, the heart is the place where decay is likely to begin. Nevertheless, the tests of beams would not, it seems to the author, bear out the conclusion that such pieces as contain the heart are always weaker than those that do not.

Another matter that claims serious consideration is the effect of seasoning upon the strength of timber. This question can only be decided by tests on full-size pieces, as the small pieces season much more rapidly and uniformly than full-size pieces.

In this regard, the observation should be made, that practically our buildings and other constructions are built with green lumber; i.e., lumber which has been cut from three months to a year. Unless it can be shown that the seasoning which the lumber receives while in use imparts to it a greater strength, it will only be proper to consider its strength the same as that of green lumber. Not very much evidence has thus far been obtained upon this point; but, such as it is, it will be noted here.

1°. We have, on p. 653, the results of the tests of a lot of old mill columns; and, while some of them did exhibit a greater strength than green ones, a perusal of this set of tests will convince the reader that it would not be safe to rely upon any greater strength in these columns than in green ones. Moreover, these columns had been in a building heated by steam for a number of years, and during the seasoning process they had been subjected to the load they had to support. The writer has also observed some evidence of the same kind in connection with one of his time tests.

2°. In the case of beams, we have, in Nos. 60 and 66, examples of beams which had been seasoning, ***unloaded***, in a building heated by steam; and in these cases there was a great gain in strength. Some yellow-pine beams exhibited a similar action. On the other hand, beams Nos. 18 and 19 had been seasoning on the wharf, in the open air, for about one year; and while some yellow-pine beams which had seasoned without load, in the building, showed great strength, in other cases the increase was not so marked.

In view of the fact that the above is practically all the evidence we have in the matter, it would seem to the writer, unless future experiments shall prove the contrary to be true, that we cannot rely, in our constructions, upon having any greater strength than that of the green lumber, and that the figures to be used should be those obtained by testing green lumber.

§ 242. **Building-Stones.**—The three most important factors about a building-stone besides its beauty, are its durability, its strength, and the ease with which it can be quarried and worked. In order to be durable it must be able to withstand the deleterious influences of rain, wind, frost, fire, and of the acids that are found in the air especially in large cities, where the most common are carbonic acid and sulphur acids.

The durability of a stone is probably its most important feature, and is, perhaps, the most difficult to test thoroughly. As a rule, the greater its hardness and the less its absorptive power for moisture, the more durable will it be.

Tests of hardness are easily and frequently made. Tests of absorptive power for moisture are very often made. While the methods pursued by different people differ, they consist essentially of weighing the specimen dry, and then soaking it in either hot or cold water until it has absorbed all that it will, and weighing it again.

There are a number of methods pursued in order to determine its power of withstanding the action of frost, and the results differ, of course, according to the method pursued. Bauschinger's method consists in—

1°. Determining the compressive strength of the stone in a dry and in a saturated condition, and comparing the two.

2°. Determining the compressive strength after twenty-five freezings and thawings.

3°. Determining the loss of weight after these twenty-five freezings.

4°. Examining the specimen with a microscope for cracks after the twenty-five freezings.

Stones will not withstand the heat of a large conflagration, brick being better than any building-stone.

As to how well a stone will stand the gases in the air of a large city, a great many tests have been proposed and used, but none of them are entirely satisfactory. Of course we can get indications from a study of the chemical composition, or better from a microscopical examination, which shows also the arrangement of the different components, this being, of course, the part of the geologist or mineralogist.

After the question of durability, the strength comes in as the factor of next importance, and although the loads usually put upon stones in construction are very much smaller than the breaking-strength of the stone as shown by small specimens tested in the testing-machine, nevertheless the mortar or cement joints, the bonding, and the necessary unevenness render the real factor of safety very much less than would be at first imagined.

Building-stones are more often called upon to bear a compressive load than any other, though they are sometimes called upon to bear a transverse load, as in the case of window-lintels.

The results are very variable, partly because the stone varies very much in quality, and partly because it is only of late years that it has been recognized that in order to obtain correct results in compression tests the pressure must be evenly distributed over the surfaces of the specimen pressed upon, and that in order to accomplish this even distribution it is necessary that the faces which come in contact with the platforms of the testing-machine shall be accurate planes, and that unless the compression platforms are adjustable, the two faces pressed upon shall be parallel—provided, of course, the platforms are parallel, as they should be. Formerly it was thought that the desired result could be obtained by interposing between the surface of the specimen and the platform some soft substance, as a cushion of wood or of lead, whereas it is a fact that any such cushions only render the results smaller and more variable than the real crushing-strength of the specimens. Hence it is that a great many of the tests that have been made are of no value, because this matter was not attended to. In a rough way we may divide the most common building-stones as follows: 1°. Granites and allied stones; 2°. Limestones, including marbles; 3°. Sandstones; 4°. Slates.

The most systematic set of tests was made by Bauschinger, and is reported in the Mittheilungen, Hefte 1, 4, 5, 6, 10, 11, 18, and 19.

Besides this we may note the following references:

1°. Report on Compressive Strength, etc., of the Building-Stones in the United States, 1876. Gillmore.
2°. Compressive Resistance of Freestone, Brick Piers, Hydraulic Cements, Mortars, and Concretes, 1888. Gillmore.
3°. Masonry Construction, 1889. Baker.
4°. Testing Materials of Construction, 1888. Unwin.
5°. History of the St. Louis Bridge, 1881. Woodward.
6°. Exec. Doc. 12, 47th Congress, 1st session. Senate.
7°. Exec. Doc. 35, 49th Congress, 1st session. Senate.

Some tables of results will now be given.

The following is taken from Woodward's "History of the St. Louis Bridge:"

Material.	Length, in inches.	Diameter of Cross-section, in inches.	Modulus of Elasticity, pounds per square inch.	Breaking-Weight, per square inch.
Grafton Magnesian Limestone.	6.46	1.14	10500000	7200
" " "	5.87	1.06	8400000	8500
" " "	5.96	1.06	8500000	2000
" " "	5.99	1.07	6000000	6000
" " (5 specimens)	3.00	3 × 3	....	av. 15400
" "	8.00	2.38	12000000	10100
" "	13.00	1.13	5000000	10800
Portland Granite.............	5.88	2.36	5500000	16000
" "	5.98	2.36	6400000	18500
" "	5.97	2.38	5000000	17000
Richmond Granite............	6.00	2.30	13500000	16400
Portland Granite.............	3.00	3 × 3	....	13700
Missouri Red Granite.........	3.00	3 × 3	....	12700
" "	3.00	3 × 3	....	13000
" "	3.00	3 × 3	....	12700
" "	3.00	3 × 3	....	13600
Brown Ochre Marble.........	3.00	3 × 3	....	15000
Sandstone, St. Genevieve, Mo.	3.00	3 × 3	....	5330
" "	4.88	4.88 × 4.88	....	5500
" "	3.06	3.06 × 3.06	....	3400

In Heft 4 of the Mittheilungen, Bauschinger gives a long table of results of testing granites, limestones, and sandstones, from which the following examples are selected:

In Heft 5 of the Mittheilungen is to be found a study of the modulus of elasticity of building-stones. Bauschinger found that in this case the departure from Hooke's law is greater with small than with large loads. Heft 6 contains an experimental study of the laws of compression. Heft 10 contains an experimental investigation of the principal Bavarian building-stones. Hefte 11 and 18 contain a study of the compressive strength and wearing qualities of paving-stones. Heft 19 contains a study of the power of stones to resist frost. For all these the student is referred to the Mittheilungen.

Kind of Stone.	Place.	Tests by Compression.		Transverse Tests.	
		Crushing Strength, pounds per sq. in.	Direction of the Pressure with respect to the bed.	Modulus of Rupture, pounds per sq. in.	Direction of Breaking Section with respect to the bed.
Granites.					
Yellow, very soft, exceptional quality	Selb in Oberfranken.	7750	$\perp$		
Gray, coarse-grained	"	11300	$\perp$		
" " "	"	11720	$\parallel$		
Very light-colored, tolerably coarse-grained	Waldstein	14790	$\perp$		
Very hard, coarse-grained	St. Gotthard	11740	$\perp$	av.1940	oblique
" " " "	"	12660	$\parallel$	1309	$\parallel$
Very hard, striped	"	15650	$\perp$	2773	across
" " "	"	13230	$\parallel$		
White, very hard, tolerably fine-grained, only good for paving .	Cham	22190		1991	
Syenite, black, with a good deal of gray.	Ebendaher	19200	$\perp$		
Limestones.					
White marble	Schlanders in Tyrol	12800	$\perp$		
Muschelkalk.	Wurzburg	6260		980	
"	Kronach	22760	$\perp$		
"	"	11390	$\parallel$		
Yellowish white, soft limestone of the white Frankenjura	Kelheim				
Best quality	....	{ 6900 8390		1252	
Poorest quality.	....	3560		939	
Granitic marble	Rosenheim	{ 10600 20340 17920	$\perp$		
Dolomite.					
From the white Frankenjura. . . .	Poppenheim	{ 18490 16780	$\perp$ $\parallel$	} 2560	
" " " . . .	Herzbruch	{ 12520 11240			
Sandstone.					
Bunter sandstone, gray, with yellow and brown streaks	Kronach	4836	$\perp$		
Bunter sandstone, red, very rich in quartz, fine-grained.	Unterfranken	19270	$\perp$		
Do. do. do.	"	20550	$\parallel$		
Bunter sandstone, dark red, fine-grained	Carlsruhe	11950	$\perp$		
Do. do. do.	"	8100	$\parallel$		
Keuper sandstone, red, fine-grained.	Wurtemburg	{ 8620 9390	} $\perp$		
Do. do. do.		654c	$\parallel$		
Keuper sandstone, white, with reddish bed stripes, coarse-grained.	Ansbach	2420	$\perp$		
Keuper sandstone, white, tolerably fine-grained.	"	2490	$\perp$		
Keuper sandstone, brown	Nuremberg	4340	$\perp$		
Green sandstone, yellow, with brown layers, fine-grained.	Regensberg	5480	$\perp$		
Green sandstone, dirty green, tolerably fine-grained	"	2680	$\perp$		
Green sandstone, greenish, fine-grained	Kelheim	{ 4410 3630	$\perp$ $\parallel$		

The following table is taken from the Trans. Am. Soc. Civ. Eng. for Oct. 1886, where it is quoted from " Mechanical tests of building materials, made at the Watertown Arsenal, by the United States Ordnance Dept., at the request of the Commissioners for the erection of the Philadelphia Public Buildings :"

Kind of Stone.	Locality.	Color.	Direction of Pressure.	Total Load applied, lbs.	Crushing Strength per sq. in. applied, lbs.	Sectional Area, sq. in.	Remarks.
Marble.	Lee, Mass.	Blue.	End.	715000	20504	34.87	Burst in fragments.
	" "	White.	Bed.	800000	22370	35.16	Slight flaking.
	" "	W. & B.	End.	800000	22860	34.99	No apparent injury.
	" "	White.	"	800000	22820	35.05	" " "
	" "	Blue.	Bed.	800000	22900	34.93	Flaked, one edge.
	" "	W. & B.	"	767000	21700	35.34	Crushed suddenly.
	Montgomery Co., Pa.	Blue.	"	466300	11470	40 64	Failed suddenly.
	" "	"	End.	400000	10420	38.40	Ultimate strength.
	" "	"	Bed.	543000	13700	39.63	" "
	" "	"	End.	398000	10120	39.33	" "
	" "	"	"	347500	9590	36.24	" "
	" "	"	Bed.	434000	10940	39.67	" "
Limestone.	Conshohocken, Pa. .	....	End.	494000	14090	35.05	Ultimate strength.
	" " .	....	Bed.	566000	16340	34.63	" "
	Indiana	....	End.	377000	8530	44.22	" "
	"	....	"	320500	7190	44.56	" "
	"	....	Bed.	321000	7776	41.38	" "
	"	....	"	438300	10620	41.28	" "
Marble.	Vermont	Dove-colored	Bed.	531200	13400	39.65	Ultimate strength.
	"		End.	379800	9870	38.48	" "

Kind of Stone.	Locality.	Color.	Direction of Pressure.	Total Load applied, lbs.	Crushing Strength per sq. in. applied, ibs.	Sectional Area, Sq. In.	Remarks.
Sandstone.	Hummelstown, Pa. .	....	Bed.	528700	12810	41.28	Ultimate strength.
	" " .	....	End.	570300	13610	41.92	" "
	Ohio	Buff.	Bed.	256000	6510	39.32	" "
	"	"	End.	199500	4860	41.02	" "
	"	"	Bed.	289500	7020	41.25	" "
	"	"	End.	160000	3940	40.06	Bearings imperfect.
	"	Blue.	"	305000	7680	39.68	Ultimate strength.
	"	"	Bed.	435400	10400	41.90	" "
	"	"	End.	391800	9795	40.00	" "
	"	"	Bed.	351000	8710	40.30	" "
	"	"	End.	672100	16280	41.28	Fractured suddenly.
	"	"	Bed.	493500	12420	39.74	Ultimate strength.

§ 243. **Hydraulic Cements and Brick Piers.**—When a pure or nearly pure limestone is calcined, so as to drive off the carbonic acid, we have an oxide of lime, commonly called quick-lime, which, on the addition of water, slakes, with the development of considerable heat; and the result is a fine powder, which by the addition of more water is reduced to a paste which slowly hardens upon exposure to the air. It is this paste, mixed with sand, which forms the mortar used in cheap buildings. It is very weak, and hardens very slowly, even in the air.

On the other hand, a hydraulic lime or a hydraulic cement contains impurities, of which silica forms the principal portion, though we usually find also alumina, protoxide of iron, and magnesia; and these impurities are in so large a proportion that the slaking entirely or nearly disappears, and the addition of water after calcination causes the formation of hydrated silicates, etc., which harden under water.

While we cannot draw a sharp line of demarcation between hydraulic lime and hydraulic cement, nevertheless the essential difference is that the first contains pure lime in sufficient proportions to slake, but at the same time contains enough clay, silica, etc., to enable it to set under water; whereas hydraulic cement contains less pure lime, and hardly slakes at all, but sets more rapidly than hydraulic lime.

Hydraulic cements are known as, 1°, Portland cement, and, 2°, Natural cement. The latter is commonly called Rosendale in America, and Roman in Europe. It acquires its strength more slowly, is weaker and cheaper than Portland, and it usually sets more quickly.

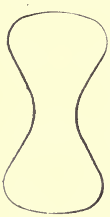
Portland cement is manufactured extensively in France, Germany, and England, and in the United States. It is made by mixing either dry or in paste, and then calcining, to the point of incipient vitrefaction, such mixtures of rocks as will give the proper chemical composition. The paste, when ready for the furnace, should contain from 76 to 81 per cent of carbonate of lime, and from 19 to 24 per cent of clay.

To give precise definitions of what constitutes Portland cement, what Natural cement, what Hydraulic lime, etc., is not an easy matter. An attempt to do so was made by the International Association for Testing Materials, but their definitions are not universally accepted, and will not be given here.

For definitions of Portland cement, and of Natural cement, which are by no means perfect, but which will answer in a general way, the reader is referred to those adopted by the Am. Soc. for Testing Materials on pp. 730 and 733. The manufacture of Natural cement dates from a very early period, and depends upon finding rocks of suitable composition; that of Portland cement dates from the early part of the nineteenth century, when Jos. Aspdin, of Leeds, made a slow-setting cement, by calcining a mixture of carbonate of lime, and clay, in suitable proportions.

TESTS OF THE STRENGTH OF CEMENTS.

While a good many tests have been made on the compressive strength of cement, and a few also on transverse or shearing strength, nevertheless the test most commonly used in order to determine its quality is the test of its tensile strength. The specimen used for this purpose is called a briquette, and



the cut shows one of its common forms, the smallest section being generally one square inch. The real reason for using the tensile instead of the compressive strength for a test is, in the opinion of the author, that inasmuch as the tensile strength of cement is very much less than its compressive strength, it follows that the machines for testing the tensile strength are cheaper, and the work of testing tensile strength is less than is the case in testing its compressive strength, although there are some who give other reasons which have some appearance of plausibility.

In order to discuss this matter intelligently, however, we should bear in mind —

Either the tensile or the compressive test is comparative and merely helps to determine the quality, and also to compare different lots of cement, but neither of them furnish us any figures which would be suitable to use in computing the allowable load on any structure which depended upon cement for its strength.

We might, therefore, conclude that, as far as the objects that can be attained by testing cement are concerned, either test would answer the purpose equally well; but inasmuch as it is also a fact that, with the best appliances thus far provided for the purposes, it is possible to obtain greater accuracy in the compressive than in the tensile test, therefore it seems to the author that the compressive and not the tensile is the test that should be used in making cement tests. Nevertheless, inasmuch as the tensile strength is most used, a brief account will be given here, showing what has been done, what we can reasonably expect from good cements, and a few precautions will be mentioned, which it is necessary to use in making the tests, in order to insure correct results.

The literature of cement testing is very extensive, but only the following will be given here

1°. Q. A. Gillmore: Practical Treatise on Limes, Hydraulic Cements, and Mortars.

2°. John Grant: Articles in the Proceedings of the British Institution of Civil Engineers, vols. xxv., xxxii., and xli.

3°. Charles Colson: Experiments on the Portland Cement used in the Portsmouth Dockyard Extension. Proc. Brit. Inst. Civ. Engrs., vol. xli.

4°. Isaac John Mann: The Testing of Portland Cement. Proc. Brit. Inst. Civ. Engrs., vol. xlvii.

5°. Wm. V. Maclay: Notes and Experiments on the Use and Testing of Portland Cement. Trans. Am. Soc. Civ. Engrs., Dec. 1877.

6° Eliot C. Clarke: Record of Tests of Cement made for the Boston Main Drainage Works, 1878–1884. Trans. Am. Soc. Civ. Engrs., April, 1885.

7°. Q. A. Gillmore: Notes on the Compressive Resistance of Freestone, Brick Piers, Hydraulic Cements, Mortars, and Concretes.

8°. J. Sondericker: How to Test the Strength of Cements. Am. Soc. Mech. Engrs. for 1888.

9°. Bauschinger: Mittheilungen aus dem Mechanisch-Technischen Laboratorium, Hefte i., vii., and viii.

10°. Exec. Doc. 12, 47th Congress, 1st session, House: Compressive Tests of Seven Cubes of Concrete.

11°. Exec. Doc. 5, 48th Congress, 1st session, Senate: Shearing Test of One Concrete Cube.

12°. Exec. Doc. 35, 49th Congress, 1st session, Senate: Tests of Neat Cement and Cement Mortars.

13°. Preliminary Report of the Committee on a Uniform System for Tests of Cement. Trans. Am. Soc. Civ. Engrs., January, 1884.

14°. Final Report of the Committee on a Uniform System for Tests of Cement. Trans. Am. Soc. Civ. Engrs., January, 1885.

15°. Behavior of Cement Mortars under various Contingencies of Use. F. Collingwood: Trans. Am. Soc. Civ. Engrs., Nov. 1885.

16°. Report of Progress by the Committee on the Compressive Strength of Cements, etc. Trans. Am. Soc. Civ. Engrs., July, 1886.

17°. Another Report of the Committee. Trans. Am. Soc. Civ. Engrs., June, 1888.

18°. E. F. Miller: Testing Cement. The Bricklayer.

19°. Candlot, E.: Ciments et Chaux hydrauliques. Fabrication, Propriétés, Emploi. Paris.

20°. Feret, R.: Note sur Diverses Expériences concernant les Ciments. Annales des Ponts et Chaussées, 1890, 1r semestre, page 313.

21°. Alexandre, Paul: Recherches expérimentales sur les Mortiers hydrauliques. Annales des Ponts et Chaussées, 1890, 2e semestre p. 227.

22°. Feret, R.: Sur la compacité de Mortier hydraulique. Annales des Ponts et Chaussées, 1892, 2e semestre, page 5.

23°. D. B. Butler: Portland Cement.

24°. Baumaterialienkunde: This is the official organ of the International Assoc. for Testing Materials, and contains many papers, and discussions on cement.

25°. Commission des méthods d'essai des matériaux de construction. Tome 1, Section B.—Essais des matériaux d'aggrégation des maçonneries. Rapport Général présenté par Paul Alexandre.

26°. Tests of Metals made at Watertown Arsenal.

27°. Mit . der Materialprüfungsanstalt in Zurich.

28°. Mitt. aus dem Mech. Tech. Lab. in Berlin.

29°. Mitt. aus dem Mech. Tech. Lab. in München.

30°. Report of Board of Engineer officers on testing hydraulic cements, 1902.

31°. Many articles in the Trans. Am. Soc. Civil Engineers.

32°. Many papers read before the Association of American Portland Cement Manufacturers.

Some quotations will be given from Candlot's treatise, including a portion of the specifications of the French Maritime Service.

Candlot says:

"The properties of Portland cement, which have given complete satisfaction for many years, being known, well defined, and absolutely constant, it ought to be sufficient, in order to determine the value of a cement, to see whether it presents, to the same degree, the qualities which characterize this list of hydraulic products."

"The tests generally made on cements have to do with their chemical composition, their density, their fineness of grinding, their time of setting, their tensile and compressive strength, and their invariability of volume."

The reason for each of these tests is so plain that no comment will be made, except to say that a cement that swells is liable to disintegrate after setting in consequence of free lime.

In France the greater part of the large manufactories are to be found in the region around Boulogne-sur-Mer; and the maritime service of the Department of "Ponts et Chaussées," which has a cement laboratory at Boulogne, has established certain specifications to which all cement used in their work must conform.

A portion of these specifications as given by Candlot will now be quoted in the following three pages :

ART. 1. The Portland cement furnished shall come exclusively from the manufactory of the one who offers it for sale. It shall be produced by grinding scorified rocks, obtained by calcining to the point of vitrification, of an intimate mixture of carbonate of lime and clay, carefully mixed, and chemically and physically homogeneous throughout.

ART. 2. The administration reserves the right, under conditions which it determines, to supervise the manufacture, the storing at the factory, and the shipping of the cement.

For this purpose the engineer or his representative shall have access at all times to all parts of the factory concerned ; and he may—

1°. Do whatever he thinks necessary to make sure of the composition of the crude pastes used.

2°. Supervise the sorting after calcining.

3°. Follow the cement after the sorting to the special cases where it is to be stored after grinding.

4°. Supervise the packing when it is taken from the cases, and also the shipping of the cement.

5°. Place special agents permanently at the factory for the above-stated purposes.

ART. 4. Every partial lot of cement, on its arrival at the storehouse of the works, must be examined as to dryness. No bag shall be allowed to enter which has been exposed to dampness, or whose contents is not entirely pulverulent throughout. Then the part allowed to enter, as far as dryness is concerned, shall be submitted to the tests prescribed for, 1°, density ; 2°, chemical composition ; 3°, time of setting ; 4°, absence of cracks after setting ; 5°, strength of briquettes of neat cement ; 6°, strength of briquettes of cement with normal sand.

The engineer, or his representative, shall take some cement from one or more bags chosen arbitrarily at such points as he shall decide, but without mixing cement from different bags. He shall then proceed to the tests, observing the precautions prescribed. Each of the specimens thus chosen must satisfy separately the conditions prescribed ; the measures to be taken in regard to the whole of a partial lot being those suitable for the specimen giving the least satisfactory result.

ART. 5 gives very elaborate instructions in regard to the determination of the minimum weight per litre of cement that has passed a sieve of 5000 meshes per square centimetre. A portion of this article is as follows, viz.:—To obtain, under conditions always comparable, an unheaped litre of the fine dust, produced by sifting cement through a sieve of 5000 meshes per square centimetre, we place on a firm support a measure of one litre capacity; above this measure we arrange a plane inclined at 45°, formed of a sheet of zinc 50 cm. long, whose horizontal lower edge shall be fixed one centimetre above the level of the upper plane of the measure; we pour, gently, the cement dust, by means of a spoon, onto the inclined plane at the top, until the measure is a little more than filled, and we remove the excess of cement by sliding over the edges of the measure a straight-edge held in a vertical plane. During this entire operation the measure must not be subjected to jar or shock. To obtain the weight of a litre, we make one single weighing of the total amount of five measures, filled with the above-described precautions.

ART. 6. Every cement in which the chemical analysis shall show more than 1% sulphuric acid or compounds of sulphur in measurable proportion shall be rejected.

ART. 7. All cement will be declared suspected in which chemical analysis shows more than 4% of oxide of iron, or which has a value less than $\frac{44}{100}$ for the ratio between the total weight of the combined silicon and aluminum, on the one hand, and the lime on the other.

ART. 8. In the tests of neat cement, the cement shall be mixed in sea-water. The water and air during mixing shall be kept as nearly as possible between 15° and 18° C.

To determine the proper proportion of water to mix with the cement we make the following preliminary test:

The mortar is obtained by taking 900 grammes of cement, and pouring on it all at once the water to be used, mixing the mortar with a trowel on a marble slab for five minutes from the moment of pouring the water.

The quantity of water used shall be considered normal if the mortar forms a firm paste, well united, brilliant and plastic, satisfying the following conditions:

1°. The consistency of the paste must not change if the mixing goes on for eight minutes instead of five.

2°. A small quantity of paste taken with the trowel and let fall on the marble from about 50 cm. must detach itself from the trowel without leaving any adhering to the trowel, and, after its fall, it must preserve approximately its form without cracking.

3°. A small quantity of paste being taken in the hand, it must be sufficient to give it some light taps to give it a rounded form and to make the water come to the surface; it must neither flatten out completely nor stick to the skin, and if the ball be let fall from half a metre, it must preserve a rounded form (slightly flattened), without cracks.

4°. With less water the paste should be dry, not well united, and should show cracks in falling. With more water it should have a muddy consistency, with adherence to the trowel.

After making a series of successive approximations we must adopt as normal proportion the greatest proportion of water tried which shall have produced a plastic and not a muddy paste satisfying the conditions stated.

ART. 9. A part of this article reads as follows: With a part of the paste thus obtained we fill a cylindrical metal box of 0.04 m. height and 0.08 m. diameter, jarring it a few seconds, and leaving the water that rises to the top. Then suspend, by a cord passing over a pulley, a Vicat needle of 300 grams weight and a square section 1 mm. on a side, and lower it gradually.

The beginning of the set is taken as the time when the needle ceases to penetrate to the bottom of the mould, and the end of the set, as the time when the needle no longer penetrates appreciably.

Times are estimated from the moment when the water is poured on the dry powder.

If the cement begins to set before thirty minutes or completes its set before three hours, the partial lot shall be rejected; the temperature during the operation having been between 15° and 18° C.

ART. 10 prescribes a form of test to guard against the presence of cracks after setting.

ART. 11. The paste for tensile tests of neat cement is obtained by mixing with a trowel, on a marble slab, during 5 minutes, 900 grammes of cement with the normal quantity of water, as already determined. Each mixing will furnish paste for 6 briquettes. Make three successive mixings to obtain 18 briquettes, which is the number to be used in each test.

The form of the briquette is prescribed, the thickness being $0^{m}.0222$, the smallest section being $0^{m}.0225$ wide; area, 5 square centimetres.

Put the moulds on a marble slab, and fill each set of six with one mixing, putting enough in each mould at once so that it shall overflow. Pack with the flat of the trowel. When the filling is complete, give little taps with the trowel handle on the side to disengage bubbles of air.

As soon as the consistency of the cement permits, smooth off the upper surface even with the mould by using the blade of a knife.

After the cement has set remove the moulds, leaving the briquettes on the slab.

During the first 24 hours the briquettes must be kept on the slab, in a damp atmosphere, free from currents of air and the direct rays of the sun, at a temperature of from 15° to 18° C.

After 24 hours immerse them in sea-water, the water to be renewed every week, and kept, as nearly as possible, at a temperature between 15° and 18° C.

For each sample of cement to be tested make 18 briquettes of neat cement, of which 6 are to be broken 7 days from the time of mixing, 6 at the end of 28 days, and 6 at the end of 84 days. For each series take one briquette from each mixing.

The testing-machine prescribed is one where the tension is obtained by pouring a jet of grains of lead into a vase at the end of a second lever. Among the six results in each series, choose the three highest; the mean of these three shall be considered to be the strength of the sample tested at that time.

ART. 12. *b*. The resistance of briquettes of neat cement at the end of the 7th day must be at least 20 kilogrammes per square centimetre. It must be at least 35 kg. at the end of the 28th day. Every partial lot whence comes a sample not satisfying these two conditions shall be rejected.

ART. 13. The strength per square cm. at the end of 28 days must be at least 5 kg. greater than that at the end of 7 days; otherwise the partial lot shall be suspected, the suspicion not to be removed unless the strength at the end of 28 days is at least 55 kg.

ART. 14. The strength per square cm. at the end of 84 days must be at least 45 kg. It must also exceed the strength at the end of 28 days when the latter was not at least 55 kg. Every partial lot not satisfying these conditions to be rejected.

The tests of cement mortar are made on briquettes of mortar composed of one part by weight of cement to three of normal sand, the latter being furnished by the Administration, and being such as will pass through a sieve of 64 meshes per square centimetre and be rejected by one of 144 meshes per square centimetre.

The amount of water used is 12% of the total weight of cement and sand.

Very minute directions are given in regard to the mixing and preparing the briquettes very similar to those for neat cement, and then the specifications proceed as follows, viz.: For each sample of cement we make 18 briquettes of normal sand mortar, of which 6 are to be broken at the end of 7 days, 6 at the end of 28 days, and 6 at the end of 84 days; using in each series a briquette from each of the six different mixtures in which the mortar is to be made. Of the six results in each series we take the three highest, and the mean of these is the figure admitted for the resistance of the mortar.

ART. 17. *c*. The strength of normal sand mortar at the end of seven days must be at least 8 kg. per sq. cm., and at the end of 28 days at least 15 kg. per sq. cm. Each partial lot whence comes a sample not satisfying these conditions is to be rejected.

ART. 18. The resistance at the end of 28 days must exceed that at the end of 7 days by at least 2 kilogrammes, otherwise the partial lot is to be suspected.

ART. 19. The resistance at the end of 84 days must be at least 18 kilogrammes, and it must exceed the resistance at the end of 28 days. Every partial lot whence comes a sample not satisfying these conditions should be rejected.

The German, the Swiss, and other specifications may be found in Candlot's book; but a portion of those of the American Society for Testing Materials will be quoted here.

AMERICAN SOCIETY FOR TESTING MATERIALS.

REPORT OF COMMITTEE ON STANDARD SPECIFICATIONS FOR CEMENT.

GENERAL OBSERVATIONS.

1. These remarks have been prepared with a view of pointing out the pertinent features of the various requirements and the precautions to be observed in the interpretation of the results of the tests.

2. The Committee would suggest that the acceptance or rejection under these specifications be based on tests made by an experienced person having the proper means for making the tests.

3. *Specific Gravity.*—Specific gravity is useful in detecting adulteration or underburning. The results of tests of specific gravity are not necessarily conclusive as an indication of the quality of a cement, but when in combination with the results of other tests may afford valuable indications.

4. *Fineness.*—The sieves should be kept thoroughly dry.

5. *Time of Setting.*—Great care should be exercised to maintain the test pieces under as uniform conditions as possible. A sudden

change or wide range of temperature in the room in which the tests are made, a very dry or humid atmosphere, and other irregularities vitally affect the rate of setting.

6. *Tensile Strength.*—Each consumer must fix the minimum requirements for tensile strength to suit his own conditions. They shall, however, be within the limits stated.

7. *Constancy of Volume.*—The tests for constancy of volume are divided into two classes, the first normal, the second accelerated. The latter should be regarded as a precautionary test only, and not infallible. So many conditions enter into the making and interpreting of it that it should be used with extreme care.

8. In making the pats the greatest care should be exercised to avoid initial strains due to molding or to too rapid drying-out during the first twenty-four hours. The pats should be preserved under the most uniform conditions possible, and rapid changes of temperature should be avoided.

9. The failure to meet the requirements of the accelerated tests need not be sufficient cause for rejection. The cement may, however, be held for twenty-eight days, and a retest made at the end of that period. Failure to meet the requirements at this time should be considered sufficient cause for rejection, although in the present state of our knowledge it cannot be said that such failure necessarily indicates unsoundness, nor can the cement be considered entirely satisfactory simply because it passes the test.

General Conditions.

Of these the first eight will not be quoted here.

9. All tests made in accordance with the methods proposed by the Committee on Uniform Tests of Cement of the American Society of Civil Engineers, presented to the Society, January 21, 1903, and amended January 20, 1904, with all subsequent amendments thereto. (See addendum to these specifications.)

10. The acceptance of rejection shall be based on the following requirements:

Natural Cement.

11. *Definition.*—This term shall be applied to the finely pulverized product resulting from the calcination of an argillaceous limestone at a temperature only sufficient to drive off the carbonic acid gas.

12. *Specific Gravity.*—The specific gravity of the cement thoroughly dried at 100° C., shall be not less than 2.8.

13. *Fineness.*—It shall leave by weight a residue of not more than 10 per cent on the No. 100, and 30 per cent on the No. 200 sieve.

14. *Time of Setting.*—It shall develop initial set in not less than ten minutes, and hard set in not less than thirty minutes, nor more than three hours.

15. *Tensile Strength.*—The minimum requirements for tensile strength for briquettes one inch square in cross-section shall be within the following limits, and shall show no retrogression in strength within the periods specified:*

Age.	Neat Cement.	Strength.
24 hours in moist air		50–100 lbs.
7 days (1 day in moist air, 6 days in water)		100–200 "
28 " (1 " " " " 27 " " ")		200–300 "
	One Part Cement, Three Parts Standard Sand.	
7 days (1 day in moist air, 6 days in water)		25– 75 lbs.
28 " (1 " " " " 27 " " ")		75–150 "

16. *Constancy of Volume.*—Pats of neat cement of about three inches in diameter, one-half inch thick at center, tapering to a thin edge shall be kept in moist air for a period of twenty-four hours.

(*a*) A pat is then kept in air in normal temperature.

(*b*) Another is kept in water maintained as near 70° F. as practicable.

17. These pats are observed at intervals for at least 28 days, and, to satisfactorily pass the tests, should remain firm and hard and show no signs of distortion, checking, cracking or disintegrating.

Portland Cement.

18. *Definition.*—This term is applied to the finely pulverized product resulting from the calcination to incipient fusion of an intimate mixture of properly proportioned argillaceous and calcareous materials, and to which an addition no greater than 3 per cent has been made subsequent to calcination.

* For example the minimum requirements for the 24-hour neat cement test should be some value within the limits of 50 and 100 lbs., and so on for each period stated.

19. *Specific Gravity.*—The specific gravity of the cement, thoroughly dried at 100° C., shall not be less than 3.10.

20. *Fineness.*—It shall leave by weight a residue of not more than 8 per cent on the No. 100, and not more than 25 per cent on the No. 200 sieve.

21. *Time of Setting.*—It shall develop initial set in not less than thirty minutes, but must develop hard set in not less than one hour, nor more than ten hours.

22. *Tensile Strength.*—The minimum requirements for tensile strength for briquettes one inch square in section shall be within the following limits, and shall show no retrogression in strength within the periods specified: *

Age.	Neat Cement.	Strength.
24 hours in moist air		150–200 lbs.
7 days (1 day in moist air, 6 days in water)		450–550 "
28 " (1 " " " " 27 " " ")		550–650 "
	One Part Cement, Three Parts Standard Sand.	
7 days (1 day in moist air, 6 days in water)		150–200 lbs.
28 " (1 " " " " 27 " " ")		200–300 "

23. *Constancy of Volume.*—Pats of neat cement about three inches in diameter, one-half inch thick at the center, and tapering to a thin edge, shall be kept in moist air for a period of twenty-four hours.

(*a*) A pat is then kept in air in normal temperature and observed at intervals for at least twenty-eight days.

(*b*) Another pat is kept in water maintained as near 70° F. as practicable, and observed at intervals for at least twenty-eight days.

(*c*) A third pat is exposed in any convenient way in an atmosphere of steam, above boiling water, in a loosely closed vessel for five hours.

24. These pats, to satisfactorily pass the requirements, shall remain firm and hard and show no signs of distortion, checking, cracking or disintegrating.

* For example the minimum requirement for 24-hour neat cement test should be some value within the limits of 150 and 200 lbs., and so on for each period stated.

25. *Sulphuric Acid and Magnesia.*—The cement shall not contain more than 1.75 per cent of anhydrous sulphuric acid (SO_3), nor more than 4 per cent of magnesia (MgO).

Bauschinger has also made a large number of compression tests, accounts of which may be found in Hefte 1, 7, and 8 of the Mittheilungen, but for these the student is referred to the Mittheilungen.

PRECAUTIONS TO BE OBSERVED IN TESTING CEMENTS.

The results obtained by testing different samples of the same cement will vary with—

1°. The percentage of water used in mixing.

2°. The length of time the sample has been kept under water and also the length of time it has been kept in the air before testing.

3°. The temperature of the water with which it was mixed, and also of that in which it was kept; also the temperature of the air in which it was kept.

4°. The rapidity of breaking.

Hence, in order that our results may be of value, we must take pains to regulate all these matters.

But another and all-important matter that has not received the necessary amount of attention is, that some means should be adopted for distributing the pull, in the case of a tension test, evenly over the section of the briquette, and in the case of a compression test for distributing the thrust evenly over the surface of the specimen. In the ordinary cement-testing machines to be found in the market there is generally no adequate provision for this purpose, and this is the reason why so great a variation exists in the results obtained with the same cement by so many experimenters. For a fuller account of this matter see Trans. Am. Soc. Mech. Engrs. for 1888, page 172.

The following is a summary of a part of a paper read by Mr.

James E. Howard of Watertown Arsenal, before the Assoc. of Am. Portland Cement Mfr., in April, 1905. He says:

1°. That a 100-mesh sieve has openings 0."0058 diam.

That a 200-mesh sieve has openings 0."0031 diam.

That a No. 20 bolting cloth has openings 0."0027 diam.

He advises the use of the latter for the separation of the fine from the coarse particles.

2°. That while he obtained for freshly ground Portlands specific gravities in the vicinity of 3.1, there were a number of natural cements examined, which had substantially the same values as Portlands, although some brands fell below 3.

That hydration, partial or complete, lowers the specific gravity. That hydration begins at once, goes on more quickly in the finer particles and more slowly in the coarser ones. That, in some cases, hydration was not complete at the end of five days. Hence, that the usual arbitrary methods of determining the beginning and the end of the set do not show the beginning and end of hydration.

That, as a rule, the compressive strength of the cement will not be diminished, if a period of about eight hours intervene between gauging and use.

3°. That exposure to high temperatures is liable to lead to ultimate disintegration.

4°. That a number of compression specimens were moulded under pressures varying from 7000 to 14000 pounds per square inch, continued for 40 hours or more, and were subsequently tested, at ages of 1 and 2 months. They developed phenomenal strength, a sample of neat cement showing a strength of 22050 pounds per square inch at the age of 57 days, and 1 to 1 mortar 19120 pounds per square inch at the age of 1 month.

Setting under high pressures admits of the use of smaller quantities of water in gauging; in one case only 5 per cent having been used.

TESTS OF FULL-SIZE PIECES.

Inasmuch as cement and mortar are almost always used as binding materials, tests of full-size pieces in which they enter are those of some form of masonry. Of such tests the number is not large, and those that will be quoted here are some tests of reenforced concrete, and some of brick pieces.

Concrete is composed of mortar and some hard material, as gravel, broken stone, cinder, etc., the general plan being to so proportion them that the cement shall approximately fill the voids in the sand, and that the mortar shall approximately fill the voids in the broken stone, or other hard material used.

Reenforced concrete, which is made by imbedding in the concrete, iron or steel bars, wire mesh or expanded metal, etc., is now attracting a great deal of attention.

COLUMNS.

An extensive series of tests of columns of reenforced concrete is now being carried on at the Watertown Arsenal, and the following table, which gives a summary of the tests of this kind already published, is quoted from "Tests of Metals for 1904."

TABULATION OF COMPRESSIVE STRENGTH OF CONCRETE AND MORTAR COLUMNS, PLAIN AND REENFORCED WITH LONGITUDINAL STEEL BARS.

Nominal dimensions 12″ by 12″ square by 8 feet high.

No. of Test.	Brand of Cement.	Composition. Cement.	Composition. Sand.	Composition. Stone or Cinder.	Composition. Kind of Stone or Cinder.	Age. Months.	Age. Days.	Reenforcing Bars. No. and Kind.	Reenforcing Bars. Per Cent of Material in Column.	Sectional Area in Square Inches. Gross.	Sectional Area in Square Inches. Concrete.	Sectional Area in Square Inches. Bars.	Weight of Concrete in Lbs. per Cu. Ft.	Compressive Strength. Total. Lbs.	Compressive Strength. Lbs. Per Sq. In. on Gross Area.	Remarks.
1618	Vulcanite	1	1	...	....	6	...	None	.....	159.64	......	....	135.0	800000	5011	Not broken.
1611	"	1	1	2	½″ to 1½″ pebbles	7	28	None	.....	158.00	......	....	144.1	271000	1720	
1613	"	1	1	2	"	8	...	4 ¾″ twisted	1.46	158.25	155.94	2.31	144.1	458000	2890	
1609	"	1	2	3	"	7	25	None	.....	157.12	......	....	143.2	278000	1769	
1612	"	1	2	3	"	7	28	4 ¾″ twisted	1.44	160.40	158.09	2.31	142.2	322000	2010	
1583	"	1	2	4	"	3	17	None	.....	159.30	......	....	147.6	272000	1710	
1580	"	1	2	4	"	3	13	4 ¾″ twisted	1.43	157.62	155.37	2.25	143.7	313000	1990	
1582	"	1	2	4	"	3	16	4 ⅝″ corrugated	0.97	156.76	158.21	1.55	145.7	349000	2180	
1581	"	1	2	4	"	3	14	4 ¾″ Thatcher	1.03	159.64	157.99	1.65	145.4	318000	1990	
1585	"	1	2	4	"	3	15	8 ¾″ twisted	2.86	157.50	153.00	4.50	143.5	497000	3160	
1584	"	1	2	4	"	3	14	8 ⅝″ corrugated	1.94	159.52	156.43	3.09	146.9	452000	2830	
1579	"	1	2	4	"	3	12	8 ¾″ Thatcher	2.09	158.50	155.19	3.31	146.4	438000	2760	
1610	"	1	2	4	"	7	26	4 ¾″ twisted	1.45	159.39	157.08	2.31	143.1	289800	1820	
1615	"	1	2	4	¼″ to 1½″ trap-rock	5	10	None	.....	159.26	......	....	151.2	278705	1750	
1614	"	1	2	4	Cinders	5	16	None	.....	159.39	......	....	103.7	138900	871	
1616	"	1	2	4	"	5	16	4 ¾″ twisted	1.45	159.39	157.08	2.31	106.3	334000	2095	
1607	"	1	3	6	½″ to 1½″ pebbles	7	23	None	.....	160.27	......	....	139.8	74100	462	
1608	"	1	3	6	"	7	24	4 ¾″ twisted	1.44	160.00	157.09	2.31	139.5	218600	1370	
1617	"	1	3	6	¼″ to 1½″ trap-rock	5	10	8 ⅝″ corrugated	1.94	159.39	156.30	3.09	148.8	365000	2290	
1639	Atlas	1	3	6	"	7	...	None	.....	159.63	......	....	155.4	75200	471	
1640	"	1	3	6	"	7	...	8 ⅝″ corrugated	1.93	160.27	157.18	3.09	154.0	192324	1200	Not broken.
1639	"	1	3	6	"	...	...	....	.....	......	......	....	.....	215000	1350	After rest of 4 months.
1640	....	...	...	...	....	...	...	....	.....	......	......	....	.....	425000	2650	After rest of 4 months.

The following table gives the results of a set of tests made in the Laboratory of Applied Mechanics of the Mass. Institute of Technology, upon reenforced concrete columns, the concrete consisting of 1 part Portland cement (Star brand), 2 parts sand, and 6 parts trap-rock.

Numbers.	Age in days.	Dimensions of Section. In. In.	Length in Feet.	No. of Rods.	Side of Square of Rod.	Plain, P; Twisted, T.	Breaking-Load.	Remarks.
1	30	8×8	17	1	1	P	107000	Crushed at end.
2	30	"	17	1	1	T	127000	Buckled, then crushed at end.
3	29	"	12	1	1	P	100000	Buckled, then crushed at end.
4	28	"	12	1	1	T	126000	Crushed at end. Poorly made. Crushed portion cut off; the rest bore 150000 lbs. at 40 days.
5	32	"	6	1	1	P	138000	Crushed at middle, then sheared off along rod to end.
6	31	"	6	1	1	T	133000	Crushed at end.
7	31	"	17	1	1¼	P	136000	Crushed at end, shearing obliquely.
8	25	"	17	1	1¼	T	154000	Crushed at end, breaking off 3 feet.
9	35	"	17	4	¾	P	182000	Crushed at end.
10	34	"	17	4	¾	T	167000	Crushed at end, concrete rather poor and rough at that end.
11	31	"	12	4	¾	T	147000	Crushed and split open at end.
12	32	"	12	4	¾	P	153000	Crushed and split open at end.
13	29	"	6	4	1	T	158000	Crushed and split open at end.
14	31	"	6	4	1	P	244000	Crushed at end.
15	35	10×10	17	1	1	P	215000	Broke off clean for 3 or 4 feet at end.
16	35	"	6	1	1	P	240000	Sheared diagonally at end.
17	45	"	6	1	1	T	228400	Sheared diagonally at end, and broke back for half the length.
18	31	"	12	1	1¼	T	262000	Sheared diagonally near end.
19	29	"	12	1	1¼	P	257000	Crushed at end.
20	28	"	12	4	¾	T	300000	Not broken.
21	29	"	12	4	¾	P	274000	Crushed at end. Wedge-shaped piece forced in between rods.

REENFORCED CONCRETE BEAMS.

While more or less theorizing has been done by different people regarding suitable formulæ for use in the case of reenforced concrete beams, the writer believes that more tests are needed before such theorizing can be placed upon a permanent basis. Some results of tests of full-size reenforced concrete beams are given in the following tables:

SOME TESTS MADE IN THE LABORATORY OF APPLIED MECHANICS OF THE MASSACHUSETTS INSTITUTE OF TECHNOLOGY.

Concrete was of the same composition as in the case of the columns. Size of beams 8″×12″. Span 11′. When load was at two points they were 44″ apart, and symmetrical with reference to the center.

No. of Beam.	Age in Days.	No., Size, and Kind of Bars near Bottom.			Manner of Loading at Time of Fracture.	Weight of Beams in Lbs.	Breaking Load, Exclusive of Weight of Beam, in Lbs.	Maximum Bending-Moment at Fracture in In.-lbs.
		No.	Side of, in Sq. Ins.	Plain, P, or Twisted, T.				
1	40	..	..	..	Center	1198	1302	62733
2	40	1	1/4	T	"	1200	1300	62700
3	39	1	1/2	T	At two points	1205	10095	241973
4	38	1	3/4	T	Center	1160	13680	470580
5	50	1	7/8	T	"	1290	14710	506715
6	50	1	1	T	"	1204	15796	541134
7	41	1	1 1/4	T	"	1195	12805	442283
8	41	2	1	T	"	1240	18760	639540
9	42	2	1 1/4	T	"	1274	23105	783486
10*	42	2	1 1/4	T	"	1279	21105	717569
11†	45	2	1 1/4	T	"	1294	23105	783816
12	30	2	1 1/4	T	At two points	1282	24200	553553
13	31	2	1 1/4	T	" " "	1292	29200	663718
14†	30	2	1 1/4	T	" " "	1341	24200	554527
15	53	1	1	P	" " "	1292	15250	356818
16	49	1	1	T	" " "	1211	16500	382982
17	43	2	3/4	P	" " '	1271	15950	370700
18	40	2	3/4	T	" " "	1200	19600	438807
19	35	4	1/2	P	" " "	1261	17500	378065
20	33	4	1/2	T	" " "	1213	20000	433329
21	57	1	1/4	P	" " "	1213	12500	295015
22	54	2	7/8	T	" " "	1248	22250	510092
23	57	2	7/8	P	" " "	1221	20250	465647
24	47	4	5/8	T	" " "	1203	19250	443350
25	50	4	5/8	P	" " "	1192	15250	355168
26	40	2	3/4	T	" " "	1215	24250	553548
27	49	1	1	T	" " "	1222	21750	498688

* Also one bar ½″ square near top.
† Also two bars ¾″ square near top.

In beams Nos. 12 and 13 there were, on each side of the middle of the span, eight pieces of ¼-inch twisted steel wire, bent in the form of a U enclosing the two reenforcing rods. In No. 13 they were vertical, and in No. 12 they were inclined at 45° to the horizon, sloping upward away from the middle. In No. 14 the wire pieces were in the form of a square, vertical, and enclosing all four of the bars. Nos. 26 and 27, each contained, in addition to the bars, a vertical layer of expanded metal, extending throughout their length and height.

SOME TESTS MADE AT THE ENGINEERING EXPERIMENT STATION, UNIVERSITY OF ILLINOIS.

PLAIN CONCRETE.

Beams were 12″ wide by 13½″ deep. Mixture by volume was 1 part Chicago A A Portland cement; 3 parts clean, sharp sand; 6 parts broken limestone (¾″–1½.″).

Beam No.	Length.		Age, days.	Span.		Maximum Applied Load.	Modulus of Rupture.
	Ft.	Ins.		Ft.	Ins.		
8	15	4	64	14		3600	412
11	15	4	65	14		2600	337
18	15	4	64	14		2400	322
26	12		62	10	8	5500	390
30	12		62	10	8	4800	355
23	9	6	61	8	6	6355	347
31	9	6	62	8	6	8000	422
24	6		61	5		10240	299
25	6		64	5		10200	299

REENFORCED CONCRETE.

Size of beams: Length, 15′ 4″, Span 14′, breadth 12″, depth 13½″, center of metal reenforcement 12″ below top surface of beam. Loads applied at the one-third points of beam.

Amount and Kind of Reenforcement.	Area of Metal, Sq. Ins.	Maximum Load, Lbs.
3 ½″ plain round	.59	9000
3 ½″ " "	.59	9200
3 ½″ " square	.75	9900
3 ½″ " "	.75	10000
4 ¾″ " "	2.25	26900
3 ½″ Ransome	.75	22800
3 ¾″ Thatcher	1.20	18400
3 ¾″ "	1.20	16600
3 ¾″ Kahn	2.40	24400
5 ½″ "	2.00	23000
4 ½″ "	1.60	17200
3 ½″ "	1.20	15000
6 ¾″ Johnson	2.19	34300
7 ½″ "	1.40	29000
5 ½″ "	1.00	20900
5 ½″ "	1.00	20600
3 ½″ "	.60	14000
3 ½″ "	.60	14000

BRICKS AND BRICK PIERS.

In this connection two sets of tests of brick piers, made at the Watertown Arsenal, will be quoted here. The first is taken from Tests of Metals for 1886, and the second from Tests of Metals for 1904.

The tabulation of the second series, which comprises 26 piers, is given in the table on page 742.

The first series comprises 53 piers, in the construction of which two kinds of brick were used, viz., common hard-burned bricks and face-bricks, laid on bed, with joints broken every course. The tabulation follows:

FACE-BRICK PIERS.

Weight per Cubic Foot.	Actual Dimensions.				Sectional Area.	Ultimate Strength.		
	Height.		Cross-section.			Total.	Per Sq. In.	Per Sq. Ft.
lbs.	ft.	in.	in.	in.	sq. in.	lbs.	lbs.	tons.
132.7	2	0.	7.63	7.61	58.06	141000	2428	174.81
134.4	2	0.27	7.64	7.63	58.29	123400	2117	152.42
130.2	3	11.95	7.75	7.68	59.52	122016	2050	147 60
129.7	4	0.	7.75	7.70	59.68	116000	1944	139.97
127.6	6	0.37	7.75	7.75	60.06	117117	1950	140.4
129.6	6	0.26	7.85	7.75	60.84	106470	1750	126.0
125.2	8	0.56	7.78	7.75	60.30	102000	1691	121.75
....	9	11.27	7.80	7.70	60.06	100749	1677	120.77
126.8	10	0.37	7.82	7.80	61.00	110500	1811	130.39
126.4	*5	11.68	11.60	11.60	134.56	257300	1912	137.66
129.0	5	11.27	11.55	11.50	132.82	258100	1943	139.89
130.3	5	11.	11.55 4.20	11.50 4.10	115.61	219659	1900	136.8
129.5	5	10.09	15.45	15.40	237.93	499653	2100	151.2
126.0	5	11.81	15.45	15.35	237.16	468700	1976	142.27

COMMON-BRICK PIERS.

120.8	1	10.87	7.65	7.52	57.53	161000	2798	201.45
123.3	1	11.13	7.65	7.60	58.14	157800	2714	195.40
125.4	4	0.37	7.60	7.55	57.38	111891	1950	140.40
124.6	3	11.62	7.68	7.58	58.21	101867	1750	126.00
121.5	6	1.62	7.65	7.60	58.14	144300	2481	178.63
123.3	6	1.18	7.60	7.58	57.61	132503	2300	165.60
121.4	8	1.50	7.65	7.63	58.37	90474	1550	111.60
121.4	7	11.98	7.60	7.55	57.38	90800	1582	113.90
121.0	10	0.93	7.60	7.55	57.38	86070	1500	108.00
123.1	10	1.31	7.60	7.55	57.38	104200	1815	130.68
123.5	1	11.06	11.60	11.40	132.24	307800	2327	167.54
125.8	1	10.75	11.38	11.35	129.16	318500	2466	177.55
124.9	3	11.58	11.55	11.50	132.83	224100	1687	121.46
125.1	3	11.81	11.40	11.30	128.82	251199	1950	140.40
123.2	6	0.75	11.45	11.45	131.10	222870	1700	122.40
121.7	6	0.75	11.48	11.45	131.45	216200	1644	118.36
121.6	8	1.37	11.45	11.40	130.53	190700	1461	105.19
120.8	8	0.75	11.50	11.40	131.10	211100	1610	115.92
119.5	10	1.00	11.55	11.45	132.25	178200	1347	96.98

* Core built of common brick.

COMMON-BRICK PIERS—*Continued.*

Weight per Cubic Foot.	Actual Dimensions.				Sectional Area.	Ultimate Strength.		
	Height.		Cross-section.			Total.	Per Sq. In.	Per Sq. Ft.
lbs.	ft.	in.	in.	in.	sq. in.	lbs.	lbs.	tons.
126.2	1	11.00	11.40 / 4.55	11.40 / 4.50	109.48	271500	2480	178.56
127.7	2	1.12	11.45 / 4.90	11.40 / 4.55	108.23	265400	2452	176.54
127.3	4	0.18	11.35 / 4.70	11.35 / 4.70	106.73	198100	1856	133.63
127.7	3	11.98	11.45 / 4.80	11.40 / 4.70	107.97	215200	1993	143.49
118.8	6	0.75	11.35 / 4.90	11.35 / 4.60	106.28	162100	1525	110.52
124.3	8	3.18	11.50 / 4.90	11.50 / 4.80	108.73	184841	1700	122.40
125.1	10	2.27	11.50 / 4.80	11.40 / 4.80	108.06	157500	1457	104.90
123.0	6	1.18	15.50	15.45	239.48	358200	1495	107.64
121.8	6	1.37	15.70	15.65	245.71	356600	1451	104.49
123.7	9	11.00	15.50	15.40	238.70	230200	964	69.41
120.1	9	11.98	15.70	15.60	244.92	247400	1010	72.72
124.3	6	0.68	15.45 / 8.60	15.45 / 8.30	167.32	270100	1614	116.21
125.6	6	0.68	15.50 / 8.70	15.45 / 8.60	164.66	260200	1580	113.76
123.4	9	11.25	15.40 / 8.40	15.40 / 8.30	167.44	212100	1267	91.22
128.9	9	10.56	15.45 / 8.70	15.40 / 8.60	163.11	202000	1238	89.14
122.3	*12	6.5	11.60	11.60	134.56	217200	1622	116.78
125.0	†12	6.5	11.55	11.40	131.67	193300	1468	105.69
117.1	† 4	0.	7.90	7.90	62.41	72300	1158	83.37
124.0	‡ 3	11.25	8.00	8.00	64.00	105800	1654	119.09
124.5	§ 6	1.1	11.55 / 4.20	11.55 / 4.20	115.76	60800	525	37.80

* Laid with bond stones 4 feet apart. † Common-brick pier grouted.
‡ Face-brick pier grouted. § Face-brick pier, laid without mortar.

The mortar was composed of Rosendale cement 1, sand 2. The piers were 21 months old when tested. In this series the mortar was kept purposely the same throughout, so that the variation in strength should be due to the variation in dimensions of the piers. The mortar, however, was found to be much stronger in some places than in others.

The tabulation of the second series, which comprises 26 piers, is given on page 742.

Nominal dimensions 12″×12″×8′ high. Piers laid in neat Portland (Vulcanite brand) cement mortar, and lime mortar. The hard and light hard common were sand-struck bricks.

No. of Test.	Description of Bricks.	Pier Load in	Age.		W'ght per Cubic Foot.	Sec-tional Area.	Compressive Strength.		Moduli of Elasticity between Loads per Sq. In. of			Permanent Set on Gauged Length of 50″ after Loads in Lbs. per Sq. In. of		
							Total.	Per Sq. In.	100 and 600.	600 and 1000.	1000 and 2000.	600	1000	2000
			Mo.	Da.	Lbs.	Sq. In.	Lbs.	Lbs.	Lbs.	Lbs.	Lbs.	Inch.	Inch.	Inch.
1594	Face, dry pressed.	Neat cement	1	...	137.3	139.48	402000	2880	2717000	2410000	2283000	.0005	.0012	.0027
1637		1 cement, 3 sand	6	...	130.0	142.68	342432	2400	2137000	2041000	1894000	.0017	.0028	.0082
1628		1 lime, 3 sand	6	...	129.0	138.42	210000	1517	1101000	1070000	....	.0709	.1479	
1634	Face, repressed, mud bricks.	Neat cement	5	23	124.4	144.96	279000	1925	1190000	1105000	....	.0048	.0086	
1625		1 cement, 3 sand	5	22	120.5	149.08	249000	1670	1190000	1070000	....	.0030	.0059	
1630		1 lime, 3 sand	6	...	120.5	143.76	181000	1260	689000	673000	....	.0821	.1925	
1599	Face, wire cut, mud bricks.	1 cement, 2 sand	...	25	133.7	142.09	342000	2410	2212000	2151000	2016000	.0012	.0022	.0077
1636		Neat cement	5	23	133.6	141.49	569000	4021	2381000	2273000	2165000	.0015	.0025	.0056
1629		1 lime, 3 sand	6	...	130.3	138.53	196700	1420	1064000	1163000	....	.0886	.1761	
1596	Hard, W. Cambridge, Mass.	Neat cement	1	...	135.5	132.60	623600	4700	2403000	2326000	2336000	.0014	.0027	.0058
1597		1 cement, 3 sand	1	...	126.9	141.49	254682	1800	1736000	1575000	....	.0022	.0053	
1627		1 lime, 3 sand	5	27	123.8	130.64	129900	994	806000	....	....	.2437		
1595	Light hard, W. Cambridge Mass.	Neat cement	1	...	117.6	149.57	226000	1510	874000	758000	....	.0042	.0084	
1619		1 cement, 3 sand	6	...	111.6	142.09	215800	1519	1012000	881000	....	.0025	.0051	
1631		1 lime, 3 sand	6	...	105.2	147.99	108300	732	379000	....	....	.2070		
1620*		1 lime, 3 sand	5	14	105.6	148.47	120100	809						
1632	Hard, E. Brookfield Mass.	Neat cement	5	20	116.6	147.01	289500	1969	1029000	1047000	....	.0060	.0095	
1633		1 cement, 3 sand	6	...	114.0	147.62	265716	1800	1295000	1176000	....	.0021	.0042	
1622		1 lime, 3 sand	5	26	106.9	147.62	108200	733	506000	....	....	.1854		
1621*		1 lime, 3 sand	5	14	109.9	143.88	127500	886	597000	....	....	.1858		
1623	Light hard, E. Brookfield, Mass.	Neat cement	5	17	109.7	151.53	160700	1061	661000	606000	....	.0092	.0172	
1635		1 cement, 3 sand	6	...	107.8	152.02	186000	1224	731000	629000	....	.0060	.0125	
1598		1 lime, 3 sand	...	24	105.9	148.72	69200	465						
1624	Hard, Mechanicsville, N. Y.	Neat cement	5	16	106.4	144.12	201768	1400	1012000	1000000	....	.0054	.0103	
1638		1 cement, 3 sand	6	...	103.8	147.38	208000	1411	1208000	1081000	....	.0037	.0066	
1626		1 lime, 3 sand	6	...	99.8	148.23	106500	718	529000	....	....	.1536		

* Panels of bricks filled with neat Portland cement.

CHAPTER VIII.

CONTINUOUS GIRDERS.

§ 244. **Fundamental Principles.** — A continuous girder is one that is continuous over one or more supports; i.e., one that has at least one support in addition to those at the ends. The principle of continuity is, that the neutral line is throughout a continuous curve over the supports, the tangent to one branch of the curve at the support being a prolongation of the tangent to the other branch.

Whereas, in the girder supported at the ends, the bending-moment at the support is zero, in the continuous girder there is a bending-moment at the support, where the girder is continuous. There is also a shearing-force at each side of the support, the sum of the shearing-forces on the two sides of any one support forming the supporting-force.

In this chapter will be given the general methods of determining the bending-moments, slopes, and deflections of continuous girders.

1°. When the loads are distributed.

2°. When the loads are all concentrated.

3°. When there are both distributed and concentrated loads.

It is believed that the reader will thus have the means of solving all cases of continuous girders, and that, whenever it is desirable to have a set of simplified formulæ for a small but definite number of spans, or for some special proportions or distribution of the load, he will be able to deduce such simplified formulæ from the more general ones.

§ 245. **Distributed Loads.** — In this case we assume that all the loads are distributed, whether they are uniformly distributed or not. The first step to be taken is, to find the bending-moment over each support: this is done by using what is known as the "*three-moment equation*," which we shall now proceed to deduce; and, in the course of the reasoning by which we deduce it, we shall derive a number of useful equations, expressing bending-moment, shearing-force, slope, deflection, etc., at various points.

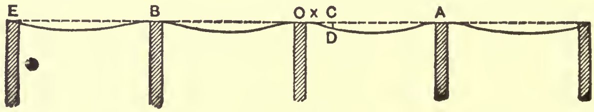


FIG. 247.

For the purpose in view, let us assume our origin at O (Fig. 247), and let

M_1 = bending-moment at B.
M_2 = bending-moment at O.
M_3 = bending-moment at A.
l_1 = OA.
l_{-1} = OB.
F_0 = shearing-force just to the right of O.
F_{-0} = shearing-force just to the left of O.
F_1 = shearing-force at distance x to the right of origin.
F_{-1} = shearing-force at distance x to the left of origin.

Shear is taken as positive when the tendency is to slide the part remote from the origin downwards.

If S_0 = supporting-force at O,

$$S_0 = F_0 + F_{-0}.$$

Beginning, now, by taking O as origin, and x positive to the right, —

Let $OC = x$.

$CD = v =$ deflection at distance x from origin.

$w =$ load per unit of length (either constant, or variable with x).

We shall then have, from the principles of the common theory of beams,

$$F_1 = F_0 - \int_0^x wdx; \tag{1}$$

i.e., the shearing-force at a distance x to the right of O is found by subtracting from the shearing-force just to the right of O the sum of the loads between the section at x and the support; and this sum is

$$\int_0^x wdx.$$

In a similar manner, if we were to take origin at O, and x positive to the left, we should have

$$F_{-1} = F_{-0} - \int_0^x wdx. \tag{2}$$

In § 204 we found the equation

$$\frac{dM}{dx} = F_1,$$

$$\therefore \quad \frac{dM}{dx} = F_0 - \int_0^x wdx.$$

Hence, integrating between $x = 0$ and $x = x$, and observing, that, when $x = 0$, $M = M_2$, we have

$$M - M_2 = F_0 x - \int_0^x \int_0^x wdx^2,$$

which reduces to

$$M = M_2 + F_0 x - \int_0^x \int_0^x w dx^2; \qquad (3)$$

or, in words, —

The bending-moment at a distance x to the right of O is equal to the bending-moment over the support at the origin, plus the product of the shearing-force just to the right of the origin by the distance of the section from the origin, minus the sum of the moments of the loads between the section and the support about the section.

Observe that this sum of the moments of the loads between the section and the support about the section has, for its mathematical equivalent, the expression

$$\int_0^x \int_0^x w dx^2;$$

and, as a particular instance, it may be noted, that when the load is *uniformly* distributed, and hence w is constant, this will reduce to

$$\frac{wx^2}{2} = (wx)\frac{x}{2},$$

wx being the load between the section and the support, and $\frac{x}{2}$ being the leverage of its resultant.

Now write, for brevity,

$$\int_0^x \int_0^x w dx^2 = m;$$

then

$$M = M_2 + F_0 x - m. \qquad (4)$$

Now, from § 194, we have

$$\frac{d^2v}{dx^2} = \frac{M}{EI}.$$

Let α_1 = slope at distance x to the right of the origin.
α_{-1} = slope at distance x to the left of the origin.
α_0 = value of α_1 when $x = 0$.
α_{-0} = value of α_{-1} when $x = 0$.

Then

$$\tan \alpha_1 = \frac{dv}{dx} = \int_0^x \frac{M}{EI} dx + c,$$

where c is an arbitrary constant, to be determined from the conditions of the problem.

If, now, we substitute for M its value $M_2 + F_0 x - m$, we shall have

$$\tan \alpha_1 = \frac{dv}{dx} = M_2 \int_0^x \frac{dx}{EI} + F_0 \int_0^x \frac{xdx}{EI} - \int_0^x \frac{mdx}{EI} + c.$$

To determine c, observe, that, when $x = 0$, $\alpha_1 = \alpha_0$;

$$\therefore \quad c = \tan \alpha_0$$

$$\therefore \quad \tan \alpha_1 = \frac{dv}{dx} = \tan \alpha_0 + M_2 \int_0^x \frac{dx}{EI}$$

$$+ F_0 \int_0^x \frac{xdx}{EI} - \int_0^x \frac{mdx}{EI}. \quad (5)$$

Integrate again, and observe, that, when $x = 0$, $v = 0$, and we obtain

$$v = x \tan \alpha_0 + M_2 \int_0^x \int_0^x \frac{dx^2}{EI}$$

$$+ F_0 \int_0^x \int_0^x \frac{xdx^2}{EI} - \int_0^x \int_0^x \frac{mdx^2}{EI}. \quad (6)$$

Now write, for the sake of brevity,

$$m = \int_0^x \int_0^x wdx^2, \quad m_1 = \int_0^{l_1} \int_0^x wdx^2, \quad m_{-1} = \int_0^{l-1} \int_0^x wdx^2,$$

$$n = \int_0^x \int_0^x \frac{dx^2}{EI}, \quad n_1 = \int_0^{l_1} \int_0^x \frac{dx^2}{EI}, \quad n_{-1} = \int_0^{l-1} \int_0^x \frac{dx^2}{EI},$$

$$q = \int_0^x \int_0^x \frac{xdx^2}{EI}, \quad q_1 = \int_0^{l_1} \int_0^x \frac{xdx^2}{EI}, \quad q_{-1} = \int_0^{l-1} \int_0^x \frac{xdx^2}{EI},$$

$$V = \int_0^x \int_0^x \frac{mdx^2}{EI}, \quad V_1 = \int_0^{l_1} \int_0^x \frac{mdx^2}{EI}, \quad V_{-1} = \int_0^{l-1} \int_0^x \frac{mdx^2}{EI},$$

the last four being derived by taking x positive to the left. We shall have

$$v = x \tan \alpha_0 + M_2 n + F_0 q - V; \qquad (7)$$

and, if v_1 = deflection at A = vertical height of A above O, we shall have, by substituting l_1 for x in (7),

$$v_1 = l_1 \tan \alpha_0 + M_2 n_1 + F_0 q_1 - V_1.$$

Now, if we assume any horizontal datum line entirely below all the points of support, and let the height of B above this line be y_b, that of A, y_a, and that of O, y_o, etc., we shall have

$$y_a - y_o = l_1 \tan \alpha_0 + M_2 n_1 + F_0 q_1 - V_1. \qquad (8)$$

And, if we put $x = l_1$ in (4), we shall have

$$M_3 = M_2 + F_0 l_1 - m_1$$

$$\therefore \quad F_0 = \frac{M_3 - M_2 + m_1}{l_1}: \qquad (9)$$

and, if we substitute this value of F_0 in (8), we obtain, by reducing,

$$y_a - y_o = l_1 \tan \alpha_0 + M_2\left(n_1 - \frac{q_1}{l_1}\right) + \frac{M_3 q_1}{l_1} + \frac{m_1 q_1}{l_1} - V_1;$$

and, solving for $\tan \alpha_0$, we obtain

$$\tan \alpha_0 = \frac{y_a - y_0}{l_1} + M_2\left(\frac{q_1}{l_1^2} - \frac{n_1}{l_1}\right) - M_3\frac{q_1}{l_1^2} - \frac{m_1 q_1}{l_1^2} + \frac{V_1}{l_1}. \quad (10)$$

This expression gives us the tangent of the slope at O in span OA; and equation (9) gives us the shearing-force just to the right of O in span OA, in terms of M_2, M_3, and known quantities.

If we were to take the origin at O, as before, and x positive to the left instead of the right, we should have, in place of (4),

$$M = M_2 + F_{-0}x - m; \quad (11)$$

in place of (9),

$$F_{-0} = \frac{M_1 - M_2 + m_{-1}}{l_{-1}}; \quad (12)$$

and in place of (10),

$$\tan \alpha_{-0} = \frac{y_b - y_0}{l_{-1}} + M_2\left(\frac{q_{-1}}{l_{-1}^2} - \frac{n_{-1}}{l_{-1}}\right)$$
$$- M_1\frac{q_{-1}}{l_{-1}^2} - \frac{m_{-1}q_{-1}}{l_{-1}^2} + \frac{V_{-1}}{l_{-1}}. \quad (13)$$

But, since the girder is continuous, we must have the tangent at O to the left-hand part, a prolongation of the tangent at O to the right-hand part, as shown in Fig. 248.

Hence we must have

$$\alpha_{-c} = -\alpha_0$$

$$\therefore \quad \tan \alpha_{-0} + \tan \alpha_0 = 0.$$

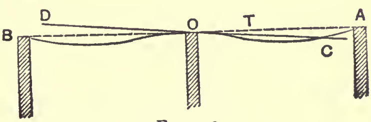


Fig. 248.

Hence, adding (10) and (13), we have

$$\frac{y_a - y_0}{l_1} + \frac{y_b - y_0}{l_{-1}} + M_2\left\{\frac{q_1}{l_1^2} - \frac{n_1}{l_1} + \frac{q_{-1}}{l_{-1}^2} - \frac{n_{-1}}{l_{-1}}\right\} - M_3\frac{q_1}{l_1^2}$$
$$- M_1\frac{q_{-1}}{l_{-1}^2} - \frac{m_1 q_1}{l_1^2} - \frac{m_{-1}q_{-1}}{l_{-1}^2} + \frac{V_1}{l_1} + \frac{V_{-1}}{l_{-1}} = 0; \quad (14)$$

and this is the "*three-moment equation*" for the case of a distributed load, whether it be *uniformly* distributed or otherwise.

CASE WHEN SUPPORTS ARE ON THE SAME LEVEL.

When the supports are all on the same level, then $y_a = y_b = y_o$, and the three-moment equation becomes

$$M_2\left\{\frac{q_1}{l_1^2} - \frac{n_1}{l_1} + \frac{q_{-1}}{l_{-1}^2} - \frac{n_{-1}}{l_{-1}}\right\} - \frac{M_3 q_1}{l_1^2} - \frac{M_1 q_{-1}}{l_{-1}^2}$$
$$- \frac{m_1 q_1}{l_1^2} - \frac{m_{-1} q_{-1}}{l_{-1}^2} + \frac{V_1}{l_1} + \frac{V_{-1}}{l_{-1}} = 0. \quad (15)$$

MANNER OF USING THE THREE-MOMENT EQUATION.

When the dimensions and load of the girder are known, all the quantities in the three-moment equation, whether we use (14) or (15), are known, except the three bending-moments, M_1, M_2, and M_3.

Suppose, now, the girder to have any number of (say, seven) points of support; then, by taking the origin at B (Fig. 247), we obtain one equation between the bending-moments at E, B, and O, the first of which, if E is an end support, is zero. Next take the origin at O, and we obtain one equation between the three bending-moments at B, O, and A; and so, continuing, we obtain five equations between five unknown quantities.

Solving these, we obtain the bending-moments over the supports; and from these bending-moments, after they are found, we can obtain the shearing-forces, bending-moments, slopes, and deflections, by using the equations deduced in the course of the reasoning for the three-moment equation, as equations (4), (5), (7), (9), and (10).

SPECIAL CASE,

when, the supports being all on the same level, the load on any one span is uniformly distributed over that span, and when the girder is of uniform section throughout.

Let w_1 = load per unit of length on span OA, origin at O.
w_{-1} = load per unit of length on span OB, origin at O.
I = the constant moment of inertia of the section
$y_a = y_b = y_0$.

Then

$$m_1 = \frac{w_1 l_1^2}{2}, \qquad m_{-1} = \frac{w_{-1} l_{-1}^2}{2};$$

$$n_1 = \frac{l_1^2}{2EI}, \qquad n_{-1} = \frac{l_{-1}^2}{2EI};$$

$$q_1 = \frac{l_1^3}{6EI}, \qquad q_{-1} = \frac{l_{-1}^3}{6EI};$$

$$V_1 = \frac{w_1 l_1^4}{24EI}, \qquad V_{-1} = \frac{w_{-1} l_{-1}^4}{24EI}.$$

With these substitutions, the three-moment equation, either (14) or (15), becomes

$$M_1 l_{-1} + 2M_2(l_{-1} + l_1) + M_3 l_1 + \tfrac{1}{4}(w_1 l_1^3 + w_{-1} l_{-1}^3) = 0. \quad (16).$$

This is a simpler form of the three-moment equation, applicable to this particular case only.

EXAMPLE I. — Suppose we have a continuous girder of uniform section, uniformly loaded, and of three equal spans,

FIG. 249.

to find M_B and M_C, also the supporting-forces, shearing-forces, bending-moments, slopes, and deflections throughout.

Solution. — Take the origin at B, and we have

$$M_1 = 0, \qquad M_2 = M_3 = M_B = M_C;$$

since

$$l_1 = l_{-1} = l,$$

equation (16) gives

$$5M_B l + \tfrac{1}{2}wl^3 = 0 \qquad \therefore \quad M_B = M_C = -\frac{wl^2}{10}.$$

Next, to find the shearing-forces, we have, from (9),

$$F_B = F_{-C} = \frac{-\frac{wl^2}{10} + \frac{wl^2}{10} + \frac{wl^2}{2}}{l} = \frac{wl}{2},$$

equals shearing-force just to the right of B or left of C.

Shearing-force just to the right of C or left of $B =$

$$\frac{+\frac{wl^2}{10} + \frac{wl^2}{2}}{l} = \tfrac{3}{5}wl.$$

Hence supporting-forces are

$$S_B = S_C = (\tfrac{3}{5} + \tfrac{1}{2})wl = \tfrac{11}{10}wl,$$

$$S_A = S_D = \tfrac{1}{2}(3wl - \tfrac{22}{10}wl) = \tfrac{2}{5}wl.$$

Bending-moment in span AB at distance x from A, or in span CD at distance x from D,

$$M = \tfrac{2}{5}wlx - \frac{wx^2}{2}.$$

Bending-moment in middle span at a distance x from B or from C,

$$M = -\frac{wl^2}{10} + \frac{wlx}{2} - \frac{wx^2}{2}.$$

Shearing-force in span AB or CD at a distance x from A or D,

$$F = \tfrac{2}{5}wl - wx.$$

Shearing-force in middle span at distance x from B or C,

$$F = \frac{wl}{2} - wx.$$

Maximum bending-moment in span BC (when $x = \frac{l}{2}$),

$$M_o = -\frac{wl^2}{10} + \frac{wl^2}{4} - \frac{wl^2}{8} = \frac{wl^2}{40}.$$

Maximum bending-moment in span AB or CD,

$$x = \tfrac{2}{5}l,$$

$$M_0 = \frac{4}{25}wl^2 - \frac{4}{50}wl^2 = \frac{2wl^2}{25}.$$

Hence the greatest bending-moment to which the girder is subjected is that at B or C, and its amount is $\dfrac{wl^2}{10}$.

Slope at B in middle span, from equation (10),

$$\tan \alpha_B = -\frac{wl^2}{10}\left(-\frac{l}{3EI}\right) + \frac{wl^2}{10}\left(\frac{l}{6EI}\right) - \frac{wl^3}{12EI} + \frac{wl^3}{24EI}$$

$$= \frac{wl^3}{EI}\left(\frac{1}{30} + \frac{1}{60} - \frac{1}{24}\right) = \frac{wl^3}{120EI},$$

which denotes an upward slope at B towards the right. In the same way, the girder slopes upwards at C towards the left. The slopes at B and C in the end spans are, of course, downwards.

Slope in the middle span at a distance x from B,

$$\tan \alpha = \frac{dv}{dx} = \frac{1}{EI}\left\{-\frac{wl^2}{10}x + \frac{wlx^2}{4} - \frac{wx^3}{6}\right\} + c.$$

When $x = 0$,

$$\tan \alpha = +\frac{wl^3}{120EI} \qquad \therefore \quad c = \frac{wl^3}{120EI}$$

$$\therefore \quad \tan \alpha = \frac{w}{EI}\left\{\frac{l^3}{120} - \frac{l^2x}{10} + \frac{lx^2}{4} - \frac{x^3}{6}\right\}$$

$$= \frac{w}{120EI}(l^3 - 12l^2x + 30lx^2 - 20x^3)$$

$$\therefore \quad \text{Deflection} = v = \frac{w}{120EI}(l^3x - 6l^2x^2 + 10lx^3 - 5x^4).$$

In order to make plain all methods of proceeding, the slope in the end spans will be found in two different ways, as follows: —

For bending-moment, slope, and deflection in left-hand span at a distance x from B (or in the right-hand span at distance x from C), we have

$$M = -\frac{wl^2}{10} + \frac{3}{5}wlx - \frac{wx^2}{2}.$$

$$\tan\alpha = \frac{dv}{dx} = \frac{1}{EI}\left\{-\frac{wl^2x}{10} + \frac{3wlx^2}{10} - \frac{wx^3}{6}\right\} + c.$$

When $x = 0$,

$$\tan\alpha = -\frac{wl^3}{120EI} \qquad \therefore \quad c = -\frac{wl^3}{120EI}$$

$$\therefore \quad \tan\alpha = \frac{dv}{dx} = \frac{w}{EI}\left\{-\frac{l^3}{120} - \frac{l^2x}{10} + \frac{3}{10}lx^2 - \frac{x^3}{6}\right\}$$

$$= \frac{w}{120EI}(-l^3 - 12l^2x + 36lx^2 - 20x^3)$$

$$\therefore \quad \text{Deflection} = v = \frac{w}{120EI}(-l^3x - 6l^2x^2 + 12lx^3 - 5x^4).$$

We may, on the other hand, accomplish the same object by finding the slope and deflection in left-hand span at distance x from A, or in right-hand span at distance x from D, as follows: —

$$\tan\alpha = \frac{dv}{dx} = \frac{1}{EI}\int\left\{\frac{2}{5}wlx - \frac{wx^2}{2}\right\}dx = \frac{w}{EI}\left\{\frac{lx^2}{5} - \frac{x^3}{6}\right\} + c.$$

When $x = l$,

$$\tan\alpha = \frac{wl^3}{120EI}$$

$$\therefore \quad \frac{wl^3}{120EI} = \frac{wl^3}{30EI} + c \qquad \therefore \quad c = -\frac{wl^3}{40EI}$$

$$\therefore \quad \tan\alpha = \frac{dv}{dx} = \frac{w}{EI}\left\{-\frac{l^3}{40}+\frac{lx^2}{5}-\frac{x^3}{6}\right\}$$

$$= \frac{w}{120EI}\left\{-3l^3+24lx^2-20x^3\right\}$$

$$\therefore \quad v = \frac{w}{120EI}\left\{-3l^3x+8lx^3-5x^4\right\}.$$

The figure shows the mode of bending of the girder.

FIG. 250.

To find the greatest deflection in either span, put the expression for the slope equal to zero, and find x by the ordinary methods for solving an equation of the third degree, and then substitute this value in the expression for the deflection.

EXAMPLE II. — Continuous girder of two equal spans, section uniform, and load uniformly distributed.

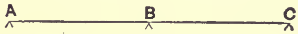


FIG. 251.

Solution. — Take origin at B.

$$M_1 = M_3 = M_A = M_C = 0, \quad M_2 = M_B, \quad l_1 = l_2 = l, \quad w_1 = w_2 = w;$$

therefore, from equation (16),

$$4M_Bl + \tfrac{1}{2}wl^3 = 0 \qquad \therefore \quad M_B = -\frac{wl^2}{8}.$$

Shearing-force either side of B =

$$F_B = F_{-B} = \frac{\dfrac{wl^2}{8}+\dfrac{wl^2}{2}}{l} = \tfrac{5}{8}wl.$$

Supporting-force at $B = \frac{5}{4}wl$.
Supporting-force at A and $C = \frac{3}{8}wl$.
Shear at distance x from A or C,

$$F = \tfrac{3}{8}wl - wx.$$

Bending-moment at distance x from A or C,

$$M = \frac{3}{8}wlx - \frac{wx^2}{2}.$$

Maximum bending-moment occurs when $x = \frac{3}{8}l$,

$$M_0 = \frac{9}{64}wl^2 - \frac{9}{128}wl^2 = \frac{9wl^2}{128}.$$

Hence greatest bending-moment to which the girder is subjected is that at B, and its magnitude is $\frac{wl^2}{8}$.

Slope at B, from equation (6),

$$\tan\alpha_B = \tan\alpha_{-B} = -\frac{wl^2}{8}\left(-\frac{l}{3EI}\right) - \frac{wl^3}{12EI} + \frac{wl^3}{24EI}$$

$$= \frac{wl^3}{EI}\left\{\frac{1}{24} - \frac{1}{12} + \frac{1}{24}\right\} = 0,$$

as was to be expected.

Slope at distance x from A in span AB,

$$\tan\alpha = \frac{1}{EI}\left(\frac{3}{16}wlx^2 - \frac{wx^3}{6}\right) + c.$$

When $x = l$, $\quad \alpha = 0$;

$$\therefore \quad C = -\frac{wl^3}{48EI}$$

$$\therefore \quad \tan\alpha = \frac{dv}{dx} = \frac{w}{EI}\left\{\frac{3}{16}lx^2 - \frac{x^3}{6} - \frac{l^3}{48}\right\}$$

$$= \frac{w}{48EI}(9lx^2 - 8x^3 - l^3).$$

Deflection,

$$v = \frac{w}{48EI}(-l^3x + 3lx^3 - 2x^4).$$

For maximum deflection, we have

$$\frac{3}{16}lx^2 - \frac{x^3}{6} - \frac{l^3}{48} = 0$$

$$\therefore \quad x = 0.44l.$$

$$\text{Maximum deflection} = \frac{wl^4}{48EI}\left\{-1 + 3(0.44)^2 - 2(0.44)^3\right\}(0.44)$$
$$= -0.0054\frac{wl^4}{EI}$$

Example III. — In order to solve a case where no simplifications enter, on account of symmetry or otherwise, we will take a continuous girder of five spans (as shown in the figure), the spans varying in length from $3l$ to $7l$; the loads being uniformly distributed, and varying in intensity from $3w$ on the longest span to $7w$ on the shortest; the beam being of uniform section.

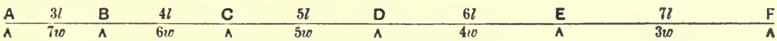


Fig. 252.

For this case we can use equation (16).

Origin at B,

$$0 + 14lM_B + 4lM_C + \tfrac{1}{4}[7w(27l^3) + 6w(64l^3)] = 0,$$

or

$$56M_B + 16M_C = -573wl^2.$$

Origin at C,

$$4lM_B + 18lM_C + 5lM_D + \tfrac{1}{4}wl^3[6(64) + 5(125)] = 0,$$

or

$$16M_B + 72M_C + 20M_D = -1009wl^2.$$

Origin at D,

$$5lM_C + 22lM_D + 6lM_E + \tfrac{1}{4}[5(125) + 4(216)]wl^3 = 0,$$

or

$$20M_C + 88M_D + 24M_E = -1489wl^2.$$

Origin at E,

$$6lM_D + 26lM_E + \frac{wl^3}{4}[4(216) + 3(343)] = 0,$$

or

$$24M_D + 104M_E = -1893wl^2.$$

The four equations are:

$$56M_B + 16M_C = \qquad\qquad - 573wl^2. \qquad (1)$$

$$16M_B + 72M_C + 20M_D = \qquad -1009wl^2. \qquad (2)$$

$$20M_C + 88M_D + 24M_E = -1489wl^2. \qquad (3)$$

$$24M_D + 104M_E = -1893wl^2. \qquad (4)$$

Eliminate M_E between (3) and (4), and we obtain

$$130M_C + 536M_D = -6839wl^2. \qquad (5)$$

Eliminate M_D between (2) and (5), and we obtain

$$2144M_B + 8998M_C = -101011wl^2. \qquad (6)$$

Eliminate M_C between (1) and (6), and we obtain

$$234792M_B = -1769839wl^2. \qquad (7)$$

$$\therefore \qquad M_B = - 7.5379wl^2,$$

$$\therefore \text{ from } (1), M_C = - 9.4299wl^2,$$

$$\text{from } (5), M_D = -10.4722wl^2,$$

$$\text{from } (4), M_E = -15.7853wl^2.$$

Shearing-force just to the right of

$$A = \frac{-0 - 7.5379 + 31.5}{3}wl = 7.9874wl,$$

$$B = \frac{-9.4299 + 7.5379 + 48}{4}wl = 11.5270wl,$$

$$C = \frac{-10.4722 + 9.4299 + 62.5}{5}wl = 12.2915wl,$$

$$D = \frac{-15.7853 + 10.4722 + 72}{6}wl = 11.1145wl,$$

$$E = \frac{15.7853 + 73.5}{7}wl = 12.7550wl.$$

Shearing-force to the left of

$$B = \frac{7.5379 + 31.5}{3}wl = 13.0126wl,$$

$$C = \frac{-7.5379 + 9.4299 + 48}{4}wl = 12.4730wl,$$

$$D = \frac{-9.4299 + 10.4722 + 62.5}{5}wl = 12.7084wl,$$

$$E = \frac{-10.4722 + 15.7853 + 72}{6}wl = 12.8855wl,$$

$$F = \frac{-15.7853 + 73.5}{7}wl = 8.2450wl.$$

Supporting-force at

$A = 7.9874wl$, $C = 24.7645wl$, $E = 25.6405wl$,
$B = 24.5396wl$, $D = 23.8229wl$, $F = 8.2450wl$.

Shearing-force at distance x to the right of

A in section $AB = 7.9874wl - 7wx$,
B in section $BC = 11.5270wl - 6wx$,
C in section $CD = 12.2915wl - 5wx$,
D in section $DE = 11.1145wl - 4wx$,
E in section $EF = 12.7550wl - 3wx$.

Bending-moment at distance x from

$$A \text{ in section } AB = + 7.9874wlx - \frac{7wx^2}{2},$$

$$B \text{ in section } BC = - 7.5379wl^2 + 11.5270wlx - \frac{6wx^2}{2},$$

$$C \text{ in section } CD = - 9.4299wl^2 + 12.2915wlx - \frac{5wx^2}{2},$$

$$D \text{ in section } DE = -10.4722wl^2 + 11.1145wlx - \frac{4wx^2}{2},$$

$$E \text{ in section } EF = -15.7853wl^2 + 12.7550wlx - \frac{3wx^2}{2}.$$

For the sections of maximum bending-moments (put shearing-force = 0), —

$$\text{In } AB,\ x = 1.1410l;$$
$$\text{In } BC,\ x = 1.9211l;$$
$$\text{In } CD,\ x = 2.4583l;$$
$$\text{In } DE,\ x = 2.7786l;$$
$$\text{In } EF,\ x = 4.2517l.$$

Hence the maximum bending-moments are respectively, in —

Section AB,

$$wx\left(-\frac{7x}{2} + 7.9874l\right) = \qquad +4.5570wl^2.$$

Section BC,

$$-7.5379wl^2 + wx(11.5270l - 3x) = 3.5347wl^2.$$

Section CD,

$$-9.4299wl^2 + wx(12.2915l - \tfrac{5}{2}x) = 5.6781wl^2.$$

Section DE,

$$-10.4722wl^2 + wx(11.1145l - 2x) = 4.9693wl^2.$$

Section EF,

$$-15.7853wl^2 + wx(12.7550 - \tfrac{3}{2}x) = 11.3297wl^2.$$

Values of $\tan \alpha_0$ = slope in every case in the span, towards the right.

$$\tan \alpha_0 = M_2\left(\frac{q_1}{l_1^2} - \frac{n_1}{l_1}\right) - \frac{M_3 q_1}{l_1^2} - \frac{m_1 q_1}{l_1^2} + \frac{V_1}{l_1}.$$

Slope at B,

$$\tan \alpha_B = -7.5379\frac{wl^3}{EI}\left(\frac{2}{3} - 2\right) + 9.4299\left(\frac{2}{3}\right)\frac{wl^3}{EI} - \frac{32wl^3}{EI} + \frac{16wl^3}{EI}$$
$$= 0.3371\frac{wl^3}{EI}.$$

Slope at C,

$$\tan\alpha_C = -9.4299\frac{wl^3}{EI}\left(\frac{5}{6} - \frac{5}{2}\right) + 10.4722\left(\frac{5}{6}\right)\frac{wl^3}{EI} - \frac{625}{12}\frac{wl^3}{EI}$$
$$+ \frac{625}{24}\frac{wl^3}{EI} = -1.5983\frac{wl^3}{EI}.$$

Slope at D,

$$\tan\alpha_D = -10.4722\frac{wl^3}{EI}(1 - 3) + 15.7853\frac{wl^3}{EI} - \frac{72wl^3}{EI} + \frac{36wl^3}{EI}$$
$$= 0.7297\frac{wl^3}{EI}.$$

Slope at E,

$$\tan\alpha_E = -15.7853\frac{wl^3}{EI}\left(\frac{7}{6} - \frac{7}{2}\right) - \frac{1029}{12}\frac{wl^3}{EI} + \frac{343}{8}\frac{wl^3}{EI} = -6.0426\frac{wl^3}{EI}.$$

The manner of bending, very much exaggerated, is shown in the accompanying figure.

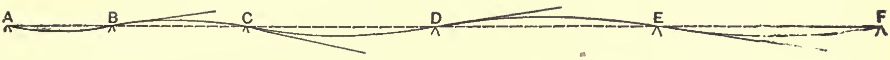


FIG. 253.

$$\text{Slope at } A = -4.096\frac{wl^3}{EI}, \quad \text{slope at } F = +23.4578\frac{wl^3}{EI}.$$

For the deduction, see what follows.

Slopes in General.

Span AB, origin at A,

$$\tan\alpha = \frac{w}{EI}\left\{3.9937lx^2 - \frac{7}{6}x^3\right\} + c.$$

When $x = 3l$, $\tan\alpha = 0.3371\frac{wl^3}{EI}$;

$$\therefore \quad \frac{wl^3}{EI}(35.9433 - 31.5) + c = 0.3371\frac{wl^3}{EI}$$

$$\therefore \quad c = -4.106\frac{wl^3}{EI}$$

$$\therefore \quad \tan\alpha = \frac{w}{EI}\left\{3.9937lx^2 - \frac{7}{6}x^3 - 4.106l^3\right\}.$$

Span BC, origin at B,

$$\tan\alpha = 0.3371\frac{wl^3}{EI} - 7.5379\frac{wl^2x}{EI} + 5.7635\frac{wlx^2}{EI} - \frac{wx^3}{EI}.$$

Span CD, origin at C,

$$\tan\alpha = \frac{w}{EI}\left\{-1.5983l^3 - 9.4299l^2x + 6.1458lx^2 - \frac{5}{6}x^3\right\}.$$

Span DE, origin at D,

$$\tan\alpha = \frac{w}{EI}\left\{0.7297l^3 - 10.4722l^2x + 5.55725lx^2 - \frac{2}{3}x^3\right\}.$$

Span EF, origin at E,

$$\tan\alpha = \frac{w}{EI}\left\{-6.0426l^3 - 15.7853l^2x + 6.3775lx^2 - \frac{wx^3}{\ }\right\}.$$

When $x = 7l$,

$$\tan\alpha = \frac{wl^3}{EI}(-6.0426 - 110.4971 + 312.4975 - 172.5)$$

$$= +23.4578\frac{wl^3}{EI}$$

Deflections.

Span AB,

$$v = \frac{w}{EI}\left\{1.3312lx^3 - \frac{7}{24}x^4 - 4.106l^3x\right\}.$$

Span BC,

$$v = \frac{w}{EI}\left\{0.3371l^3x - 3.7689l^2x^2 + 1.9211lx^3 - \frac{x^4}{4}\right\}.$$

Span CD,

$$v = \frac{w}{EI}\left\{-1.5983l^3x - 4.7149l^2x^2 + 2.0486lx^3 - \frac{5}{24}x^4\right\}.$$

Span DE,

$$v = \frac{w}{EI}\left\{0.7297l^3x - 5.2361l^2x^2 + 1.8524 + lx^3 - \frac{x^4}{6}\right\}.$$

Span EF,

$$v = \frac{w}{EI}\left\{-6.0426l^3x - 7.8927l^2x^2 + 2.1258lx^3 - \frac{x^4}{8}\right\}.$$

The maximum deflections can be obtained by putting the slopes equal to zero, as before.

§ 246. **Continuous Girder with Concentrated Loads.**— For our next general case, we will take that where there are no distributed loads, but where all the loads are concentrated at single points, and the section uniform throughout; and we will begin by assuming only one concentrated load on each span.

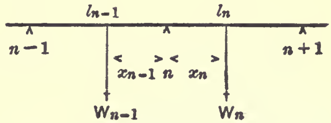


FIG. 254.

Let the support marked $n - 1$ be the $(n - 1)^{\text{th}}$ support, and the length of the $(n - 1)^{\text{th}}$ span be l_{n-1}; let the load on this span be W_{n-1}, and likewise for the other spans. Assume the origin at n, and let

F_n = shearing-force just to the right of n.
F_{-n} = shearing-force just to the left of n.
F_1 = shearing-force at distance x to the right of n.
F_{-1} = shearing-force at distance x to the left of n.

Shear is taken as positive when the tendency is to slide the part remote from the origin downwards.

If S_n = supporting-force at n,

$$S_n = F_n + F_{-n}. \qquad (1)$$

Let, also, x_n = distance from origin to point of application of load W_n, and let x_{n-1} = distance from origin to point of application of load W_{n-1}.

Take x positive to the right. Then, for

$$\left.\begin{aligned} x < x_n, \quad F_1 &= F_n; \\ x > x_n, \quad F_1 &= F_n - W_n. \end{aligned}\right\} \qquad (2)$$

Moreover, we have

$$\frac{dM}{dx} = F_1;$$

hence, by integration, for

$$x < x_n, \int_{M_n}^{M} dM = \int_0^x F_n dx;$$

$$x > x_n, \int_{M_n}^{M} dM = \int_0^x F_n dx - \int_0^x W_n dx + c;$$

the value of c being determined from the condition, that, when $x = x_n$ the two results must be identical. Hence we have, for

$$\left. \begin{array}{ll} x < x_n, & M = M_n + F_n x; \\ x > x_n, & M = M_n + F_n x - W_n(x - x_n). \end{array} \right\} \qquad (3)$$

Make $x = l_n$ in the last equation, and we have

$$M_{n+1} = M_n + F_n l_n - W_n(l_n - x_n). \qquad (4)$$

Now let $l_n - x_n = a_n$, and (4) becomes

$$M_{n+1} = M_n + F_n l_n - W_n a_n; \qquad (5)$$

hence

$$F_n = \frac{M_{n+1} - M_n + W_n a_n}{l_n}. \qquad (6)$$

Moreover, we have, as before,

$$\frac{d^2v}{dx^2} = \frac{M}{EI} \qquad \therefore \quad EI\frac{d^2v}{dx^2} = M,$$

I being a constant.

Let, as before, —

α_1 = slope at distance x to the right of origin.
α_{-1} = slope at distance x to the left of origin.
α_n = value of α_1 when $x = 0$.
α_{-n} = value of α_{-1} when $x = 0$.

Then by integration, determining the constant in the same way as in (3), we have, for

$$\left.\begin{aligned} x < x_n, \quad & EI(\tan\alpha_1 - \tan\alpha_n) = M_n x + F_n\frac{x^2}{2}; \\ x > x_n, \quad & EI(\tan\alpha_1 - \tan\alpha_n) = M_n x + F_n\frac{x^2}{2} - \frac{W_n(x - x_n)^2}{2}. \end{aligned}\right\} \quad (7)$$

Hence

$$x < x_n, \quad EI\frac{dv}{dx} = EI\tan\alpha_n + M_n x + F_n\frac{x^2}{2};$$

$$x > x_n, \quad EI\frac{dv}{dx} = EI\tan\alpha_n + M_n x + F_n\frac{x^2}{2} - \frac{W_n(x - x_n)^2}{2}.$$

Integrate again, and determine constants in the same way, and for

$$\left.\begin{aligned} x < x_n, \quad & EIv = EIx\tan\alpha_n + M_n\frac{x^2}{2} + F_n\frac{x^3}{6}; \\ x > x_n, \quad & EIv = EIx\tan\alpha_n + M_n\frac{x^2}{2} + F_n\frac{x^3}{6} - \frac{W_n(x - x_n)^3}{6}. \end{aligned}\right\} \quad (8)$$

Make $x = l_n$ in the last equation, and denote the heights of the supports above the datum line in the same way as in § 245, and we have

$$EI(y_{n+1} - y_n) = EIl_n\tan\alpha_n + M_n\frac{l_n^2}{2} + F_n\frac{l_n^3}{6} - \frac{W_n(l_n - x_n)^3}{6}. \quad (9)$$

Substitute for $l_n - x_n$, a_n, and for F_n, its value from (6), and we have

$$EI(y_{n+1} - y_n) = EIl_n\tan\alpha_n + M_n\frac{l_n^2}{3} + M_{n+1}\frac{l_n^2}{6} + \frac{W_n a_n}{6}(l_n^2 - a_n^2). \quad (10)$$

Hence

$$EI\tan\alpha_n + \frac{l_n}{6}(2M_n + M_{n+1}) + \frac{W_n a_n}{6l_n}(l_n^2 - a_n^2) = EI\frac{(y_{n+1} - y_n)}{l_n}. \quad (11)$$

Now, if we take origin at n and x positive to the left, we should obtain, instead of (11),

$$EI \tan \alpha_{-n} + \frac{l_{n-1}}{6}(2M_n + M_{n-1})$$
$$+ \frac{W_{n-1}a_{n-1}}{6l_{n-1}}(l_{n-1}^2 - a_{n-1}^2) = EI\left(\frac{y_{n-1} - y_n}{l_{n-1}}\right). \quad (12)$$

Now add (11) and (12), and observe, that, since the girder is continuous,

$$\tan \alpha_n + \tan \alpha_{-n} = 0,$$

and we obtain

$$\frac{M_n}{3}(l_n + l_{n-1}) + M_{n-1}\frac{l_{n-1}}{6} + \frac{M_{n+1}l_n}{6}$$
$$+ \frac{W_n a_n}{6l_n}(l_n^2 - a_n^2) + \frac{W_{n-1}a_{n-1}}{6l_{n-1}}(l_{n-1}^2 - a_{n-1}^2)$$
$$= EI\left\{\frac{y_{n+1} - y_n}{l_n} + \frac{y_{n-1} - y_n}{l_{n-1}}\right\}; \quad (13)$$

and this is the "three-moment equation" for the case of a single concentrated load on each span, and a uniform section.

When the supports are all on the same level, this becomes

$$\frac{M_n}{3}(l_n + l_{n-1}) + M_{n-1}\frac{l_{n-1}}{6} + \frac{M_{n+1}l_n}{6} + \frac{W_n a_n}{6l_n}(l_n^2 - a_n^2)$$
$$+ \frac{W_{n-1}a_{n-1}}{6l_{n-1}}(l_{n-1}^2 - a_{n-1}^2) = 0. \quad (14)$$

Either of these equations can be used (when it is applicable) just as the three-moment equation was used in the case of distributed loads.

CASE OF MORE THAN ONE LOAD ON EACH SPAN.

When there is more than one load on each span, the three-moment equation becomes as follows: —

$$\frac{M_n}{3}(l_n + l_{n-1}) + M_{n-1}\frac{l_{n-1}}{6} + M_{n+1}\frac{l_n}{6}$$
$$+ \Sigma\frac{W_n a_n}{6l_n}(l_n^2 - a_n^2) + \Sigma\frac{W_{n-1}a_{n-1}}{6l_{n-1}}(l_{n-1}^2 - a_{n-1}^2)$$
$$= EI\left\{\frac{y_{n+1} - y_n}{l_n} + \frac{y_{n-1} - y_n}{l_{n-1}}\right\}. \quad (15)$$

In using these equations for concentrated loads, we can determine the moments over the supports; but we must observe, that, in getting slopes and deflections, bending-moments, etc., the algebraic expressions that represent them are different on the two sides of any one load, and hence we must deduce new values whenever we pass a load, determining the constants for our integration to correspond.

Example. — Given a continuous girder of three spans, the middle span = 20 feet, each end span = 15 feet; supports on same level. The only loads on the girder are two; viz., a load of 5000 lbs. at 5 feet from the left-hand end, and one of 4000 lbs. 5 feet from the right-hand end. The supports are lettered from left to right, A, B, C, D, respectively. Find the greatest bending-moment and greatest deflection.

Solution. — Origin at B,

$$\frac{M_B}{3}(20 + 15) + \frac{M_C}{6}(20) + \frac{5000 \times 5}{6 \times 15}(225 - 25) = 0. \quad (1)$$

Origin at C,

$$\frac{M_C}{3}(20 + 15) + \frac{M_B}{6}(20) + \frac{4000 \times 5}{6 \times 15}(225 - 25) = 0. \quad (2)$$

These reduce to

$$\left.\begin{aligned} 70M_B + 20M_C + \frac{1000000}{3} &= 0 \\ 20M_B + 70M_C + \frac{800000}{3} &= 0 \end{aligned}\right\} \quad \therefore \quad \begin{aligned} M_B &= -4000 \text{ foot-lbs.} \\ M_C &= -2667 \text{ foot-lbs.} \end{aligned}$$

Shearing-forces.		Supporting-forces.	Slopes at supports.
$F_A = 3067$,	$F_{-C} = -67$.	$S_A = 3067$.	$\tan\alpha_A = -\frac{59444}{EI}$.
$F_{-B} = 1933$,	$F_C = 1511$.	$S_B = 2000$.	$\tan\alpha_B = +\frac{35556}{EI}$.
$F_B = 67$,	$F_{-D} = 2489$.	$S_C = 1444$.	$\tan\alpha_C = -\frac{31111}{EI}$.
		$S_D = 2489$.	$\tan\alpha_{-D} = -\frac{48889}{EI}$.

Span AB, origin at A,

$$x < 5, \quad M = 3067x.$$
$$x > 5, \quad M = 3067x - 5000(x - 5) = 25000 - 1933x.$$

Maximum bending-moment occurs when $x = 5$ and therefore $M_o = 15333$.

$$x < 5, \quad EI\tan\alpha_1 = -59444 + 1533x^2;$$
$$x > 5, \quad EI\tan\alpha_1 = 25000x - 967x^2 + c.$$

Determine c by condition, that, when $x = 5$, these two become equal;

$$\therefore \quad c = -121944;$$
$$\therefore \quad x > 5, \quad EI\tan\alpha_1 = -121944 + 25000x - 967x^2.$$

For deflections,

$$x < 5, \quad EIv = -59444x + 511x^3;$$
$$x > 5, \quad EIv = -121944x + 12500x^2 - 322x^3 + c.$$

Determine c from condition, that, when $x = 15$, $v = 0$;

$$\therefore \quad c = 104167;$$
$$\therefore \quad x > 5, \quad EIv = 104167 - 121944x + 12500x^2 - 322x^3.$$

For maximum deflection, equate slope to zero, and find x.

We find it at $x = 6.53$.

$$\therefore \quad EIv_0 = -249531.$$

Span BC, origin at B,

$$M = -4000 + 67x,$$

$$EI \tan \alpha_1 = 35557 - 4000x + 33x^2,$$

$$EIv = 35557x - 2000x^2 + 11x^3.$$

For maximum deflection, equate slope to zero, and find x.

We find it at $x = 9.78$.

$$\therefore \quad EIv_0 = 116740.$$

Span CD, origin at C,

$x < 10,\ M = -2667 + 1511x;$

$x > 10,\ M = -2667 + 1511x - 4000(x - 10) = 37333 - 2489x;$

$x < 10,\ EI \tan \alpha_1 = -31111 - 2667x + 756x^2;$

$x > 10,\ EI \tan \alpha_1 = -231111 + 37333x - 1245x^2.$

For deflections,

$x < 10,\ EIv = -31111x - 1334x^2 + 252x^3;$

$x > 10,\ EIv = -231111x + 18667x^2 - 415x^3 + c.$

When $x = 15,\ v = 0;$

$$\therefore \quad x > 10, \quad EIv = -37132785 - 231111x + 18667x^2 - 415x^3.$$

For maximum deflection, equate slope to zero, and find x.

We find it at $x = 8.41$.

$$\therefore \quad EIv_0 = -24506.$$

Hence greatest bending-moment and greatest deflection are both in span AB.

Observe, that, since we have used one foot as our unit of measure, all dimensions must be taken in feet, and the value of E is also 144 times that ordinarily given.

§ 247. **Continuous Girder, with both Distributed and Concentrated Loads.** — In this case we may either calculate the bending-moments, slopes, and deflections due to each separately, and then add the results with their proper signs, or we may modify the solution that was used for the case of a distributed load, so as to extend its applicability to this case.

Let W represent any one concentrated load, and let x_1 represent the distance of its point of application from the origin. Then, in the general formulæ deduced for the distributed load, make the following changes; viz., —

1°. Instead of

$$m = \int_0^x \int_0^x w dx^2,$$

put

$$m = \int_0^x \int_0^x w dx^2 + \Sigma_0^x W(x - x_1),$$

since, as was shown, m represents the sum of the moments of the loads, between the section and the support, about the section.

2°. Instead of

$$V = \int_0^x \int_0^x \left\{ \int_0^x \int_0^x w dx^2 \right\} \frac{dx^2}{EI},$$

put

$$V = \int_0^x \int_0^x \left\{ \int_0^x \int_0^x w dx^2 \right\} \frac{dx^2}{EI} + \Sigma W \int_{x_1}^x \int_{x_1}^x \frac{(x - x_1) dx^2}{EI},$$

and make the corresponding changes in the values of m_1, m_{-1}, V_1, and V_{-1}, leaving n and q just as before; then use the same three-moment equation as before, with these substitutions, i.e.,

$$M_2 \left\{ \frac{q_1}{l_1^2} - \frac{n_1}{l_1} + \frac{q_{-1}}{l_{-1}^2} - \frac{n_{-1}}{l_{-1}} \right\} - M_3 \frac{q_1}{l_1^2} - M_1 \frac{q_{-1}}{l_{-1}^2} - \frac{m_1 q_1}{l_1^2} - \frac{m_{-1} q_{-1}}{l_{-1}^2}$$
$$+ \frac{V_1}{l_1} + \frac{V_{-1}}{l_{-1}} + \frac{y_a - y_0}{l_1} + \frac{y_b - y_0}{l_{-1}} = 0.$$

SPECIAL CASE.

when the distributed load is uniformly distributed on each span, but may be different on the different spans, and when the girder is of uniform section.

Let w_1 = weight per unit of length on OA.

w_{-1} = weight per unit of length on OB.

Denote by W_1 any concentrated load on OA at distance x_1 from O.

Denote by W_{-1} any concentrated load on OB at distance x_{-1} from O.

Then we shall have

$$m_1 = \frac{w_1 l_1^2}{2} + \Sigma W_1(l_1 - x_1),$$

$$m_{-1} = \frac{w_{-1} l_{-1}^2}{2} + \Sigma W_{-1}(l_{-1} - x_{-1}),$$

$$V_1 = \frac{w_1 l_1^4}{24EI} + \Sigma \frac{W_1(l_1 - x_1)^3}{6EI},$$

$$V_{-1} = \frac{w_{-1} l_{-1}^4}{24EI} + \Sigma \frac{W_{-1}(l_{-1} - x_{-1})^3}{6EI},$$

and, as before,

$$n_1 = \frac{l_1^2}{2EI}, \qquad n_{-1} = \frac{l_{-1}^2}{2EI},$$

$$q_1 = \frac{l_1^3}{6EI}, \qquad q_{-1} = \frac{l_{-1}^3}{6EI}.$$

Making these substitutions in the three-moment equation, and clearing fractions, we obtain for the case, when the supports are all on the same level,

$$0 = M_1 l_{-1} + 2M_2(l_1 + l_{-1}) + M_3 l_1 + \tfrac{1}{4}(w_1 l_1^3 + w_{-1} l_{-1}^3)$$
$$+ \frac{1}{l_1}\Sigma\{W_1[l_1^2 - (l_1 - x_1)^2](l_1 - x_1)\}$$
$$+ \frac{1}{l_{-1}}\Sigma\{W_{-1}[l_{-1}^2 - (l_{-1} - x_{-1})^2](l_{-1} - x_{-1})\}.$$

CONCENTRATED AND DISTRIBUTED LOADS.

EXAMPLE. — Let the girder be of uniform section, of two equal spans, each being 10 feet; let the concentrated loads be as shown in the figure, the distributed load being 96 lbs. per foot. Find the value of EI, so that the deflection may nowhere exceed $\frac{1}{400}$ of the span.

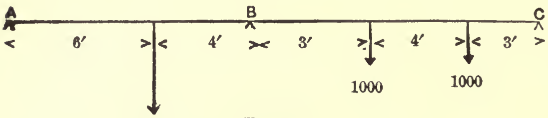


FIG. 255.

Solution. — Use equation (12); and, in deducing value of M_B, use dimensions in feet; afterwards use inches.

Origin at B, $M_A = M_C = 0$;

$$40M_B + \tfrac{96}{4}(1000 + 1000) + \tfrac{1}{10}\{2000(64)(6) + 1000(51)(7) + 1000(91)(3)\} = 0,$$

$$40M_B + 48000 + 139800 = 0, \quad \text{or} \quad M_B = -4695 \text{ foot-lbs.},$$

or

$$M_B = -56340 \text{ inch-lbs.}, \quad M_A = M_C = 0.$$

$$m_1 = 177600, \qquad m_{-1} = 201600,$$

$$n_1 = \frac{7200}{EI}, \quad \frac{n_1}{l_1} = \frac{60}{EI}. \qquad n_{-1} = \frac{7200}{EI}, \quad \frac{n_{-1}}{l_{-1}} = \frac{60}{EI}.$$

$$q_1 = \frac{288000}{EI}, \quad \frac{q_1}{l_1^2} = \frac{20}{EI}. \qquad q_{-1} = \frac{288000}{EI}, \quad \frac{q_{-1}}{l_{-1}^2} = \frac{20}{EI}.$$

$$V_1 = \frac{175680000}{EI}, \quad \frac{V_1}{l_1} = \frac{1464000}{EI}. \qquad V_{-1} = \frac{193536000}{EI}, \quad \frac{V_{-1}}{l_{-1}} = \frac{1612800}{EI}.$$

$$\text{Shear right side of middle} = \frac{0 + 56340 + 177600}{120} = 1949.5,$$

$$\text{Shear left side of middle} = \frac{0 + 56340 + 201600}{120} = 2149.5;$$

$$\text{Shear left end} = \frac{-56340 + 153600}{120} = 810.5,$$

$$\text{Shear right end} = \frac{-56340 + 177600}{120} = 1010.5;$$

$$\text{Middle supporting-force} = 4099.$$

Bending-Moments in Each Span.

Span AB, origin at A,

$$810.5x - 4x^2$$

or

$$810.5x - 4x^2 - 2000(x - 72).$$

Span BC, origin at B,

$$-56340 + 1949.5x - 4x^2,$$
$$-56340 + 1949.5x - 4x^2 - 1000(x - 36),$$
$$-56340 + 1949.5x - 4x^2 - 1000(x - 36) - 1000(x - 84).$$

To ascertain position of the greatest bending-moment, differentiate each one.

$$\begin{aligned}
&810.5 - 8x &&= 0, \quad x = 101.31;\\
&810.5 - 8x - 2000 &&= 0, \quad x = \text{a minus quantity};\\
&1949.5 - 8x &&= 0, \quad x = 243.69;\\
&1949.5 - 8x - 1000 &&= 0, \quad x = 118.69;\\
&1949.5 - 8x - 1000 - 1000 &&= 0, \quad x = \text{a minus quantity}.
\end{aligned}$$

Hence, in span AB, maximum bending is at the load, and its amount is

$$(810.5)(72) - 4(72)(72) = 37620.$$

Span BC, maximum is at right-hand load, and is

$$-56340 + 1949.5(84) - 4(84)(84) - 1000(48) = 31194.$$

SLOPES.

Slope at B,

$$T = \frac{-56340}{EI}(20 - 60) - \frac{(177600)(20)}{EI} + \frac{1464000}{EI} = \frac{165600}{EI}.$$

Slope and Deflection in Span AB.

First part,

$$\tan\alpha = \frac{1}{EI}\left\{405.25x^2 - \frac{4}{3}x^3\right\} + \tan\alpha_0,$$

α_0 being slope at A.

Second part,

$$\tan\alpha = \frac{1}{EI}\left\{405.25x^2 - \frac{4}{3}x^3 - 1000x^2 + 144000x\right\} + c.$$

When $x = 72$, α is the same in both cases;

$$\therefore \quad \frac{1}{EI}\{1000(72)^2 - (144000)(72)\} + \tan\alpha_0 - c = 0$$

$$\therefore \quad c = \tan\alpha_0 - \frac{5184000}{EI}.$$

When $x = 120$, the second value of $\tan\alpha$ becomes $\dfrac{165600}{EI}$;

$$\therefore \quad \frac{1}{EI}\left\{(405.25)(120)^2 - \frac{4}{3}(120)^3 - (1000)(120)^2 + 144000(120)\right\}$$
$$+ \tan\alpha_0 - \frac{5184000}{EI} = \frac{165600}{EI},$$

$$\tan\alpha_0 = \frac{1}{EI}\{-6411600 + 5184000 + 165600\} = -\frac{1062000}{EI},$$

$$\therefore \quad c = -\frac{6246000}{EI}.$$

Hence slope in first part (between A and the load),

$$\tan\alpha = \frac{1}{EI}\left\{-1062000 + 405.25x^2 - \frac{4}{3}x^3\right\}.$$

Second part (between B and the load),

$$\tan\alpha = \frac{1}{EI}\left\{-6246000 + 144000x - 594.75x^2 - \frac{4}{3}x^3\right\}.$$

Deflection.

First part,

$$v = \frac{1}{EI}\left\{-1062000x + 135.08x^3 - \frac{x^4}{3}\right\}.$$

Second part,

$$v = \frac{1}{EI}\left\{-6246000x + 72000x^2 - 198.25x^3 - \frac{1}{3}x^4\right\} + c.$$

When $x = 120$, $v = 0$;

$\therefore \quad c =$

$$\frac{1}{EI}\left\{(6246000)(120)-(72000)(120)^2+(198.25)(120)^3+\frac{(120)^4}{3}\right\}$$

$$= \frac{120}{EI}(1036800) = \frac{124416000}{EI}.$$

Point of greatest deflection is found by putting slope equal zero. Moreover, it is plain that the greatest deflection is in the first, and not the second, part.

Hence equation is

$$\tfrac{4}{3}x^3 - 405.25x^2 + 1062000 = 0$$

$$\therefore \quad x = 56''.77;$$

and, substituting this in the expression for the deflection, we obtain

$$v_0 = -\frac{39037720}{EI}.$$

Hence, putting $\dfrac{120}{400} = \dfrac{39037720}{EI}$, we obtain

$$EI = 130125733.$$

If $E = 1400000$, $I = 92.9$; therefore, if $b = 3$ inches, $h = 7$ inches.

Slope and Deflection in Span BC.

Portion nearest B,

$$\tan \alpha = \frac{1}{EI}\left\{165600 - 56340x + 974.8x^2 - \frac{4}{3}x^3\right\}.$$

When $x = 36$ inches, we obtain

$$\tan \alpha = \frac{1}{EI}(165600 - 2028240 + 1263341 - 62208) = -\frac{661507}{EI}.$$

Middle portion,

$$\tan \alpha = \frac{1}{EI}\left\{-20340x + 474.75x^2 - \frac{4}{3}x^3\right\} + c.$$

When $x = 36$ inches, then $\tan\alpha = \dfrac{661507}{EI}$;

$$\therefore\quad -661507 = -732240 + 615276 - 62208 + EIc$$

$$\therefore\quad c = -\frac{482335}{EI}$$

$$\therefore\quad \tan\alpha = \frac{1}{EI}\left\{-482335 - 20340x + 474.75x^2 - \frac{4}{3}x^3\right\}.$$

When $x = 84$ inches,

$$\tan\alpha = \frac{1}{EI}(-482335 - 1708560 + 3349836 - 790272)$$

$$= +\frac{368669}{EI}$$

Portion nearest C,

$$i = \frac{1}{EI}\left\{63660x - 25x^2 - \frac{4}{3}x^3\right\} + c.$$

When $x = 84$ inches, then $\tan\alpha = \dfrac{368669}{EI}$;

$$\therefore\quad 368669 = 5347440 - 176400 - 790272 + EIc$$

$$\therefore\quad c \qquad = -\frac{4012099}{EI}$$

$$\therefore\quad \tan\alpha = \frac{1}{EI}\left\{-4012099 + 63660x - 25x^2 - \frac{4}{3}x^3\right\}.$$

When $x = 120$ inches,

$$\tan\alpha = \frac{1}{EI}(-4012099 + 7639200 - 360000 - 2304000) = \frac{963101}{EI}.$$

Deflection.

Portion nearest B,

$$v = \frac{1}{EI}\left\{165600x - 28170x^2 + 324.9x^3 - \frac{1}{3}x^4\right\}.$$

When $x = 36$ inches,

$$v = \frac{1}{EI}(165600 - 1014120 + 421070 - 15552)(36)$$

$$= -\frac{(443002)36}{EI} = -\frac{15948072}{EI}.$$

Middle portion,

$$v = \frac{1}{EI}\left\{-482335x - 10170x^2 + 158.25x^3 - \frac{1}{3}x^4\right\} + c.$$

When $x = 36$ inches, then $v = -\dfrac{15948072}{EI}$;

$$\therefore \quad -15948072 = (-482335 - 366120 + 205092 - 15552) + EIc,$$

$$\therefore \quad c = -\frac{15289157}{EI}.$$

$$\therefore v = \frac{1}{EI}\left\{-15289157 - 482335x - 10170x^2 + 158.25x^3 - \frac{1}{3}x^4\right\}.$$

Greatest deflection occurs in the middle portion, and the point is given by the equation.

$$0 = -482335 - 20340x + 474.75x^2 - \tfrac{4}{3}x^3 = 0;$$

$$\therefore \quad x = 71.4.$$

Greatest deflection in span BC,

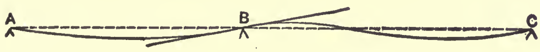


FIG. 256.

$v =$

$$\frac{1}{EI}(-15289157-34438719-51846253+57602105-8662899)$$

$$= -\frac{52634923}{EI}.$$

Hence, putting $\dfrac{120}{400} = \dfrac{52634923}{EI}$, we obtain

$$EI = 175449743;$$

therefore, if $E = 1400000$, we have

$$I = 125.3.$$

If $b = 3$ inches, $h = 8$ inches.

EXAMPLES OF CONTINUOUS GIRDERS.

1°. Let I = uniform moment of inertia of girder.
w = load per unit of length uniformly distributed.

Find expressions for

1, the bending-moment over each support,

2, the supporting-forces,

3, the greatest bending-moment,

4, the slopes at the supports,

5, the greatest deflection,

in each of the following cases: —

(*a*) Two equal spans, length l.

(*b*) Three equal spans, length l.

(*c*) Four equal spans, length l.

(*d*) Two spans, lengths l_1 and l_2 respectively.

(*e*) Three spans, lengths l_1, l_2, and l_3 respectively.

(*f*) Four spans, lengths l_1, l_2, l_3, and l_4 respectively.

(*g*) Two equal spans; loads per unit of length on each span, w_1 and w_2 respectively.

(*h*) Three equal spans; loads per unit of length on each span, w_1, w_2, and w_3 respectively.

2°. Do the same in the case where each span is loaded with a centre load W, and has no distributed load.

3°. Find greatest bending-moment and greatest deflection for a continuous girder of two spans, uniformly loaded on these two spans with load w per unit of length, and which overhang the outer supports; the overhanging parts having lengths l_{-0} and l_0 respectively, and the same distributed load per unit of length on the overhanging parts.

CHAPTER IX.

EQUILIBRIUM CURVES.—ARCHES AND DOMES.

§ 248. **Loaded Chain or Cord.**—It has been already shown (§ 126), when the form of a polygonal frame is given, that the loads must be adapted, in direction and magnitude, to that form, or else the frame will not be stable. The same is true of a loaded chain or cord, which would be realized if the frame were inverted.

If a set of loads be applied which are not consistent with the equilibrium of the frame under that form, it will change its shape until it assumes a form which is in equilibrium under the applied loads.

As to the manner of finding (when a sufficient number of conditions are given) the stresses in the different members, etc., this was sufficiently explained under the head of "Frames," and will not be repeated here, as the figures speak for themselves.

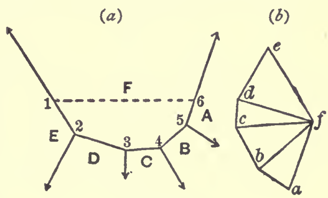


Fig. 257.

In Fig. 257 the polygon *fedcbaf* is the force polygon, while the equilibrium polygon is 123456, an open polygon. A straight line joining *e* and *a* would represent the resultant of the loads.

CHAIN WITH VERTICAL LOADS.

If all the loads are vertical, the broken line *edcba* becomes a straight line and vertical, as shown in Fig. 258*b*. Whenever the loads are concentrated at single points, as 2, 3, 4, 5, the form of the chain is polygonal; and when the load is distributed, it becomes a curve, as shown in Fig. 259.

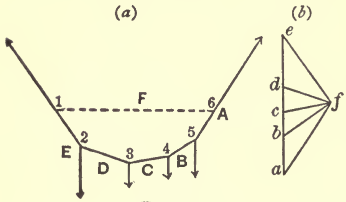


FIG. 258.

CURVED CHAIN WITH A VERTICAL DISTRIBUTED LOAD.

Given the form of the chain *AOE* supported at *A* and *E*, and the total load upon it (*bc*, Fig. 259*b*), to find the distribution of the load graphically. First lay off *bc* to scale, to represent the total load: this is balanced by the two supporting-forces at *A* and *E* respectively, as shown in the figure. Hence draw *ca* parallel to the tangent at *E*, and *ba* parallel to that at *A*, and we have the force polygon *abca;* the equilibrium curve being the chain *AOE* itself. Moreover, if the lowest point of the chain be *O*, then the load must be so distributed that the portion between *O* and *A* shall be balanced by the tension at *O* and that at *A*, and hence that its resultant shall pass through the intersection of the tangents at *O* and *A*. Its amount will be found by drawing from *a* a horizontal line; and then we shall have *ao* as the tension at *o*, *ab* as the tension at *A*, and *bo* as the load between *A* and *O*. Hence the load between *E* and *O* will be *oc*.

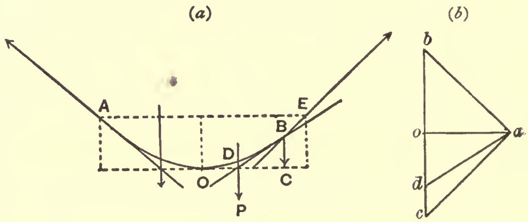


FIG. 259.

Moreover, the load between O and any point, as B, will be balanced by the tension at O, and the tension at B, and hence will be od, where ad is drawn parallel to the tangent BD, so that the load between B and E will be dc; and in this way we see that we can find the tension at any point of the chain by simply drawing a line from a, parallel to the tangent at that point, till it meets the load-line bc.

It is to be observed, that, if the tension at any point of the chain be resolved into horizontal and vertical components, the horizontal component will, when the loads are all vertical, be a constant, and the vertical component will be equal to the portion of the load between the lowest point and the point in question.

If we assume our origin at O, axis of x horizontal and axis of y vertical, and let the co-ordinates of B be x and y, and if w be the intensity of the load at the point (x, y), we shall have, for the load od between O and B,

$$P = \int_0^x w dx;$$

and, since the angle oad = angle BDC, we shall have

$$\frac{dy}{dx} = \frac{BC}{DC} = \frac{od}{oa} = \frac{P}{H}.$$

By differentiation, we shall have

$$\frac{d^2y}{dx^2} = \frac{\left(\frac{dP}{dx}\right)}{H}$$

or

$$\frac{d^2y}{dx^2} = \frac{w}{H}, \qquad (1)$$

and this is the equation for all vertically loaded cords.

From it we can find the form of the cord to suit a given distribution of the load.

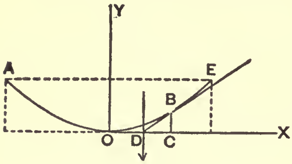


FIG. 260.

§ 249. **Chain with the Load Uniformly Distributed Horizontally.** — In this case w is a constant; and if we assume our origin at the lowest point of the chain, and use the same notation as before, we shall have

$$\frac{d^2y}{dx^2} = \frac{w}{H}.$$

Hence, integrating, and observing, that, when $x = 0, \dfrac{dy}{dx} = 0$, we have

$$\frac{dy}{dx} = \frac{wx}{H};$$

and by another integration, observing, that, when $x = 0, y = 0$, we obtain

$$y = \frac{w}{2H}x^2.$$

This is the equation of a parabola; hence a chain so loaded assumes a parabolic form.

EXAMPLE I. — Given the heights of the piers for supporting a chain so loaded, above the lowest point of the chain, as 8 and 18 feet respectively, the span being 100 feet, to find the distance of the lowest point from the foot of each pier, and the equation of the curve assumed by the chain.

Solution. — If (with the lowest point of the chain as origin) we call (x_1, y_1) the co-ordinates of the top of the first pier, and (x_2, y_2) those of the top of the second pier, we shall have, since $y_1 = 18$ and $y_2 = 8$, and since we must have

$$y = \frac{w}{2H}x^2,$$

$$18 = \frac{w}{2H}x_1^2 \qquad \text{and} \qquad 8 = \frac{w}{2H}x_2^2$$

$$\therefore \quad \frac{x_1}{x_2} = \sqrt{\frac{18}{8}} = \frac{3}{2} \qquad \therefore \quad x_1 = \frac{3}{2}x_2 \qquad \therefore \quad x_1 + x_2 = \frac{5}{2}x_2;$$

but

$$x_1 + x_2 = 100 \qquad \therefore \quad \tfrac{5}{2}x_2 = 100 \qquad \therefore \quad x_2 = 40, \qquad x_1 = 60.$$

Hence, since $18 = \frac{w}{2H}(60)^2$

$$\therefore \quad \frac{w}{2H} = \frac{18}{3600} = \frac{1}{200},$$

therefore equation of the curve is

$$y = \tfrac{1}{200}x^2.$$

EXAMPLE II. — Given the load on the above chain as 4000 lbs. per foot of horizontal length, to find the tension at the lowest point, also that at each end.

Solution.

$$\frac{w}{2H} = \frac{1}{200}, \qquad\qquad w = 4000,$$

$$\therefore \quad 2H = 800000 \qquad\qquad \therefore \quad H = 400000 \text{ lbs.}$$

Moreover, load between lowest point and highest pier = $60 \times 4000 = 240000$ lbs.

Therefore tension at highest pier =

$$\begin{aligned}\sqrt{(240000)^2 + (400000)^2} &= 10000\sqrt{(24)^2 + (40)^2} \\ &= 10000\sqrt{2176} = 466480 \text{ lbs.}\end{aligned}$$

Tension at lowest pier =

$$\begin{aligned}\sqrt{(160000)^2 + (400000)^2} &= 10000\sqrt{256 + 1600} \\ &= 10000\sqrt{1856} = 430813 \text{ lbs.}\end{aligned}$$

EXAMPLE III. — Given the span of the chain as 20 feet, and its length as 25 feet, the two points of support being on the same level, to find the position of the lowest point.

§ 250. **Catenary.** — The catenary is the form of the curve of a chain, which, being of uniform section, is loaded with its own weight only, i.e., with a load uniformly distributed along the length of the chain.

To deduce the equation of the catenary: if we assume the origin, as before, at the lowest point of the curve, we shall have still the general equation

$$\frac{d^2y}{dx^2} = \frac{w}{H};$$

but w in this case is not constant.

If we let w_1 = the load per unit of length of chain, we shall have

$$w = w_1\frac{ds}{dx} = w_1\sqrt{1 + \left(\frac{dy}{dx}\right)^2};$$

hence

$$\frac{d^2y}{dx^2} = \frac{w_1}{H}\frac{ds}{dx}.$$

Or, if we let

$$\frac{w_1}{H} = \frac{1}{m},$$

a constant,

$$\frac{d^2y}{dx^2} = \frac{1}{m}\frac{ds}{dx}, \qquad (1)$$

which is the differential equation of the catenary; and we only need to integrate it to obtain the equation itself.

To do this, we have

$$\frac{d^2y}{dx^2} = \frac{1}{m}\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \qquad \therefore \quad \frac{\dfrac{d^2y}{dx^2}}{\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}} = \frac{1}{m};$$

therefore, integrating, and observing, that, when $x = 0$, $\frac{dy}{dx} = 0$, we shall have

$$\log_e\left\{\frac{dy}{dx} + \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\right\} = \frac{x}{m}$$

$$\therefore \quad \frac{dy}{dx} + \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = e^{\frac{x}{m}} \tag{2}$$

$$\therefore \quad 1 + \left(\frac{dy}{dx}\right)^2 = e^{\frac{2x}{m}} - 2e^{\frac{x}{m}}\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2}\left(e^{\frac{x}{m}} - e^{-\frac{x}{m}}\right) \qquad \therefore \quad y = \frac{m}{2}\left(e^{\frac{x}{m}} + e^{-\frac{x}{m}}\right) + c.$$

But, when $x = 0$, $y = 0$ $\quad\therefore\quad c = -m$; hence the equation is

$$y = \frac{m}{2}\left(e^{\frac{x}{m}} + e^{-\frac{x}{m}}\right) - m, \tag{3}$$

and this is the equation of the catenary when the origin is taken at O, the lowest point of the chain.

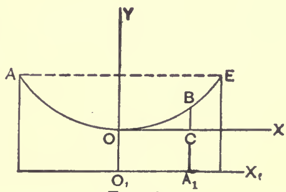


FIG. 261.

If it be transferred to O_1, where $OO_1 = m$, the equation becomes (by putting for y, $y - m$)

$$y = \frac{m}{2}\left(e^{\frac{x}{m}} + e^{-\frac{x}{m}}\right). \tag{4}$$

This is the most common form of the equation to the catenary, the origin being taken, at a distance below the lowest point of the curve equal to $m = \frac{H}{w_1}$, the horizontal tension divided by the weight per unit of length of chain.

To find x in terms of y, we have

$$e^{\frac{x}{m}} + \frac{1}{e^{\frac{x}{m}}} = \frac{2y}{m}$$

$$\therefore \quad e^{\frac{2x}{m}} + 1 = \frac{2y}{m} e^{\frac{x}{m}} \qquad \therefore \quad e^{\frac{2x}{m}} - \frac{2y}{m} e^{\frac{x}{m}} = -1.$$

Solving, we have

$$e^{\frac{x}{m}} = \frac{y}{m} \pm \sqrt{\frac{y^2}{m^2} - 1} \qquad \therefore \quad \frac{x}{m} = \log_e \left\{ \frac{y}{m} \pm \sqrt{\frac{y^2}{m^2} - 1} \right\}$$

$$\therefore \quad x = m \log_e \left\{ \frac{y}{m} \pm \sqrt{\frac{y^2}{m^2} - 1} \right\}. \qquad (5)$$

To find the length of the rope: from the equation

$$y = \frac{m}{2}\left(e^{\frac{x}{m}} + e^{-\frac{x}{m}}\right)$$

we obtain

$$\frac{dy}{dx} = \frac{1}{2}\left(e^{\frac{x}{m}} - e^{-\frac{x}{m}}\right)$$

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{1}{4}\left(e^{\frac{2x}{m}} - 2 + e^{-\frac{2x}{m}}\right)}$$

$$= \sqrt{\frac{1}{4}\left(e^{\frac{2x}{m}} + 2 + e^{-\frac{2x}{m}}\right)} = \frac{1}{2}\left(e^{\frac{x}{m}} + e^{-\frac{x}{m}}\right)$$

$$\therefore \quad \frac{ds}{dx} = \frac{1}{2}\left(e^{\frac{x}{m}} + e^{-\frac{x}{m}}\right) \qquad (6)$$

$$\therefore \quad s = \frac{1}{2}\int_0^x \left(e^{\frac{x}{m}} + e^{-\frac{x}{m}}\right)dx = \frac{m}{2}\left(e^{\frac{x}{m}} - e^{-\frac{x}{m}}\right). \qquad (7)$$

To find the area OO_1A_1B, we have

$$\text{Area} = \int_0^x y\,dx = \frac{m}{2}\int_0^x \left(e^{\frac{x}{m}} + e^{-\frac{x}{m}}\right)dx = \frac{m^2}{2}\left(e^{\frac{x}{m}} - e^{-\frac{x}{m}}\right). \qquad (8)$$

But

$$\text{arc } OB = \frac{m}{2}\left(e^{\frac{x}{m}} - e^{-\frac{x}{m}}\right) = s;$$

hence, area $OO_1A_1B = ms.$

This shows, that, if the load should be distributed in such a way as to be like a uniformly thick sheet of metal, having for one side the catenary and for the other the straight line O_1A_1, the equilibrium curve would be a catenary.

It may be convenient to have the development of $e^{\frac{x}{m}}$ and $e^{-\frac{x}{m}}$; hence they will be written here:—

$$e^{\frac{x}{m}} = 1 + \frac{x}{m} + \frac{x^2}{m^2\underline{|2}} + \frac{x^3}{m^3\underline{|3}} + \frac{x^4}{m^4\underline{|4}} + \text{etc.},$$

$$e^{-\frac{x}{m}} = 1 - \frac{x}{m} + \frac{x^2}{m^2\underline{|2}} - \frac{x^3}{m^3\underline{|3}} + \frac{x^4}{m^4\underline{|4}} + \text{etc.}$$

EXAMPLE I.—Given a rope 90 feet long, spanning a horizontal distance of 75 feet; find the equation of the catenary, the sag of the rope, and the inclination of the rope at each support, supposing these to be on the same level.

§ 251. **Transformed Catenary.**—We have just seen that the catenary is the form of chain suited to a load which may be represented by a uniformly thick sheet of metal, with a horizontal extrados, provided the distance OO_1 is equal to m, a definite quantity. A more general case, however, would be that of a chain loaded with a load which might be represented by a uniformly thick sheet of metal, where the length OO_1 is any given quantity whatever. A chain so loaded is called a *transformed catenary*, and the catenary itself becomes a particular case of the transformed catenary.

We may deduce its equation as follows:—

Let the chain be represented by ACB, and let it be so loaded that the load on CD is represented by w times area $OCDE$, so that w = weight per unit of area; then we shall have, for this load,

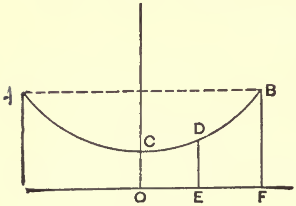


FIG. 262.

$$P = w\int_0^x ydx$$

Hence, from what we have already seen,

$$\frac{dy}{dx} = \frac{P}{H} = \frac{w}{H}\int_0^x ydx$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{w}{H}y \qquad \therefore \quad \frac{dy}{dx}\frac{d^2y}{dx^2}dx = \frac{w}{H}y\frac{dy}{dx}dx.$$

nce, integrating, we have

$$\left(\frac{dy}{dx}\right)^2 = \frac{w}{H}y^2 + c.$$

But, when $\frac{dy}{dx} = 0$, $y = a$;

$$\therefore \qquad c = -\frac{w}{H}a^2$$

$$\therefore \quad \left(\frac{dy}{dx}\right)^2 = \frac{w}{H}(y^2 - a^2).$$

Or, if we write, for brevity, $\frac{H}{w} = m^2$, we have

$$\left(\frac{dy}{dx}\right)^2 = \frac{y^2 - a^2}{m^2} \qquad \therefore \quad \frac{dy}{dx} = \frac{1}{m}\sqrt{y^2 - a^2} \qquad \therefore \quad \frac{dy}{\sqrt{y^2 - a^2}} = \frac{dx}{m}$$

$$\therefore \quad \log(y + \sqrt{y^2 - a^2}) = \frac{x}{m} + c.$$

But, when $x = 0, y = a$;

$$\therefore \quad \log(a) = c \qquad \therefore \quad \log\left\{\frac{(y + \sqrt{y^2 - a^2})}{a}\right\} = \frac{x}{m}$$

$$\therefore \quad \frac{y + \sqrt{y^2 - a^2}}{a} = e^{\frac{x}{m}} \qquad \therefore \quad y^2 - a^2 = a^2 e^{\frac{2x}{m}} - 2aye^{\frac{x}{m}} + y^2$$

$$\therefore \quad \frac{2y}{a} = e^{\frac{x}{m}} + e^{-\frac{x}{m}} \qquad \therefore \quad y = \frac{a}{2}(e^{\frac{x}{m}} + e^{-\frac{x}{m}}),$$

which is the equation of the transformed catenary. This becomes the catenary itself whenever $a = m$.

EXAMPLE. — Given a chain loaded so that the load on CD is proportional to the area $OEDC$. Let $OC = 5$ feet, $BF = 8$ feet, $OF = 4$ feet; weight per unit of area $= 80$ lbs. Find the equation of the transformed catenary, also the tension at C and that at B.

§ 252. **Linear Arch.** — In all the preceding cases, the chain or cord is called upon to resist a tensile stress arising from a load that is hung upon it. If, now, the cord be inverted, we have the proper equilibrium curve for a load placed upon it, distributed in the same manner as before; only in this latter case the cord would be subjected to direct compression throughout its whole extent. The equilibrium curve is, then, sometimes called a *linear arch.* The general equation of the equilibrium curve remains just as before,

$$\frac{d^2y}{dx^2} = \frac{w}{H},$$

the axes being so chosen that OX is horizontal and OY vertical.

Thus, if it were required to find the form of the equilibrium curve or linear arch, with the upper boundary of the loading horizontal, we should obtain a *transformed catenary.*

§ 253. **Arches.** — In the case of arches composed of a series of blocks, as in stone or brick arches, the mathematical treatment generally used for determining the proper form and proportions of the arch has been quite different from that used for the determination of the proper form and proportions of the iron arch, whether made in one piece, or two pieces hinged together, or of a lattice.

In the case of the iron arch, the treatment involves necessarily a determination of the stresses acting in all its parts, and an adaptation of its form and dimensions to the load, so that at no point shall the stress exceed the working-strength of the material.

In the case of the stone arch, it is still a question under discussion whether it would not be best to adopt the same method, although it would lead to a great deal of complexity, on account of the joints.

Nevertheless, the question usually raised is one merely of stability; i.e., as to the proper form and dimensions to prevent, not the crushing of the stone, though this must also be taken into account if there is any danger of exceeding it, but more especially the overturning about some of the joints.

The question of the stability of the stone arch may present itself in either of the two following ways: —

1°. Given the arch and its load, to determine whether it is stable or not.

2°. Given the distribution of the load, to determine the suitable equilibrium curve, and hence the form of arch, suited to bear the given load with the greatest economy of material.

§ 254. **Modes of giving Way of Stone Arches.** — An arch may yield, (1°) by the crushing of the stone, (2°) by sliding of the joints, (3°) by overturning around a joint. The following figures show the modes of giving way of an arch by the last two methods. The first two show the dislocation of the arch by the slipping of the voussoirs. In the former case the haunches of the arch slide out, and the crown slips down; in the other case the reverse happens. The second two figures show the two methods by which an arch may give way by rotation of the voussoirs around the joints.

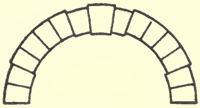


FIG. 263.

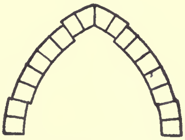


FIG. 264.

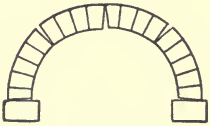


FIG. 265.

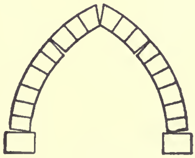


FIG. 266.

Before proceeding farther with the problem of the arch, two or three matters of a more general nature will be treated, which will be necessary in its discussion.

§ 255. **Friction.**—Let AB be a plane inclined to the horizon at an angle θ. Let D be a body resting on the plane, of weight $DG = W$. Resolve W into two components, DE and DF respectively, perpendicular and parallel to the plane. The component $DE = W \cos \theta$ is entirely neutralized by the re-action of the plane; while $DF = W \sin \theta$, on the other hand, is the only force tending to make the body slide down the plane. It is an experimental fact, that when the angle θ is less than a certain angle ϕ, called the angle of repose, the body does not slide; when $\theta = \phi$, the body is just on the point of sliding; and when θ is greater than ϕ, the body slides down the plane with an accelerated motion, showing that in this case an unbalanced force is acting. This

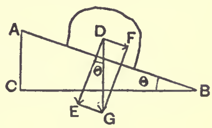


FIG. 267.

angle ϕ depends upon the nature of the material of the plane and of the body, and on the nature of the surfaces. Hence, in the first and second cases, the friction actually developed by the normal pressure DE just balances the tangential component DF; whereas, in the third case, when the angle of inclination of the plane to the horizon is greater than ϕ, the tangential component DF is only partially balanced by the friction.

Let ab be the plane when inclined to the horizon at an angle ϕ. The body is then just on the point of sliding, hence the component $df = W \sin \phi$ is just equal to the friction developed between the two surfaces. Moreover, if we represent by N the normal pressure $de = W \cos \phi$ on the plane, we shall have

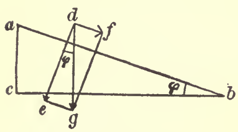


FIG. 268.

$$df = N \tan \phi.$$

Now, it is an experimental fact, that the friction developed between two given surfaces depends only on the normal pressure, i.e., that the friction bears a constant ratio to the normal pressure; and since, in this case, the friction just balances the tangential component $df = N \tan \phi$, the friction due to the normal pressure N is

$$N \tan \phi.$$

Now, it makes no difference what be the position of the plane surface: if a normal pressure N be exerted, the friction that is capable of being exerted to resist any force F tangential to the plane, tending to make the bodies slide upon each other, is $N \tan \phi$; and if the force F is greater than $N \tan \phi$, the bodies will slide, but if F is less than $N \tan \phi$, they will not slide. The quantity $\tan \phi$ is called **the co-efficient of friction,** and will be denoted by f.

From the preceding it is evident, that, if the resultant pressure on the body makes with the normal to the plane an angle less than the angle of repose, the sliding will not take place; whereas, if the resultant force makes with the normal to that plane an angle greater than the angle of repose, the body will slide.

§ 256. **Stability of Position.**—To determine under what conditions the stability of the block $DGHF$ is secure against turning around the edge D: if the resultant of the weight of the block and the pressure thereon pass outside the edge D, as OR_1, then the block will overturn; the moment of the couple tending to overturn it being $OR_1 \times DE$. If, on the other hand, it pass within the edge, as OR_2, the block will not overturn, since the force has a tendency to turn it the opposite way around D. Hence, in order that a block may not overturn around an edge at a plane joint, the resultant pressure must cut the joint within the joint itself.

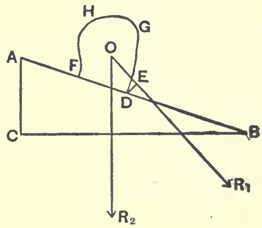


FIG. 269.

In any structure composed of blocks united at plane joints, we must have both stability of position and stability of friction at each joint, in order that the structure may not give way.

§ 257. **Stability of a Buttress about a Plane Joint.**—Let $DCEF$ be a vertical section of a buttress, against which a strut rib or piece of framework abuts, exerting a thrust $P = ZX = OR$. In order that the buttress may not give way, it must fulfil the conditions of stability at each joint. Let AB be a joint. Should several pressures act against the buttress, the force P in the line ZO may be taken to represent the resultant of all the thrusts which act on the buttress above the joint AB. Let G be the centre of gravity of the part $ABEF$, and let $W = OL$ be the weight of that part of the buttress. Let O be the point of intersection of the line of direction of the thrust, and of the weight W. Draw the parallelogram $ORNL$. Then will ON be the resultant pressure on the joint AB: and the conditions of stability require that the resultant pressure should cut the joint AB at some point between A and B, and that its line of direction should make with the normal to AB an angle less than the angle of repose, ϕ; and, in order that the buttress may not give way, these conditions must be fulfilled at each and every joint.

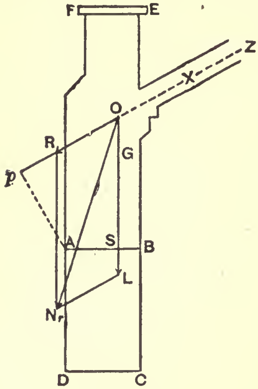


FIG. 270.

Another way of expressing this condition is as follows: The force tending to overturn the upper part of the buttress around A is the force $F = OR$; and its moment around A is $F(Ap) = Fp$ if we let $Ap = p$, whereas the moment of the weight which resists this is $W(AS) = Wq$ if we let $AS = q$. Now, when ON passes through A, we have $Fp = Wq$; when ON passes inside of A, we have $Wq > Fp$; when ON passes outside of A, we have $Wq < Fp$. Hence the conditions of stability require that

$$Wq \geqq Fp \qquad \text{or} \qquad Fp \leqq Wq.$$

EXAMPLE. — Given a rectangular buttress 8 feet high, 1 foot wide, and 4 feet thick; the weight of the material being 100 lbs. per cubic foot, the buttress being composed of 8 rectangular blocks $1 \times 4 \times 1$ foot. On this buttress is a load of 500 lbs., whose weight acts through K, where $OK = 3$ feet. Find the greatest horizontal pressure P that can be applied along the line OK, consistent with stability, against overturning around each of the edges a, b, c, d, e, f, g, h.

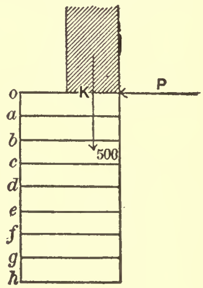


FIG. 271.

Solution. — The weight of each block will be 400 lbs. Hence we shall have the following equations: —

$$\text{Stability about } a, \text{ max } P = \frac{1500 + 400 \times 2}{1} = 2300.$$

$$\text{“ “ } b, \text{ “ } = \frac{1500 + 800 \times 2}{2} = 1550.$$

$$\text{“ “ } c, \text{ “ } = \frac{1500 + 1200 \times 2}{3} = 1300.$$

$$\text{“ “ } d, \text{ “ } = \frac{1500 + 1600 \times 2}{4} = 1175.$$

$$\text{“ “ } e, \text{ “ } = \frac{1500 + 2000 \times 2}{5} = 1100.$$

$$\text{“ “ } f, \text{ “ } = \frac{1500 + 2400 \times 2}{6} = 1050.$$

$$\text{“ “ } g, \text{ “ } = \frac{1500 + 2800 \times 2}{7} = 1014.$$

$$\text{“ “ } h, \text{ “ } = \frac{1500 + 3200 \times 2}{8} = 987.$$

The least of these being 987 lbs., it follows that the greatest pressure consistent with stability against overturning is 987 lbs.

§ 258. **Line of Resistance in a Stone Arch.** — In order to solve any problem involving the stability of a stone arch, it is necessary that the student should be able to draw a line of resistance. To make plain the meaning of the term, the following solution of an example is given. The method of drawing the line of resistance employed in this solution is given purely for purposes of illustration, and is not recommended for use in practice, as a suitable method will be given later.

EXAMPLE.—Given three blocks of stone of the form shown in the figure (Fig. 272), their common thickness (perpendicular to the plane of the paper) being such that the weight per square inch of area (in the plane of the paper) is just one pound.

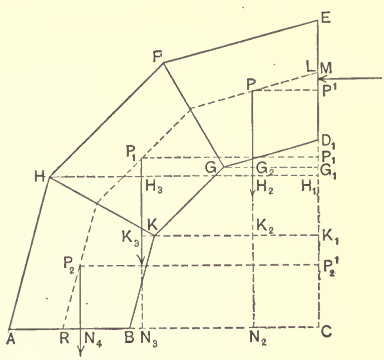


FIG. 272.

Given $AC = 13$ inches, $BC = 8$ inches. Suppose these three blocks to be kept from overturning by a horizontal force applied at the middle of DE. Find the least value of this horizontal force consistent with stability about the inner joints, also its greatest value consistent with stability about the outer joints.

Solution.

$$BK = 16 \sin 15^\circ = 4.14112.$$
$$AH = 26 \sin 15^\circ = 6.72932.$$
$$CR = \frac{2}{3}\left\{\frac{(13)^3 - (8)^3}{(13)^2 - (8)^2}\right\} = 10.7.$$

Altitude of each trapezoid $= 5 \cos 15^\circ \quad = 4.8296.$
Area of each trapezoid $= \frac{105}{2} \sin 30^\circ \quad = 26.25$ sq. in.
Weight of each stone $= 26.25$ lbs.

$$GG_2 = 8 \sin 30^\circ - 10.7 \cos 15^\circ \sin 15^\circ = 1.325.$$
$$KK_2 = 8 \cos 30^\circ - 10.7 \cos 15^\circ \sin 15^\circ = 4.253.$$
$$KK_3 = 10.7 \cos 15^\circ \cos 45^\circ - 8 \cos 30^\circ = 0.380.$$
$$BN_2 = 8 - 10.7 \cos 15^\circ \sin 15^\circ = 5.325.$$
$$BN_3 = 8 - 10.7 \cos 15^\circ \cos 45^\circ = 0.692.$$
$$BN_4 = 10.7 \cos^2 15^\circ - 8 = 1.983.$$
$$HH_2 = 13 \cos 30^\circ - 10.7 \cos 15^\circ \sin 15^\circ = 8.583.$$
$$HH_3 = 13 \cos 30^\circ - 10.7 \cos 15^\circ \cos 45^\circ = 3.950.$$

$$\begin{aligned}
AN_2 &= 13 - 10.7 \cos 15^\circ \sin 15^\circ &&= 10.325.\\
AN_3 &= 13 - 10.7 \cos 15^\circ \cos 45^\circ &&= 5.692.\\
AN_4 &= 13 - 10.7 \cos^2 15^\circ &&= 3.017.\\
G_1M &= 10.5 - 8 \cos 30^\circ &&= 3.572.\\
K_1M &= 10.5 - 8 \sin 30^\circ &&= 6.500.\\
CM &= 10.5 &&= 10.500.\\
H_1M &= 10.5 - 13 \sin 30^\circ &&= 4.000.
\end{aligned}$$

Let us represent the thrust at M by T. Then, to find what is the thrust required to produce equilibrium about G, we take moments about G, and likewise for the other joints. We may proceed as follows:—

INNER JOINTS.

Stability about G,

$$T(G_1M) = (26.25)(GG_2)$$

or

$$T(3.572) = (26.25)(1.325) \qquad \therefore \; T = 9.74.$$

Stability about K,

$$T(K_1M) = (26.25)(KK_2 - KK_3)$$

or

$$T(6.500) = (26.25)(4.253 - 0.380) \qquad \therefore \; T = 15.64.$$

Stability about B,

$$T(CM) = (26.25)(BN_2 + BN_3 - BN_4) \quad \therefore \; T = 10.08$$

OUTER JOINTS.

Stability about H,

$$T(H_1M) = (26.25)(HH_2 + HH_3) \qquad \therefore \; T = 82.25.$$

Stability about A,

$$T(CM) = (26.25)(AN_2 + AN_3 + AN_4) \quad \therefore \; T = 47.59$$

It is plain, therefore, that, in order to have equilibrium, the thrust at M must be between 15.64 lbs. and 47.59 lbs.: for, if it is less than 15.64 lbs., the arch will turn about an inner joint; and if it is greater than 47.59 lbs., it will turn around an outer joint.

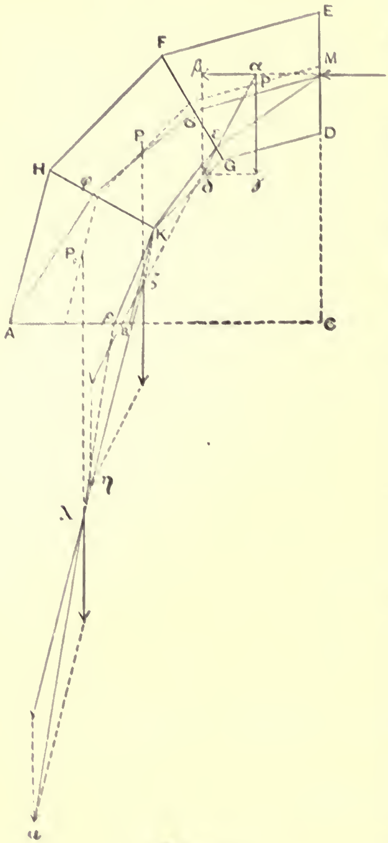


FIG. 273.

If, now, we draw through M a horizontal line to meet the vertical drawn through the centre of gravity of the first stone, and lay off $\alpha\beta = 15.64$, and $\alpha\gamma = 26.25$, then will the resultant of this thrust $\alpha\beta$ and the weight of the first stone $\alpha\gamma$ be $\alpha\delta$; this being the resultant pressure on the joint FG, its point of application being ϵ. Next, prolong this line $\alpha\delta$ to meet the vertical through the centre of gravity of the second stone, and combine $\alpha\delta$ with the weight of the second stone, thus obtaining, as resultant pressure on the joint KH, the force $\zeta\eta$, whose point of application is at K. Compounding, now, $\zeta\eta$ with the weight of the third stone, we obtain, as final resultant pressure on AB, the force $\lambda\mu$ applied at ρ. Now, joining $M\epsilon K\rho$ by a broken line, we have the *Line of Resistance* corresponding to the thrust 15.64, or the *minimum* horizontal thrust at M. If, now, we construct a line of resistance with 47.59 lbs., we obtain the line $M\sigma\phi A$, corresponding to maximum horizontal thrust at M.

If the arch is in equilibrium, and if the horizontal thrust is applied at M, it is plain that the actual thrust would either be one of these two or else somewhere between these two, and hence, that, if the requisite thrust is furnished at M to keep the arch in equilibrium, the true line of resistance cannot lie outside of these two; viz., the line corresponding to maximum and that corresponding to minimum horizontal thrust at M.

If the separate stones supported loads, it would be necessary to take into account these loads, in addition to the weights of the stones, in determining the horizontal thrust, and drawing the lines of resistance.

§ 259. **Arches with Symmetrical Distribution of the Load.** — Before considering the conditions of stability of an arch, we shall proceed to some propositions about lines of resistance corresponding to maximum and minimum horizontal thrust. If, in an arch, we draw a line of resistance AB through the point A of the crown, and then, by changing the horizontal thrust, we change the line of resistance continuously till it touches the extrados of the arch at C', we shall evidently have, in the line $AC'B'$, a line of resistance which has the greatest horizontal thrust of any line that passes through A, and lies wholly within the arch-ring. If, on the other hand, we decrease gradually the horizontal thrust until the line touches the intrados at D', then we have in this line the line of minimum horizontal thrust that passes through A. By lowering the point A, however, and keeping the point C the same, we should obtain new lines of resistance with greater and greater horizontal thrust; the greatest being attained when the line comes to have one point in common with the intrados. Hence a line of maximum horizontal thrust will have one point in common with the extrados and one point in common with the intrados, the latter being above the former.

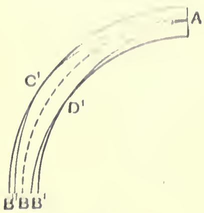


Fig. 274.

On the other hand, by retaining the point D' the same, and raising the point A, we should decrease the horizontal thrust, and thus obtain lines of resistance with less and less horizontal thrust; the least being attained when the line of resistance comes to have a point in common with the extrados. Hence the minimum line of resistance has a point in common with the extrados and one in common with the intrados, the latter being below the former.

These cases are exhibited in the following figures: —

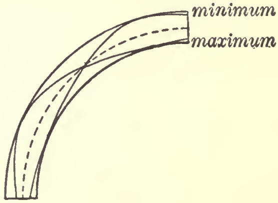


FIG. 275.

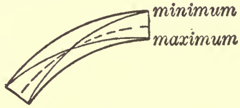


FIG. 276.

§ 260. **Conditions of Stability.** — The question of the stability of an arch must depend upon the position of its true line of resistance. If this true line of resistance lies within the arch-ring, the arch will be stable provided the material of which it is made is incompressible. If this is not the case, the stability of the arch will depend upon how near the true line of resistance approaches the edge of the joints; for the nearer it approaches the edge of a joint, the greater the intensity of the compressive stress at that joint, and the greater the danger that the crushing-strength of the stone will be exceeded at that joint. Thus, if the true line of resistance cuts any given joint at its centre of gravity, the stress upon that joint will be uniformly distributed over the joint. If, however, it cuts the joint to one side of its centre of gravity, the intensity of the stress will be greater on that side than on the opposite side; and, if it is carried far enough to one side, we may even have tension on the other side.

§ 261. **Criterion of Safety for an Arch.** — There are two criteria of safety for an arch, that have been used :—

1°. That the line of resistance should cut each joint within such limits that the crushing-strength of the stone should not be exceeded by the stress on any part of the joint.

2°. That, inasmuch as the joint is not suited to bear tension at any point, there should be no tension to resist.

The distribution of the stress is assumed to be uniformly varying from some line in the plane of the joint. The three following figures will, on this supposition, represent the three cases :—

1°. When the stress is wholly compression.

2°. When the stress becomes zero at the edge *B*.

3°. When the stress becomes negative or tensile at ***B***.

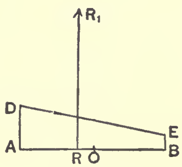


Fig. 277.

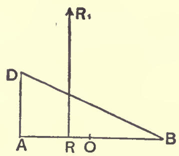


Fig. 278.

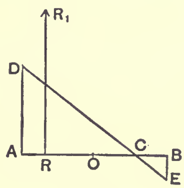


Fig. 279.

In all three figures, *AB* represents the joint which is assumed to be rectangular in section, *AD* represents the intensity of the stress at *A*, and *BE* that of the stress at *B;* while *R* represents the point of application of the resultant stress, RR_1 representing that resultant.

Proposition. — If the stress on a rectangular joint vary uniformly from a line parallel to one edge, the condition that there shall be no tension on any part requires that the resultant of the compressive stress shall be limited to the middle third of the joint.

Proof. — Let *AB* (Fig. 278) represent the projection of the joint on the plane of the paper. It is assumed that the stress is uniformly varying; and, if there is to be no tension anywhere, the intensity at one edge must not have a value less than zero, hence at the limiting case the value must be zero; hence this limiting case is correctly represented by the figure, and the resultant of the compression will be for this case at the centre of stress. Thus, if AD represent the greatest intensity of the stress, then we shall have, if B be the origin and BA the axis of x, if the axis of y be perpendicular to AB at B, and if we let a = intensity of stress at a unit's distance from B, that $RR_1 = a\int\int xdxdy$, and $(BR)(RR_1) = a\int\int x^2dxdy$;

$$\therefore \quad BR = \frac{\int\int x^2dxdy}{\int\int xdxdy} = \frac{\frac{bh^3}{3}}{\frac{bh^2}{2}} = \tfrac{2}{3}h,$$

if b = breadth, and $h = BA$ = height of rectangle.

Hence, if the resultant of the compression be nearer A than R, there will be tension at B; and, on the other hand, if it be nearer B than $\frac{1}{3}h$, there will be tension at A. Hence follows the proposition as already stated.

While the above is probably the condition most generally used to determine the stability of an arch, at the same time, if there is any danger that the intensity of the stress at any part of any joint may exceed the working compressive strength of the stone, this ought to be examined, and hence a formula by which it may be done will be deduced.

Let AB (Fig. 279) be the joint, and let, as before, b be its breadth, and $h = AB$ = depth; then, suppose the pressure to be uniformly varying, $DA = f$ = the working-strength per unit of area = greatest allowable intensity of compression; then the entire stress on the joint will be represented by the triangle ACD, for the joint is incapable of resisting tension.

Hence

$$AR = \tfrac{1}{3}AC \qquad \therefore \quad AC = 3AR;$$

but

$$P = \frac{fb(AC)}{2} = \frac{3}{2}fb(AR) \qquad \therefore \quad AR = \frac{2}{3}\frac{P}{fb},$$

and this is the least distance from the outer edge at which the resultant should cut the joint.

We thus obtain, in terms of the pressure on any joint, and of the working-strength of the material, the limits within which the line of resistance should pass, in order that the working-strength of the stone may not be exceeded.

§ 262. **Position of the True Line of Resistance.** — The question of the most probable position of the true line of resistance involves the discussion of the properties of the elastic arch. This discussion will be given later; but, for the present, the statement only of the following proposition, due to Dr. Winkler, will be given: —

"*For an arch of constant section, that line of resistance is approximately the true one which lies nearest to the axis of the arch-ring, as determined by the method of least squares.*"

From this it will follow: —

1°. That, if a line of resistance can be drawn in the arch-ring, then the true line of resistance will lie in the arch-ring; and

2°. That, if a line of resistance can be drawn within the middle third of the arch-ring, then the true line of resistance will lie in the middle third.

But, before proving this proposition, the proposition will be used, and the method explained, for determining whether a line of resistance can be drawn within the arch-ring: for, if it can, then the true line of resistance must lie within the arch-ring; and if no line of resistance can be drawn within the arch-ring, then the true line of resistance cannot pass within the arch-ring, and the arch would necessarily be unstable, even if the materials were incompressible.

By following the same method, we could determine whether it was possible to draw a line of resistance within the middle third of the arch-ring; and, if this is found to be possible, we should know that the true line of resistance will pass within the middle third of the arch-ring.

Hence our most usual criterion of the stability of a stone arch is, whether a line of resistance can be passed within the middle third of the arch-ring.

If the condition be used, that the working-strength of the stone for compression be not exceeded, then, instead of the middle third, we shall have some other limits.

In what follows, an explanation will be given of Dr. Scheffler's method (that most commonly employed) of determining whether a line of resistance can be drawn within the arch-ring, inasmuch as the same method can be employed to determine whether such a line can be drawn within the middle third or within any other given limits.

§ 263. **Preliminary Proposition referring to Arches Symmetrical in Form and Loading.** — *An arch and its load being given, a line of resistance can always be made to pass through any two given points; hence, if any two points of a line of resistance are given, the line is determined.*

Proof. — Let the arch be that shown in Fig. 281; and let us consider first the special case when the two given points are A, the top of the crown-joint, and G_4, the foot of the springing-joint. In this case, the only quantity to be determined is the thrust at A. Let this thrust be denoted by T; let P be the total weight of the half-arch and its load; let a be the perpendicular distance of the point G_4 from a vertical line through the centre of gravity of the entire half-arch and its load; let h be the vertical depth of G_4 below A. Then, taking moments about G_4, we must have

$$Th = Pa$$

$$\therefore\ T = \frac{Pa}{h}; \qquad (1)$$

and the line of resistance can then be drawn with this thrust, as has been done in the figure. Next take the general case, when the given points are not in these special positions. Let them be any two points, as A_2 and G_3.

In this case, the point of application of the thrust at the crown is not necessarily A, but may be some other point of the crown-joint: hence the quantities to be determined are two; viz., the thrust T at the crown, and the distance x of its point of application below A. Let the combined weight of the first two voussoirs and their load be P_1, and the horizontal distance of A_2 from a vertical line through the centre of gravity of P_1 be a_1.

Let P_2 be the combined weight of the first three voussoirs and their load, and let a_2 be the horizontal distance of G_3 from a vertical line through the centre of gravity of P_2.

Let the vertical depth of A_2 below A be h_1, and that of G_3 below A be h_2. Then, taking moments about A_2 and G_3 respectively, we shall have

$$T(h_1 - x) = P_1 a_1 \quad \text{and} \quad T(h_2 - x) = P_2 a_2,$$

two equations to determine the two unknown quantities T and x, which can easily be solved in any special case; and the resulting line of resistance can be drawn, which will pass through the two given points.

§ 264. **Scheffler's Method.**—In using Scheffler's method of determining whether it is possible to pass a line of resistance within a given portion of the arch-ring as the middle third or not, we should proceed as follows; viz.,—

First pass a line of resistance through 1, the top of the middle third of the crown-joint (Fig. 280), and e, the inside of the middle third of the springing-joint. If this line lies wholly within the middle third, it proves that a line of resistance can be drawn within the middle third.

If this line of resistance does not pass entirely within the middle third, proceed as follows: Suppose the line thus drawn to be 1*abcde*, passing without the middle third on both sides, as shown in the figure. Then from *a*, the point where it is farthest from the extrados of the middle third, draw a normal to this extrados, and find the point where this normal cuts this extrados: in this case, 2 is the point in question. In this way determine also the point 7, where the normal from *d* cuts the intrados of the middle third; then pass a new line of resistance through the points 2 and 7, determining the thrust and its point of application. If this new line of resistance lies within the middle third, then it is plain that it is possible to draw a line of resistance within the middle third; if not, it is not at all probable that it is possible to draw such a line.

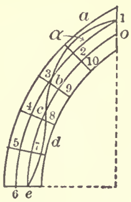


Fig. 280.

If the line of resistance drawn through 1 and *e* goes outside the middle third only beyond its extrados, as at *a*, we should draw our second line of resistance through 2 and *e*; if, on the other hand, it goes outside only below the intrados of the middle third, as at *d*, we should draw our second line through 1 and 7.

In the construction, we make use of a slice of the arch included between two vertical planes a unit of distance apart; and we take for our unit of weight the weight of one cubic unit of the material of the voussoirs, so that the number of units of area in any portion of the face of the arch shall represent the weight of that portion of the arch.

We next draw, above the arch, a line (DD_4 in Fig. 281), straight or curved, such that the area included between any portion of it, as D_1D_2, the two verticals at the ends of that portion, and the extrados of the arch-ring, shall represent by its area the load upon the portion of the arch immediately below it. This line will limit the load itself whenever this is of the same material as the voussoirs; otherwise it will not. We shall always call it, however, the extrados of the load.

The mode of procedure will best be made plain by the solution of examples; and two will be taken, in the first of which only one trial is necessary to construct a line of resistance that shall lie wholly within the arch-ring, and, in the second, two trials are necessary.

EXAMPLE. — The half-arch under consideration is shown in Fig. 281, GG_4 being the intrados, AA_4 the extrados of the arch,

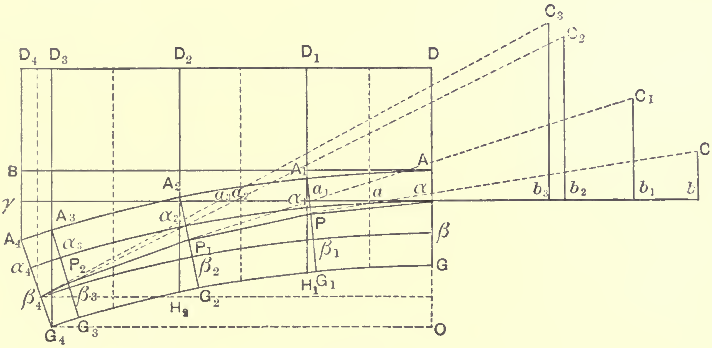


FIG. 281.

and DD_4 the extrados of the load. The arcs GG_4 and AA_4 are concentric circular arcs. The data are as follows: —

Span	$= 2(G_4O)$	$=$	6.00 feet,
Rise	$= GO$	$=$	0.50 foot,
Thickness of voussoirs $= AG$	$= A_4G_4$	$=$	0.75 foot,
Height of extrados of load above A	$= AD$	$=$	0.80 foot.

The position of the joints is not assumed to be located. We therefore draw through A a horizontal line AB, and divide this into lengths nearly equal, unless, as is usual near the springing, there is special reason to the contrary. Thus, we make the first three lengths each equal to 1 foot, and thus reach a vertical through G_4; and then the last division has a length of 0.24 foot. We have thus divided the half-arch and its load into four parts; viz., GDD_1H_1, $H_1D_1D_2H_2$, $H_2D_2D_3G_4$, and $G_4D_3D_4A_4$, the loads on these respective portions being represented by their areas respectively. We assume the centre of gravity of each load to lie on its middle vertical; and we then proceed to determine the numerical values of the several loads, the distances of their centres of gravity from a vertical through the crown, also the amount and centre of gravity of the first and second loads together, then of the first, second, and third, etc.

The work for this purpose is arranged as follows:—

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
Number of Voussoir.	Length.	Height.	Area.	Lever Arm.	Moment.	Partial Sums.	Area.	Moment.	Lever Arm.
1	1.00	1.57	1.570	0.50	0.785	1	1.570	0.785	0.500
2	1.00	1.68	1.680	1.50	2.520	1+2	3.250	3.305	1.017
3	1.00	1.90	1.900	2.50	4.750	1+2+3	5.150	8.055	1.563
4	0.24	1.72	0.413	3.12	1.287	1+2+3+4	5.563	9.342	1.680
-	3.24	-	5.563	-	9.342	-	-	-	-

Column (1) shows the number of the voussoir.
" (2) gives the horizontal lengths of the several trapezoids.
" (3) gives the middle heights of the trapezoids.
" (4) gives the areas of the trapezoids, and is obtained by multiplying together the numbers in (2) and (3).
" (5) gives the distances from A to the middle lines of the trapezoids.
" (6) gives the products of (4) and (5), giving the moments of the respective loads about an axis through A perpendicular to the plane of the paper.

Column (7) merely indicates the successive combinations of voussoirs.

" (8) has for its numbers, —

1°. The area representing the first load.

2°. The area representing the first two loads.

3°. The area representing the first three.

4°. The area representing the first four.

" (9) has for its numbers, —

1°. The moment of the first load about A.

2°. The moment of the first and second loads about A.

3°. The moment of the first, second, and third loads about A.

4°. The moment of the first, second, third, and fourth loads about A.

" (10) is obtained by dividing column (9) by column (8); the quotients being respectively the distance from A to the centres of gravity of the first, of the first and second, of the first, second, and third, and of the first, second, third, and fourth loads.

The calculation thus far is purely mathematical, and merely furnishes us with the loads and their points of application; in other words, furnishes us the data with which to begin our calculation of the thrust. Before passing to this, it should be said, however, that we now assume the joints to be drawn through the points A_1, A_2, A_3, and A_4, and generally normal to the extrados of the arch.

In this proceeding, we, of course, make an error which is very small near the crown and increases near the springing of the arch; this error, in the case of voussoir $A_1A_2G_1G_2$, amounts to the difference of the two triangles $A_2G_2H_2$ and $A_1G_1H_1$. A manner of making a correction by moving the joint will be explained later; but now we will complete our example, as the errors are not serious in this example. We now pass a line of resistance through α, the upper point of the middle third of the crown-joint, and β_4, the lowest point of the middle third of the springing. For this purpose take moments about β_4; and we shall have, if $T =$ thrust at the crown,

$$0.75T = (5.563)(3.075 - 1.68).$$

since 5.563 is the whole weight, and 3.075 — 1.68 is its leverage about β_4.

Hence

$$0.75T = (5.563)(1.395) = 7.760.$$

$$\therefore \quad T = 10.35.$$

Hence we proceed to draw a line of resistance through α, assuming, as the horizontal thrust, 10.35. To do this we proceed as follows: From a, the point of intersection of $\alpha\gamma$ with the vertical through the centre of gravity of the first trapezoid, we lay off ab to scale equal to 10.35, and then lay off bC vertically to scale equal to 1.57, the first load; then will Ca be the resultant pressure on joint A_1G_1, and its point of application will be P, which gives us one point in the line of resistance. To obtain the point P_1, we lay off $\alpha a_1 = 1.017$, the lever arm of the first two loads; then lay off $a_1b_1 = 10.35$, the thrust; then lay off b_1C_1 equal to 3.25, the weight of the first two loads. Then will C_1a_1 be the pressure on the second joint; and the point P_1, or its point of application, is at the intersection of C_1a_1 with A_2G_2.

Then lay off $\alpha a_2 = 1.563$, $a_2b_2 = 10.35$, $b_2C_2 = 5.150$; and P_2, the next point of the line of resistance, is the intersection of C_2a_2 with A_3G_3. Then lay off $\alpha a_3 = 1.680$, $a_3b_3 = 10.35$, $b_3C_3 = 5.563$; and C_3a_3 is the pressure on the springing, and this will intersect A_4G_4 at β_4 unless some mistake has been made in the work. Then is $\alpha PP_1P_2\beta_4$ the line of resistance through α and β_4, and this lies entirely within the middle third.

Hence we conclude that it is possible to draw a line of resistance within the arch-ring without having recourse to another trial.

§ 265. **Scheffler's Mode of Correcting the Joints.** — The following is the approximate construction given by Scheffler for correcting the joint: Let *DCG* be the side of the trapezoid, and *CH* the uncorrected joint. From *b*, the middle point of *GH*, draw *Db*; then draw *Gc* parallel to *bD*, and *ch* parallel to *CH*. Then will *ch* be the corrected joint.

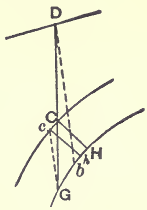


FIG. 282.

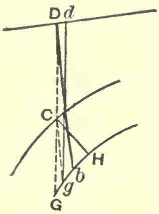


FIG. 283.

Conversely, having given the joint *CH*, to find the side of the trapezoid which limits the portion of the load upon it: through *C* draw *DG* vertical; join *D* with *b*, the middle point of *GH*; then draw *Cg* parallel to *Db*; then, from *g*, drawing *dg* vertical, we thus have the desired side of the trapezoid.

§ 266. **Another Example.** — Another example will now be solved, which necessitates two trials, and where some of the joints have to be corrected. It is practically one of Scheffler's. The dimensions of the arch are as follows: —

Half-span	32.97 feet.
Rise	24.74 feet.
Thickness of ring	5.15 feet.
Height of load at crown	8.24 feet.
Height of load at springing	33.50 feet.

The arch may be drawn by using, for the intrados, two circular arcs. Beginning at the springing, draw a 60° arc with a radius of one-fourth the span; then, with an arc tangent to this arc, continue to the crown, the proper rise having been previously laid off. The work for drawing a line of resistance through the top of the middle third of crown-joint and the inside of the middle third of the springing will be given without comment. It is as follows:

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
Number of the Voussoir.	Length.	Height.	Area.	Lever Arm.	Moment.	Partial Sums.	Area.	Moment.	Lever Arm.
1	8.24	14.1	116.18	4.12	479	1	116.18	479	4.12
2	8.24	16.4	135.14	12.36	1670	1+2	251.32	2149	8.55
3	8.24	18.3	150.79	20.60	3106	1+...+3	402.11	5255	13.07
4	4.13	22.6	93.34	26.79	2500	1+...+4	495.45	7755	15.65
5	4.13	27.1	111.92	30.92	3461	1+...+5	607.37	11216	18.46
6	5.14	34.7	178.36	35.55	6341	1+...+6	785.73	17557	22.34
–	–	–	785.73	–	17557	–	–	–	–

$$28.5T = (785.73)(34.9 - 22.34),$$

$$28.5T = 9868.77;$$

$$\therefore \quad T = 346.27.$$

Hence we construct the line of resistance passing through the top of the middle third of crown-joint and the inside of the middle third of the springing, using the thrust 346.27.

The construction is shown in the figure, and is entirely similar to that previously used. The student will readily identify this first line of resistance, and will see that it goes outside the middle third both above and below, being farthest above the extrados at the first joint from the crown, and farthest inside of the intrados opposite the first joint from the springing. Hence we proceed to pass a new line of resistance through the top of the middle third of the first joint from the crown, and the inside of the middle third of the first joint from the springing.

For this purpose we do not need to make out a new table, as it is not necessary to insert any new joints. We need only two more dimensions, i.e., the vertical depth of each of these

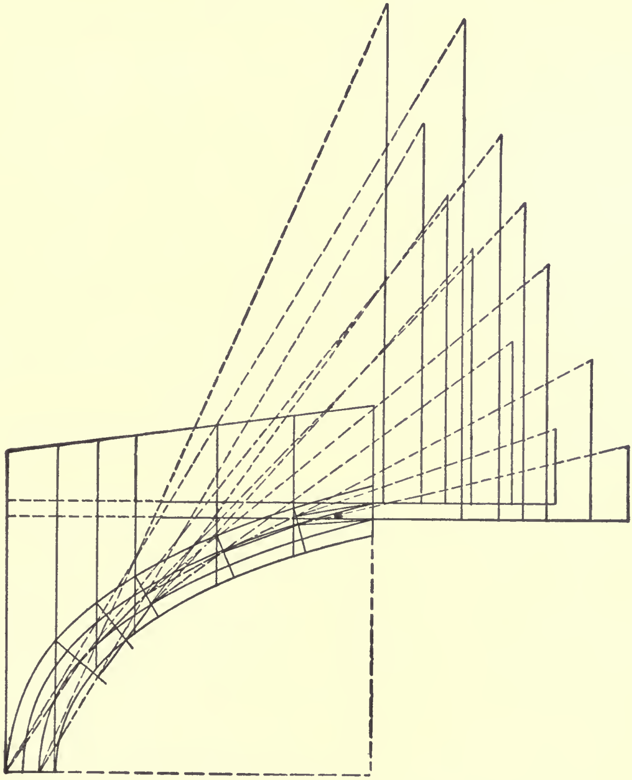


FIG. 284.

points below the top of the crown: these depths are respectively 2.10 and 16.8.

Hence we proceed as follows:

Let T = thrust at the crown;

x = distance of its point of application below the top of the crown-joint.

1°. Take moments about the upper one of the two points, and we have

$$T(2.10 - x) = (116.18)(8 - 4.12) = 450.778.$$

2°. Take moments about the lower one of the two points, and we have

$$T(16.8 - x) = (607.37)(30.5 - 18.46) = 7312.734.$$

Solving these equations, we obtain

$$T = 466.8, \quad x = 1.134.$$

Hence through a point on the crown-joint at a distance 1.134 below the top of the middle third of the crown-joint draw a horizontal line, this line being the line of action of the thrust. Then, making the construction for a new line of resistance just as before, only using this new point of application of the thrust, and using for thrust 466.8, we shall obtain a new line of resistance, which passes through the desired points.

Another method of drawing a line of resistance frequently pursued is the following: —

After determining the loads on the successive voussoirs, and also the thrust for the particular line of resistance which we wish to draw, lay off these loads and thrust to scale in their proper order and directions, and construct a force polygon (see § 126), then construct the corresponding frame (equilibrium polygon), which shall have its apices on the vertical lines passing through the centres of gravity of the loads as drawn in the figure of the arch. The intersections of the lines of the frame with the joints give us points of the line of resistance, and the line of resistance can be drawn by joining them.

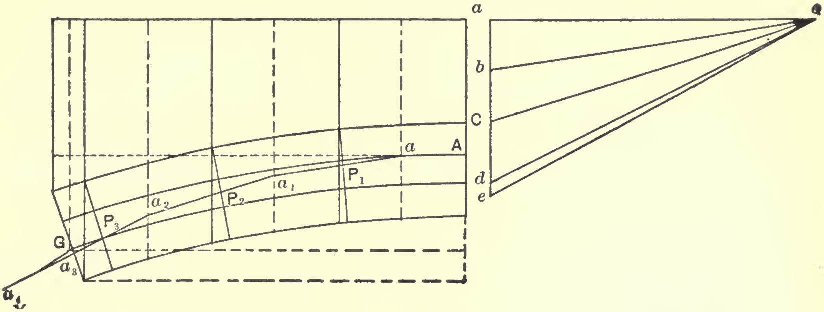
Thus, applying the solution to Fig. 281, we lay off

$$ab = 1.570, \qquad bc = 1.680,$$
$$cd = 1.900, \qquad de = 0.413.$$

Also, lay off

$$oa = 5.87,$$

and draw the lines ob, oc, od, and oe.


a
b
C
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A
P₁
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a₂
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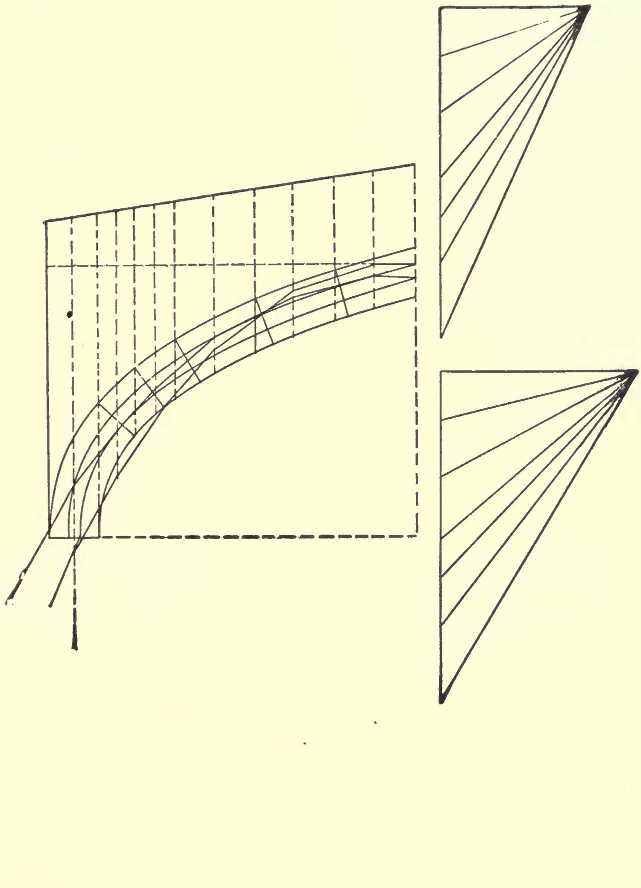
G
a₃



Then from A draw Aa parallel to oa, then draw aa_1 parallel to ob, a_1a_2 parallel to oc, a_2a_3 parallel to od, and a_3a_4 parallel to oe, this last prolonged backwards of course passes through G_4; then will the line of resistance be obtained by joining the points $AP_1P_2P_3G$.

The last figure on preceding page shows the same method applied to Fig. 284.

§ 267. **Examples.**—Four more examples will now be given to be worked out by the student. The dimensions are approximately those given in some of Scheffler's examples.

EXAMPLE I.—Half-span $= CD = 65.16$ feet, rise $= FD = 13.85$ feet, $AF = 5.32$ feet, $AE = 6.40$ feet. The arcs CF and AG are concentric circular arcs. Given width of first five horizontal divisions of line AB, counting from A, each 10.66 feet; width of sixth division, 11.86 feet; of seventh, 2.2 feet. Determine the possibility of drawing a line of resistance in the arch-ring.

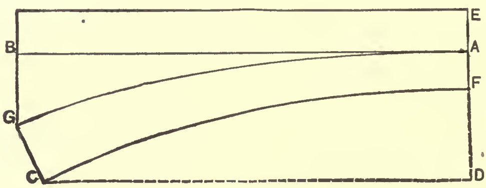


FIG. 285.

EXAMPLE II.—Half-span $= 63.98$ feet, rise $= FD = 31.99$ feet, $AF = CG = 5.32$ feet, $AE = 2.13$ feet. The intrados and extrados of this arch are seven centred ovals, both drawn from the same centres. Beginning at the springing, an arc with a radius of 21 feet is drawn, subtending 39°; the curve is continued by a curve subtending 24°, and having a radius of 35.55 feet. From F an arc subtending 10° is drawn from a centre on FD produced, and with a radius of 152 feet; the curve is completed by an arc connecting the second and last.

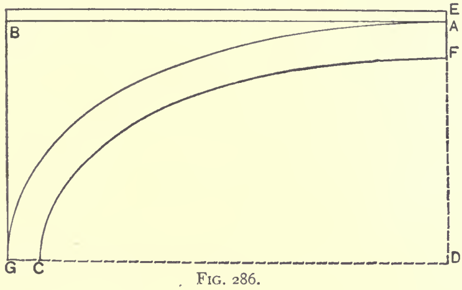


FIG. 286.

Given horizontal width of each of first six divisions, counting from A, 10.66 feet; horizontal width of seventh division, 5.32 feet. Determine the possibility of drawing a line of resistance in the arch-ring.

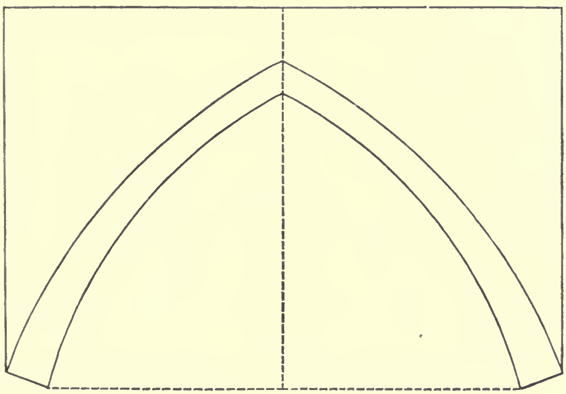


FIG. 287.

EXAMPLE III. — Given span = 74.18 feet; rise = 45.83

feet; radius of intrados = 82.42 feet; radius of extrados = 91.18 feet; height of load at crown = 8.24 feet; width of each of five divisions nearest crown = 8.24 feet; width of sixth stone = 4.13 feet. Determine the possibility of drawing a line of resistance within the arch-ring.

EXAMPLE IV. — Given span = 37.07 feet; thickness of ring = AB = 3.08 feet; height of load = BC = 82.42 feet. Determine the possibility of drawing a line of resistance within the arch-ring.

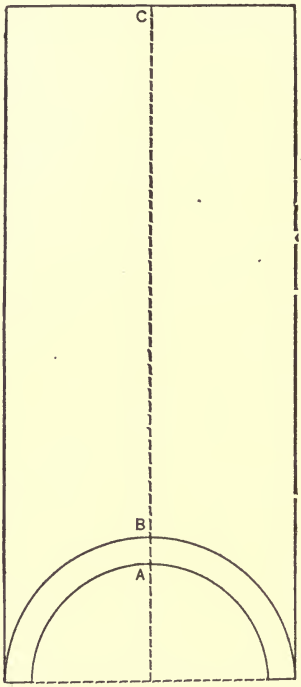


FIG. 288.

§ 268. **Criterion of Stability.** — It has already been stated, that, if a line of resistance can be drawn within the arch-ring, then the true line of resistance will lie within the arch-ring.

With those who, like Scheffler, consider the material of the voussoirs incompressible, the criterion of stability of an arch is, that it should be possible to draw a line of resistance within the arch-ring.

On the other hand, Rankine would decide upon the stability of an arch by determining whether a line of resistance can be drawn within the middle third of the arch-ring.

Other limits have been adopted instead of the middle third. In some cases the only reason for deciding upon what these limits should be has been custom or precedent.

They might also be determined so that there should be no danger of exceeding the crushing-strength of the stone.

It is needless to say that the first method is incorrect; for the material of the voussoirs is never incompressible, and an arch where the true line of resistance touches the intrados or extrados could not stand, as the stone would be crushed.

Nevertheless, no example will be solved here, where we determine the possibility of drawing a line of resistance within any other limits than the middle third, as the method of procedure is entirely similar to what we have done, the computation of the entire table being the same in all cases, the only difference occurring in the computation of the thrust and its point of application, and the consequent construction of the line of resistance. The method to be pursued is, as before, by taking moments about the points through which it is desired that the line of resistance shall pass.

§ 269. **Unsymmetrical Arrangement.** — When the arch is unsymmetrical, either in form or loading, the same criterion as to being able to pass a line of resistance within the middle third or other limits of the arch-ring will serve to determine its stability. The method of procedure differs, however, from the fact, that whereas we have heretofore found it necessary to study only the half-arch and its load, and have had the advantage of knowing, from the symmetry of arch and load, that the thrust at the crown is horizontal, we have not that advantage here, and hence we must study the entire arch, and we must assume that the thrust at the crown may be oblique, and hence have a vertical as well as a horizontal component.

In this case it will be necessary to have three instead of two points given, in order to determine a line of resistance.

If we assume (Fig. 289) a vertical joint at the crown, and let $P =$ vertical component of the thrust at the crown, $A =$ horizontal component of the thrust at the crown, $x =$ distance of point of application of thrust at the crown below upper point of crown-joint, we have thus three unknown quantities, and we shall therefore need three equations to determine them.

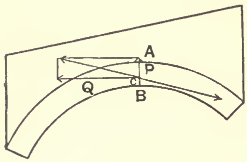


FIG. 289.

In this case, therefore, we must have three points of the line of resistance given, in order to determine it; and a reasoning similar to that pursued in § 263 would show that a line of resistance can always be passed through any three given points.

In performing the work, we should need to make out a table for the part of the arch on each side of the crown-joint, showing the loads, and centres of gravity of the loads, on each voussoir, and on combinations of the first two, first three, etc.; this portion of the work being entirely similar to that done in the case of arches of symmetrical form and loading, only that we require a separate table for the parts on each side of the crown-joint.

When these two tables have been worked out, we next proceed to impose the conditions of equilibrium by taking moments about each of the three points given.

Thus, suppose that (as is usually done first) we pass a line of resistance through the top of the middle third of the crown-joint and the inside of the middle third of each springing-joint, we then have only two unknown quantities to determine:

viz., P and Q, inasmuch as x becomes zero. Hence we take moments about the inner edge of the middle third of each of the springing-joints.

In taking moments about the inner edge of the middle third of the left-hand springing-joint, we impose the conditions of equilibrium upon the forces acting on that part of the arch that lies to the left of the crown-joint. These forces are: (1°) its load and weight, which tend to cause right-handed rotation; (2°) the horizontal component of the thrust exerted *by* the right-hand portion *upon* the left-hand portion; (3°) the vertical component P of the thrust exerted *by* the right-hand portion *upon* the left-hand portion.

It is necessary to adopt some convention, in regard to the sign of P, to avoid confusion: and it will be called positive when the vertical component of the thrust exerted *by* the right-hand portion *on* the left-hand portion is upwards; when the reverse is the case, it is negative.

We next take moments about the inner edge of the middle third of the right-hand springing-joint, and impose the conditions of equilibrium upon the forces acting upon the right-hand portion of the arch. In doing this, we must observe that we have for these forces, (1°) the weight and load which tend to cause left-handed rotation; (2°) the horizontal component Q of the thrust exerted *by* the left-hand portion *upon* the right-hand portion,—this acts towards the right; (3°) the vertical component P of the thrust exerted by the left-hand portion upon the right-hand portion; and this, when positive, acts downwards.

Having determined the values of Q and P, we next proceed to draw the line of resistance, by the use of either of the methods employed, with symmetrical arches, observing only that the thrust, i.e., the resultant of P and Q, is now oblique, and that it acts in opposite directions on the two sides of the crown-joint.

Having drawn this line of resistance, if we find that it passes outside of the middle third, we draw normals through the points where it is farthest from the middle third, and thus obtain three points through which to draw a line of resistance: then, taking moments about each of these three points, we determine, from the three resulting equations, values of Q, P, and x, and proceed to draw our new line of resistance; and, if this does not pass entirely within the middle third, it is not at all probable that a line of resistance can be drawn within the middle third. All the above will be made clearer by the following example:

Example. — Given an unsymmetrical circular arch, shown in the figure, the intrados and extrados being concentric circles, $EM = 4'$, $HF = 1'.85$, radius of $EHF = 6$, $AH = 0'.5$, $AK = 0'.8$, to determine the possibility of drawing a line of

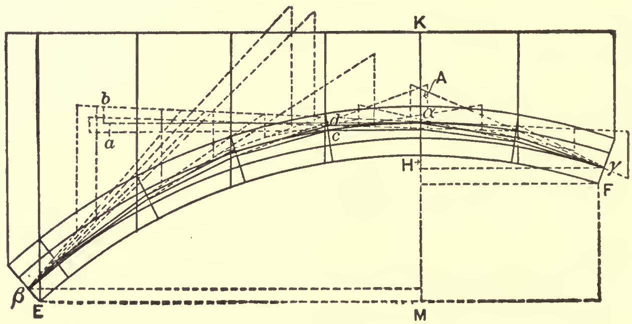


Fig. 290.

resistance in the arch-ring. The tables following show the mode of dividing up the load, and getting the centres of gravity, also the mode of arranging the work for this purpose.

LEFT-HAND PORTION.

Number of Voussoir.	Width.	Height.	Area.	Lever Arm.	Moment.	Partial Sums.	Area.	Moment.	Lever Arm.
1	1.00	1.32	1.32	0.50	0.660	1	1.32	0.660	0.50
2	1.00	1.48	1.48	1.50	2.220	1 + 2	2.80	2.880	1.03
3	1.00	1.84	1.84	2.50	4.600	1 + 2 + 3	4.64	7.480	1.61
4	1.00	2.42	2.42	3.50	8.470	1 + ... + 4	7.06	15.950	2.26
5	0.33	2.63	0.87	4.17	3.628	1 + ... + 5	7.93	19.578	2.47
–	–	–	7.93	–	19.578	–	–	–	–

RIGHT-HAND PORTION.

Number of Voussoir.	Width.	Height.	Area.	Lever Arm.	Moment.	Partial Sums.	Area.	Moment.	Lever Arm.
1	1.00	1.32	1.32	0.50	0.660	1	1.32	0.660	0.50
2	1.00	1.48	1.48	1.50	2.220	1 + 2	2.80	2.880	1.03
–	–	–	2.80	–	2.880	–	–	–	–

Now take moments about the inner edge of the middle third of the left-hand springing, and we have

$$1.73Q - 4.10P = 7.93(4.10 - 2.47) = 12.9259.$$

Then take moments about the inner edge of the middle third of the right-hand springing, and we have

$$0.47Q - 1.90P = 2.80(1.90 - 1.03) = 2.4360.$$

Solving these two equations gives us

$$Q = 6.246.$$

$$P = 0.263.$$

If R represents the resultant of P and Q, we have

$$R = \sqrt{P^2 + Q^2} = 6.251;$$

hence we proceed, as follows, to pass a line of resistance through the top of the middle third of the crown-joint and the inner edge of the middle third of each springing:

Through the top of the middle third draw a horizontal line. Lay off $\alpha a = 6.626$ and $ab = 0.263$, and draw αb; then $\alpha b = 6.635$ represents, in direction and magnitude, the thrust at the crown. Using this thrust in the same way as we did the horizontal thrust in the case of symmetrical arches, we obtain the line of resistance which is farthest outside of the arch at d; hence, drawing a normal to the arch from d, we obtain c, the upper edge of the middle third of the first joint from the crown. Hence we proceed to pass a new line of resistance through B, c, and γ.

To do this we must assume Q, P, and x all unknown.

1°. Take moments about B, and we have

$$(1.73 - x)Q + 4.1P = 12.9259.$$

2°. Take moments about γ, and we have

$$(0.47 - x)Q - 1.9P = 2.436.$$

3°. Take moments about c, and we have

$$(0.078 - x)Q + P = (1.32)(0.45) = 0.594.$$

Solving these three equations, we obtain

$$Q = 6.905,$$
$$P = 0.297,$$
$$x = 0.035.$$

Hence

$$R = \sqrt{P^2 + Q^2} = 6.91.$$

Hence, if we lay off a distance 0.035 below α, we shall have the point on the crown-joint at which the thrust is applied;

and making the same kind of construction as we just made, only using this point instead of A, and these new values of Q and P, we construct the second line of resistance. The construction is omitted in order not to confuse the figure; but the line of resistance is drawn, and the student can easily make the construction for himself. It will be seen, that, in this case, this new line of resistance lies entirely within the arch-ring.

§ 270. **General Remarks.** — Whenever there are also horizontal external forces acting upon the arch, these should be taken into account in imposing the conditions of equilibrium.

It will be noticed, that, in the preceding discussion, it has always been assumed that the load upon any one voussoir is the weight of the material directly over that voussoir. This is the assumption usually made in computing bridge arches: and it may be nearly true when the height of the load above the crown is not great; but even then it is not strictly true, and when this depth becomes great, as would be the case with an arch which supports the wall of a building, it is far from true, as the distribution of the load actually coming upon different parts of the arch must vary with, and depend upon, the bonding of the masonry, and also upon the co-efficient of friction of the material. Thus, in the case of an arch supporting a part of the wall of a building, it is probable that the only part of the load that comes upon the arch is a small triangular-shaped piece directly over the arch, and that above this the material of the wall is supported indcpendently of the arch. This will be plain when we consider, that, were such an arch removed, the wall would remain standing, only a few of the bricks near the arch falling down; and though the number of bricks that would fall would be greater while the mortar is green, still even then only a few would drop out.

In regard to these matters, we need experiments; but thus far we have none that are reliable.

Then, again, we have arches supporting a mass of sand or gravel; and then the mutual friction of the particles on each other comes into play, and it is not true in this case that the load on any voussoir is the weight of the material directly above that voussoir. In some cases this has been accounted as a mass of water pressing normally upon the arch, but we cannot assert that such a course is correct.

On the other hand, there are cases where we know that an arch is subjected to horizontal as well as to vertical forces, and sometimes we cannot tell how great these horizontal forces are. Thus, the forms of sewers are an arch for the top and an inverted arch for the bottom; but in this case the sides of the ditch in which the sewer is laid when building it, are capable of furnishing whatever horizontal thrust is needed to force the line of resistance into the arch-ring, provided that a *horizontal thrust* is what is needed to force it in. Hence it is, that, were the attempt made to pass a line of resistance within the arch-ring of almost any successful sewer, accounting the load as the weight of the earth above it, the line would almost invariably go outside; but the earth on the sides is capable of furnishing the necessary horizontal thrust to force it inside, unless a careless workman has omitted to ram it tight, or unless some other cause has loosened it on the sides of the sewer.

If we know, in any case, the actual law of the distribution of the load, we can determine the proper form for the arch by the methods of the first part of this chapter, as was done in the case of the parabola and of the catenary. Scheffler's method is, however, the one almost always used for determining the stability of any stone arch against overturning around the joints.

Should there ever arise a case where there was danger that the resultant pressure on any joint made an angle with the joint greater than the angle of friction, this could be remedied by merely changing the inclination of the joint.

§ 271. **General Theory of the Elastic Arch.** — In the case of the iron arch, the loads upon the arch are all definitely known; and it is necessary to ascertain with certainty the stress in all parts of the structure, and to so proportion the different members as to bear with safety their respective stresses.

The general discussion of the method used in calculating such arches will now be given; the method used being practically that followed by Dr. Jacob J. Weyrauch, and explained more at length in his "Theorie der Elastigen Bogenträger."

This discussion is also necessary in order to prove the proposition already enunciated in § 262; viz., that "for an arch of constant cross-section, that line of resistance is approximately the true one which lies nearest to the axis of the arch-ring as determined by the method of least squares."

In this discussion the following definitions are adopted: —

1°. The axis of the arch is a plane curved line passing through the centres of gravity of all its normal sections.

2°. The plane of this axis is called the plane of the arch.

3°. The axial layer of the arch is a cylindrical surface perpendicular to the plane of the arch, and containing its axis.

4°. A section normal to the axis is called a cross-section.

5°. The length of the axis between two sections is called the length of arch between the sections.

The loads may be single isolated loads, or they may be distributed loads.

We shall, in this discussion, assume in the plane of the arch a pair of rectangular axes, OX and OY, positive to the right and upwards respectively.

We will, then, assuming any point on the axis of the arch before the loads are applied, call x, y, the co-ordinates of that point, s the length of axis from some arbitrary fixed point, ϕ the angle made by the tangent line at that point with OX, r the radius of curvature of the axis at that point, $x + dx, y + dy$, $s + ds$, and $\phi + d\phi$, the corresponding quantities for a point very near the first before the load is applied; also we will denote by η the perpendicular distance of any fibre from the axial layer, by s_η the length of arc measured to that point where this fibre cuts the cross-section through (x, y), and $s_\eta + ds_\eta$ the length of arc measured on this fibre to the next cross-section, so that ds will be the distance apart of the cross-sections measured on the axis, and ds_η on the other fibre. All this is done before the load is applied, and is shown in Fig. 291; while the changes brought about by the application of the loads combined with change of temperature are denoted by Δ's, and shown in Fig. 292. Thus, x, y, s, and ϕ become respectively $x + \Delta x$, $y + \Delta y$, $s + \Delta s$, and $\phi + \Delta\phi$.

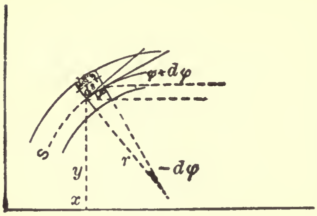


FIG 291.

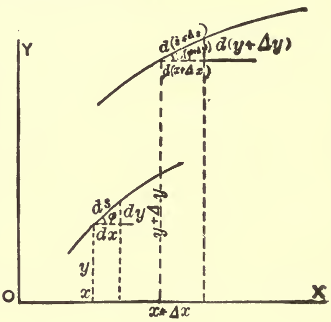


FIG. 292.

Now the course we are to follow in the discussion is, to imagine a cross-section dividing the arch into two parts, and to impose the conditions of equilibrium between the external forces acting on the part to one side of the section, and the forces exerted by the other part upon this part at the section. These latter forces may be reduced to the three following:—

1°. A normal thrust T_x uniformly distributed over the section, the resultant acting at the centre of gravity of the section.

2°. A shearing-force S_x at the section.

3°. A bending-couple at the section; this comprising a stress varying uniformly from the axial layer, and amounting to a statical couple, tension below, and compressions above, the axial layer.

Moreover, (1) and (3) combined amount to a uniformly varying stress, the magnitude of whose resultant is T_x, its point of application not being at the centre of gravity of the section; this sort of composition having been already exhibited in the case of the short strut (§ 207).

Now, let r be the radius of curvature of the axial layer at the section; and we have, from Fig. 291, by similar sectors,

$$ds_\eta = ds + \eta(-d\phi) = ds - \eta d\phi. \qquad (1)$$

But

$$r(-d\phi) = ds \qquad \therefore \quad \frac{d\phi}{ds} = -\frac{1}{r}$$

$$\therefore \quad ds_\eta = ds\left(1 + \frac{\eta}{r}\right) = ds\left(\frac{r + \eta}{r}\right). \qquad (2)$$

Now, if the loads are applied, and the changes take place that are indicated in Fig. 292, we shall have, by suitable substitutions in (1),

$$d(s_\eta + \Delta s_\eta) = d(s + \Delta s) - \eta d(\phi + \Delta\phi); \qquad (3)$$

and, combining this with (1) and (2), we obtain

$$\frac{d\Delta s_\eta}{ds_\eta} = \left(\frac{d\Delta s}{ds} - \eta\frac{d\Delta\phi}{ds}\right)\frac{r}{r + \eta}. \qquad (4)$$

Now, the change of length of fibre from ds_η to $d(s_\eta + \Delta s_\eta)$ is due to two causes: (1) the change of temperature, (2) the stress acting on the fibre normal to the section.

Let ϵ = co-efficient of expansion per degree temperature.
τ = difference of temperature, in degrees.
p_η = intensity of stress along the fibre at section.
E = modulus of elasticity of the material.

Then

$$\epsilon\tau - \frac{p_\eta}{E} = \frac{d\Delta s_\eta}{ds_\eta} = \left(\frac{d\Delta s}{ds} - \eta\frac{d\Delta\phi}{ds}\right)\frac{r}{r + \eta}. \qquad (5)$$

Hence, solving for p_η, we have

$$p_\eta = E\left(\eta\frac{d\Delta\phi}{ds} - \frac{d\Delta s}{ds}\right)\frac{r}{r + \eta} + E\epsilon\tau, \qquad (6)$$

this being the expression for the stress per square inch on the fibre whose distance is η from the axial layer.

Hence we shall have, by summation, if elementary area $=$ dA,

$$T_x = \Sigma p_\eta dA = E\left[\frac{d\Delta\phi}{ds}\Sigma\frac{r\eta dA}{r+\eta} - \frac{d\Delta s}{ds}\Sigma\frac{rdA}{r+\eta} + \epsilon\Sigma\tau dA\right]; \quad (7)$$

and for the moment M_x we have, by taking moments about the neutral axis of the section (i.e., horizontal line through its centre of gravity),

$$M_x = \Sigma p_\eta \eta dA = E\left[\frac{d\Delta\phi}{ds}\Sigma\frac{r\eta^2 dA}{r+\eta} - \frac{d\Delta s}{ds}\Sigma\frac{r\eta dA}{r+\eta} + \epsilon\Sigma\tau\eta dA\right]. \quad (8)$$

Let $\Sigma dA = A$, $r\Sigma\dfrac{\eta^2 dA}{r+\eta} = \Omega$, and observe that $\Sigma\eta dA = 0$, since the axis passes through the centre of gravity of the section, and we have

$$\Sigma\frac{r\eta dA}{r+\eta} = \Sigma\eta dA - \Sigma\frac{\eta^2 dA}{r+\eta} = -\frac{\Omega}{r},$$

$$\Sigma\frac{rdA}{r+\eta} = \Sigma dA - \frac{1}{r}\Sigma\eta dA + \frac{1}{r}\Sigma\frac{\eta^2 dA}{r+\eta} = A + \frac{\Omega}{r^2}.$$

Making these substitutions, we have

$$\frac{T_x}{E} = -\left(\frac{d\Delta\phi}{ds} + \frac{1}{r}\frac{d\Delta s}{ds}\right)\frac{\Omega}{r} - \left(\frac{d\Delta s}{ds} - \epsilon\tau\right)A,$$

$$\frac{M_x}{E} = \left(r\frac{d\Delta\phi}{ds} + \frac{d\Delta s}{ds}\right)\frac{\Omega}{r}.$$

Hence, solving for $\dfrac{d\Delta s}{ds}$ and $\dfrac{d\Delta\phi}{ds}$, we have

$$\frac{d\Delta s}{ds} = -\left(T_x + \frac{M_x}{r}\right)\frac{1}{EA} + \epsilon\tau = Y, \quad (9)$$

$$\frac{d\Delta\phi}{ds} = \left(T_x + \frac{M_x}{r}\right)\frac{1}{EAr} + \frac{M_x}{E\Omega} - \frac{\epsilon\tau}{r} = X. \quad (10)$$

Now, from Fig. 292, we have

$$d(x + \Delta x) = d(s + \Delta s) \cos (\phi + \Delta\phi),$$
$$d(y + \Delta y) = d(s + \Delta s) \sin (\phi + \Delta\phi);$$

but, if we write $\cos \Delta\phi = 1$, and $\sin \Delta\phi = \Delta\phi$,

$$\cos(\phi + \Delta\phi) = \cos\phi - \Delta\phi \sin\phi = \frac{dx}{ds} - \Delta\phi\frac{dy}{ds},$$

$$\sin(\phi + \Delta\phi) = \sin\phi + \Delta\phi\cos\phi = \frac{dy}{ds} + \Delta\phi\frac{dx}{ds}.$$

Hence

$$d\Delta x = -\Delta\phi dy + \frac{d\Delta s}{ds}dx - \left(\frac{d\Delta s}{ds}dy\Delta\phi\right),$$

$$d\Delta y = +\Delta\phi dx + \frac{d\Delta s}{ds}dy + \left(\frac{d\Delta s}{ds}dx\Delta\phi\right):$$

or, omitting the last terms, and integrating,

$$\Delta x = -\int\Delta\phi dy + \int Ydx, \qquad (11)$$

$$\Delta y = \int\Delta\phi dx + \int Ydy; \qquad (12)$$

and, integrating (9) and (10),

$$\Delta s = \int Yds, \qquad (13)$$

$$\Delta\phi = \int Xds. \qquad (14)$$

In these four equations we have

Δx = horizontal deflection due to the loads,
Δy = vertical deflection due to the loads,
Δs = change of length of arc due to the loads,
$\Delta\phi$ = change of slope due to the loads.

The three equations which we shall have occasion to use are (11), (12), and (14), and if we make the integrations between the limits x and 0, they become, by changing their order,

$$\Delta\phi = \Delta\phi_0 + \int_{x=0}^{x=x} Xds, \qquad (15)$$

$$\Delta x = - \int_{x=0}^{x=x} \Delta\phi dy + \int_{x=0}^{x=x} Y dx, \tag{16}$$

$$\Delta y = \int_{x=0}^{x=x} \Delta\phi dx + \int_{x=0}^{x=x} Y dy, \tag{17}$$

where $\Delta\phi_0$ is the change of slope for $x = 0$.

If, now, we write

$$M_1 = \frac{M_x}{E\Omega} + \frac{M_x}{EAr^2} + \frac{T_x}{EAr}, \tag{18}$$

$$P_1 = \frac{M_x}{EAr} + \frac{T_x}{EA}, \tag{19}$$

we shall have

$$X = M_1 - \frac{\epsilon\tau}{r}, \tag{20}$$

$$Y = - P_1 + \epsilon\tau; \tag{21}$$

or if we neglect the effect of temperature, we may write

$$X = M_1, \tag{22}$$

$$Y = - P_1. \tag{23}$$

Moreover, we may with very little error substitute the moment of inertia I for Ω in the value of M_1, i.e., writing this

$$M_1 = \frac{M_x}{EI} + \frac{M_x}{EAr^2} + \frac{T_x}{EAr}. \tag{24}$$

§ 272. **Manner of using the Fundamental Equations to Determine the Stresses in an Iron Arch.** — In order to be able to determine the stresses in all the members of an iron arch with any given loading, we need to determine the three quantities T_x, S_x, and M_x for each section.

Now, if we let R_x represent the thrust at the section, we shall have

$$R_x = \sqrt{T_x^2 + S_x^2}\,; \tag{1}$$

and, if we let H_x and V_x represent the horizontal and vertical components of R_x respectively, we have that we need to determine the three quantities H_x, V_x, and M_x for each section.

Let us suppose the arch to be subjected to vertical loads only, and let

H = horizontal component of thrust at all points,
V = vertical component of left-hand supporting force,
V_1 = vertical component of right-hand supporting force,
M = bending-moment at left-hand support,
M' = bending-moment at right-hand support.

Assume origin of co-ordinates at left-hand support, and x + to the right, and y + upwards, and impose the conditions of equilibrium upon the forces acting on the part of the arch between the section and the left-hand support; then we have, if W is any one load, and a the x of its point of application,

$$H_x = H, \tag{2}$$
$$V_x = V - \Sigma_0^x W, \tag{3}$$
$$M_x = M + Vx - Hy - \Sigma_0^x W(x - a). \tag{4}$$

Hence it is plain that the three quantities which we need to determine are H, V, and M.

Now these are also the three unknown quantities which will, by suitable reductions, become the three unknown constant quantities in equations **(11) to (14)**. The determination of these three quantities requires three conditions; what these conditions are depends upon the manner of building the arch, as will be seen from the following three special cases: —

CASE I. — Let the arch be jointed at three points, viz., the two supports, and one other point whose co-ordinates are $x = x_1$ and $y = y_1$. Then we know, that, for all points where there is a hinge, there can be no bending-moment. Hence

$$M = 0, \qquad M' = 0, \qquad \text{and} \qquad M_{x_1} = 0,$$

which are the three required conditions; and, if these be imposed, it is easy to obtain H_x, V_x, and M_x, for every section.

CASE II.—Let the arch be jointed only at the ends. Then $M = M' = 0$ gives us two conditions: and for the third we have $\Delta l = 0$; i.e., if we put l for x in equation (16), § 271, after having made the integrations, we have the third equation, as this expresses simply the condition that the supports remain at the same horizontal distance apart after the load is put on as before. With these three conditions we can determine H_x, V_x, and M_x for all sections.

CASE III.—Let the arch be fixed in direction at the ends. We must now have three conditions. These will be as follows:—

1°. $\Delta l = 0$; i.e., the supports remain at the same horizontal distance apart after the load is applied as before.

2°. $\Delta h = 0$ (h being the difference of level of the supports); i.e., the supports remain at the same vertical distance apart after as before the load is applied.

3°. $\Delta\phi_1 = 0$; i.e., the tangents at the ends make the same angle with each other after as before the load is applied.

The value of $\Delta\phi_1$ is obtained by integrating (15), § 271, and then substituting l for x, or h for y, observing that $\Delta\phi_0 = 0$.

The value of Δl is obtained by integrating (16), § 271, and then substituting l for x.

The value of Δh is obtained by integrating (17), § 271, and then substituting l for x, or h for y.

In this case, if we neglect the effect of temperature, write I for Ω, omit all terms containing $\frac{1}{r}$, and also neglect T_x in (15), (16), and (17) of § 271, we shall obtain by making one integration,

$$\Delta x = -\int \frac{M_x}{EI} yds, \tag{5}$$

$$\Delta y = +\int \frac{M_x}{EI} xds, \tag{6}$$

$$\Delta \phi = \int \frac{M_x}{EI} ds. \tag{7}$$

While it has often been proposed to use these as approximately true, nevertheless the degree of approximation is too coarse to render them suitable to use in practice.

CIRCULAR ARCH, UNIFORM SECTION, AND VERTICAL LOADS.

We will next deduce expressions for $\Delta\phi$, Δx, and Δy, for a circular arch of constant cross-section and loaded vertically, and thence deduce the equations from which to determine the three quantities M, M', and H in any such case, and also the expression for the horizontal thrust in an arch hinged at the two springing-points, and symmetrical in form. We will write in place of Ω the moment of inertia I, and will neglect terms containing $\frac{1}{r}$, in equations (15), (16), and (17), but will not neglect T_x.

Take the origin at the left-hand springing-point, and the axis of x horizontal.

Observe that if ϕ represent the angle the tangent line to the arch at the point (x, y) makes with the axis of x, it also represents the angle subtended by the radius drawn through the point (x, y) with the vertical radius, i.e., that through the crown.

Let ϕ_0 be the value of ϕ at the origin, and let α be the value of ϕ at the point of application of any concentrated load W, the co-ordinates of this point being (a, b).

Let the co-ordinates of the centre of the circle be

$$g = r \sin \phi_0, \qquad -k = -r \cos \phi_0;$$

of the crown be $\quad g = r \sin \phi_0, \qquad f = r - r \cos \phi_0;$

of the point of application of W be $a = r(\sin \phi_0 - \sin \alpha), \quad b = r(\cos \alpha - \cos \phi_0);$

and of any point on arch, $\quad x = r(\sin \phi_0 - \sin \phi), \quad y = r(\cos \phi - \cos \phi_0).$

The following is a list of relations which can be easily proved, and which are needed for use in the work that follows them.

$$g - x = r \sin \phi;$$
$$k + y = r \cos \phi;$$
$$x - a = r(\sin \alpha - \sin \phi);$$
$$g - a = r \sin \alpha;$$
$$y - b = r(\cos \phi - \cos \alpha);$$
$$ds = - r d\phi;$$
$$dx = - r \cos \phi d\phi;$$
$$dy = - r \sin \phi d\phi;$$

$$T_x = V_x \sin \phi + H \cos \phi;$$
$$M_x = M + Vx - Hy - \sum_0^x W(x - a);$$
$$V_x = V - \sum_0^x W;$$
$$M' = M + Vl - Hc - \sum_0^l W(l - a);$$

where M' = bending-moment at right-hand springing-point.

Also

$$X = \frac{M_x}{EI}, \qquad Y = \epsilon \tau - \frac{T_x}{EA}.$$

By making the substitutions indicated, and also the integrations, we obtain from (15), (16), and (17) the following:—

$$\Delta\phi = \Delta\phi_0 + \frac{r}{EI}\{(\phi_0 - \phi)(M + Vr \sin \phi_0 + Hr \cos \phi_0) - Vr(\cos \phi - \cos \phi_0) - Hr(\sin \phi_0 - \sin \phi) + \sum_0^x Wr(\cos \phi - \cos \alpha) - \sum_0^x Wr(\alpha - \phi) \sin \alpha\}, \qquad (8)$$

$$\Delta x = \epsilon \tau x - \frac{r}{2EA}\{V(\sin^2 \phi_0 - \sin^2 \phi) + H[\sin \phi_0 \cos \phi_0 - \sin \phi \cos \phi + (\phi_0 - \phi)] - \sum_0^x W(\sin^2 \alpha - \sin^2 \phi)\} - y\Delta\phi_0$$
$$- \frac{r^2}{EI}\{M + Vr \sin \phi_0 + Hr \cos \phi_0)[(\phi_0 - \phi) \cos \phi$$

$$- (\sin \phi_0 - \sin \phi)] - \frac{Vr}{2} (\cos \phi - \cos \phi_0)^2 - \frac{Hr}{2} [-(\phi_0 - \phi)$$
$$+ \cos \phi (\sin \phi_0 - \sin \phi) + \sin \phi_0 (\cos \phi - \cos \phi_0)]$$
$$+ \tfrac{1}{2} r \sum_0^x W (\cos \phi - \cos \alpha)^2 - r \cos \phi \sum_0^x W (\alpha - \phi) \sin \alpha$$
$$+ r \Sigma W \sin \alpha (\sin \alpha - \sin \phi) \}; \qquad (9)$$

$$\Delta y = \epsilon \tau y - \frac{r}{2EA} \{ H (\sin^2 \phi_0 - \sin^2 \phi) - V [\sin \phi_0 \cos \phi_0$$
$$- \sin \phi \cos \phi - (\phi_0 - \phi)] - \sum_0^x W [\sin \phi \cos \phi - \sin \alpha \cos \alpha$$
$$+ (\alpha - \phi)] \} + x \Delta \phi_0 + \frac{r^2}{EI} \{ (M + Vr \sin \phi_0$$
$$+ Hr \cos \phi_0) [(\cos \phi - \cos \phi_0) - (\phi_0 - \phi) \sin \phi]$$
$$- \tfrac{1}{2} Hr (\sin \phi_0 - \sin \phi)^2 - \frac{Vr}{2} [- \cos \phi_0 (\sin \phi_0 - \sin \phi)$$
$$- \sin \phi (\cos \phi - \cos \phi_0) + (\phi_0 - \phi)] + \sin \phi \sum_0^x Wr (\alpha$$
$$- \phi) \sin \alpha - \sum_0^x Wr \sin \alpha (\cos \phi - \cos \alpha)$$
$$- \tfrac{1}{2} \sum_0^x Wr [\cos \alpha (\sin \alpha - \sin \phi) + \sin \phi (\cos \phi - \cos \alpha)$$
$$- (\alpha - \phi)] \}. \qquad (10)$$

We will next write out the same values as applied to the right-hand springing-joint of a symmetrical arch.

In this case we have the value of ϕ for the right-hand end, or ϕ_1 equal to $-\phi_0$; and if we make this substitution, observing that x becomes l and y becomes zero, and if we substitute for V the value

$$V = \tfrac{1}{2} \frac{M' - M}{r \sin \phi_0} + \tfrac{1}{2} \frac{\Sigma Wr (\sin \phi_0 + \sin \alpha)}{r \sin \phi_0},$$

then we obtain the following :—

$$\Delta \phi_1 = \Delta \phi_0 + \frac{r}{EI} \{ (M' + M) \phi_0 - 2Hr (\sin \phi_0 - \phi_0 \cos \phi_0)$$
$$- \tfrac{1}{2} \sum_0^l Wr [2\alpha \sin \alpha - 2\phi_0 \sin \phi_0 + 2 (\cos \alpha - \cos \phi_0)] \}; \qquad (11)$$

$$\Delta l = \epsilon\tau l - \frac{r}{2EA}\{H(2\phi_0 + 2\sin\phi_0\cos\phi_0) + \sum_0^l W(\sin^2\phi_0 - \sin^2\alpha)\}$$

$$+ \frac{r^2}{2EI}\{(M' + M)(2\sin\phi_0 - 2\phi_0\cos\phi_0) - Hr(4\phi_0\cos^2\phi_0$$

$$- 6\sin\phi_0\cos\phi_0 + 2\phi_0) + \sum_0^l Wr[2\alpha\cos\phi_0\sin\alpha$$

$$+ 2\cos\phi_0\cos\alpha - 2\phi_0\sin\phi_0\cos\phi_0 + \sin^2\phi_0 - \sin^2\alpha$$

$$- 2\cos^2\phi_0]\}; \qquad (12)$$

$$\Delta c = l\Delta\phi_0 + \frac{r^2}{2EI}\{(M' + M)(2\phi_0\sin\phi_0) - 4Hr(\sin^2\phi_0$$

$$- \phi_0\sin\phi_0\cos\phi_0) - (M - M')(\cos\phi_0 - \frac{\phi_0}{\sin\phi_0}$$

$$- \tfrac{1}{2}\sum_0^l Wr\left[2\phi_0\left(\frac{\sin\alpha}{\sin\phi_0} - 2\sin^2\phi_0\right) + 2\alpha(2\sin\phi_0\sin\alpha - 1)\right.$$

$$- 4\sin\phi_0\cos\phi_0 + 2\sin\alpha\cos\phi_0 - 2\sin\alpha\cos\alpha$$

$$+ 4\cos\alpha\sin\phi_0]\} - \frac{1}{2EA}\left\{(M - M')\left(\cos\phi_0 - \frac{\phi_0}{\sin\phi_0}\right)\right.$$

$$\left. + \tfrac{1}{2}\sum_0^l Wr\left[\frac{2\phi_0}{\sin\phi_0}\sin\alpha - 2\alpha - 2\sin\alpha(\cos\alpha - \cos\phi_0)\right]\right\}. \qquad (13)$$

SPECIAL CASES OF SYMMETRICAL ARCHES.

1°. *Three-hinged Arch.*—In this case we do not need these equations to find the horizontal thrust: the proper ones can be used subsequently if we wish the deflections or slopes.

2°. *Arch hinged at the two springing-points.*—In this case $M = M' = 0$; and by making these substitutions in (12) and solving for H, we obtain

$$H = \frac{\begin{array}{l}\Sigma W[2\alpha\cos\phi_0\sin\alpha + 2\cos\phi_0\cos\alpha - 2\phi_0\sin\phi_0\cos\phi_0 \\ \quad + \sin^2\phi_0 - \sin^2\alpha - 2\cos^2\phi_0] \\ \quad - \frac{I}{Ar^2}\Sigma W(\sin^2\phi_0 - \sin^2\alpha) - \frac{2EI}{r^3}(\Delta l - \epsilon\tau l)\end{array}}{4\phi_0\cos^2\phi_0 - 6\sin\phi_0\cos\phi_0 + 2\phi_0 + \frac{I}{Ar^2}(2\phi_0 + 2\sin\phi_0\cos\phi_0)}. \qquad (14)$$

This formula gives the thrust when the value of Δl is known, i.e., the amount of relative yielding of the supporting points.

If the abutments do not yield at all, then $\Delta l = 0$, and that term should be omitted from the numerator; so, also, if we neglect a consideration of the temperature, then $\epsilon\tau l$ vanishes in addition.

Formula (14) gives the thrust for a set of concentrated loads, each equal to W.

For a distributed load, we should substitute for W, wdx, and integrate between the proper limits, w being the intensity of the load per unit of horizontal length, and being constant or variable according to the distribution of the load.

The formula for the thrust in the case where the load is uniformly distributed horizontally (i.e., when w is a constant) and when it covers the entire arch will now be given, but will not be worked out here, as it is easily obtained from (14).

In this formula the letters have the same meanings as heretofore, and we use also $A_1 = \int_0^l y dx =$ area of segment of arch; and let $m = \dfrac{I}{Ar^2}$.

The formula is as follows:—

$$H = w \frac{\left(1 - \dfrac{I}{Ar^2}\right)\dfrac{l^3}{6} + k\left\{2A_1 + \dfrac{r^2}{2}\sin 2\phi_0 - \phi_0\,(l^2 + r^2\cos 2\phi_0)\right\}}{(4k^2 + 2r^2)\phi_0 - 3kl + m\,(2r^2\phi_0 + kl)}. \quad (15)$$

In a similar way formulæ are easily obtained for the thrust when half or a quarter, or some other portion of the arch, is loaded.

3°. *Arch with no hinges.*—In this case, if we know $\Delta\phi$, Δl, and Δc, or if these are zero, we can obtain from (11), (12), and (13) the three quantities M, M', and H, and then the solution of the arch follows.

This will not be done here, however, as the arches usually built are of the other kinds.

EXAMPLES.

1. Given a semicircular arch jointed at each springing-joint and at the crown, radius r. Trace out the effect of a single load W acting upon it at the extremity of a radius making 45° with the horizontal.

Solution.

The presence of three joints gives us the bending-moments at each of these joints equal to zero, the co-ordinates of these joints being respectively $(0, 0)$, (r, r), and $(2r, 0)$.

Hence, using equation (4), we obtain

1°. $M = 0$,
2°. $Vr - Hr - W(0.70711r) = 0$,
3°. $V(2r) - W(1.70711r) = 0$.

Solving, we have, therefore,

$$V = 0.85355\,W = \text{left-hand supporting-force},$$

and

$$H = 0.14645\,W = \text{horizontal component of thrust.}$$

Hence $V_1 = 0.14645\,W =$ right-hand supporting-force.

Hence, for a section whose co-ordinates are (x, y),

$$x < 0.29289r, \qquad V_x = 0.85355\,W;$$
$$x > 0.29289r, \qquad V_x = -0.14645\,W.$$

Hence equation (1) gives, for

$$x < 0.29289r, \qquad R_x = W\sqrt{(0.85355)^2 + (0.14645)^2} = 0.86602\,W,$$
$$x > 0.29289r, \qquad R_x = W\sqrt{(0.14645)^2 + (0.14645)^2} = 0.20711\,W.$$

Now, the angle made by R_x with the horizontal is, for

$$x < 0.29289r, \qquad \alpha_1 = \tan^{-1}\left(\frac{0.85355}{0.14645}\right) = 80°\ 15'\ 51'',$$

$$x > 0.29289r, \qquad \alpha_1 = \tan^{-1}\left(\frac{0.14645}{0.14645}\right) = 45°.$$

Knowing, now, the angle made by R_x with the horizontal, we can find, for the point (x, y), the angle made by a tangent to the circle with the horizon, or $\alpha_2 = \tan^{-1}\left(\frac{r - x}{y}\right)$. Then resolve R_x into two components, respectively tangent to the arch and normal to it at the point x, y, and the tangential component is the direct thrust T_x, while the normal is the shearing-force S_x.

Then, for the bending moment, we have, from (4),

$$x < 0.29289r, \quad M_x = 0.85355\,Wx - 0.14645\,Wy;$$

$$x > 0.29289r, \quad M_x = 0.85355\,Wx - 0.14645\,Wy - W(x - 0.29289r).$$

Hence we determine the direct thrust, the shearing-force, and the bending-moment at any section, and can hence obtain the stresses at all points.

2. Given the same arch with a load W distributed uniformly over the circular arc, find stresses at all points.

3. Given the same arch jointed only at the two springing-points, find stresses at all points.

§ 273. **Position of True Line of Resistance in a Stone Arch.** — The proof will now be given of the proposition already referred to in regard to the position of the true line of resistance; viz., —

"For an arch of constant section, that line of resistance is approximately the true one which lies nearest to the axis of the arch-ring, as determined by the method of least squares."

PROOF. — If we denote by y the ordinate of the axis of the arch for an abscissa x, and by μ that of the line of resistance for the same abscissa, then $\mu - y$ is the vertical distance between the two curves for abscissa x. Now, the condition that the line of resistance should be as near the arch-ring as possible, is, that the sum of the $(\mu - y)^2$ shall be a minimum, or

$$\int (\mu - y)^2 ds = \text{minimum.} \qquad (1)$$

But (x, μ) are the co-ordinates of the point of application of the actual thrust, and hence $(\mu - y)$ is the distance of the point at which the resultant thrust acts from the centre of gravity of the section. Hence we have

$$\mu - y = \frac{M_x}{H}.$$

Hence (1) becomes

$$\int \left(\frac{M_x}{H}\right)^2 ds = \text{minimum.} \qquad (2)$$

But H is constant for the same line of resistance, though it varies for different lines: hence we can place H outside of the integral sign. Hence we may write

$$u = \frac{1}{H^2}\int M_x{}^2 ds = \text{minimum.} \qquad (3)$$

Now, from (4), § 272, we have

$$M_x = M + Vx - Hy - \Sigma_0{}^x W(x - a) = \phi(M, V, H);$$

M, V, and H being constants for the same line of resistance, but varying for different lines. Hence, by differentiating (3), we have

$$\frac{du}{dM} = \frac{du}{dM_x}\frac{dM_x}{dM} = \frac{2}{H^2}\int M_x ds = 0 \qquad \therefore \quad \int M_x ds = 0, \qquad (4)$$

$$\frac{du}{dV} = \frac{du}{dM_x}\frac{dM_x}{dV} = \frac{2}{H^2}\int M_x x ds = 0 \qquad \therefore \quad \int M_x x ds = 0 \qquad (5)$$

$$\frac{du}{dH} = \frac{du}{dM_x}\frac{dM_x}{dH} = -2H^{-3}\int M_x{}^2 ds - 2H^{-2}\int M_x y ds$$

$$= -H^{-1}\left\{\frac{2}{H^2}\int M_x{}^2 ds + 2\frac{1}{H}\int M_x y ds\right\} = 0.$$

But the first term must be very small: hence we may write approximately,

$$\int M_x y ds = 0. \qquad (6)$$

Now, the three expressions (4), (5), and (6) are identical with (5), (6), and (7) of § 272; and the conditions that these shall be zero are, with the degree of approximation there stated, the conditions that hold in the case of an arch fixed in direction at the ends. Hence it follows that the condition that the line of resistance shall fall as near the centre of the arch as possible is the condition which, in an elastic arch fixed in direction at the ends, gives us its true position. Hence it would seem that the most probable position for the true line of resistance is the nearest possible to the axis of the arch.

This is the conclusion reached by Winkler; and a more detailed discussion of the matter is to be found in an article by Professor Swain in "Van Nostrand's" for October, 1880.

§ 274. **Domes.** — The method to be used for determining the stability of a dome differs essentially from that used in the case of an arch, for there is no thrust at the crown in a dome. Indeed, the most general case is that of the dome open at the top. we will, therefore, consider this case first in studying the action of the forces required to preserve equilibrium.

Fig. 293 shows a meridional section of an open dome. Suppose that this dome had been entirely built, except the upper ring-course of stones, represented by *LKGH*. Then, suppose that one of the stones only of this course were placed in position without any auxiliary support, its own weight would evidently overturn it, since the line *ab*, along which the weight acts, does not cut the joint; but, if the whole ring-course is put in place, the stones keep each other in position. The way in which this is accomplished is as follows: they press laterally against each other; and the resultant of the

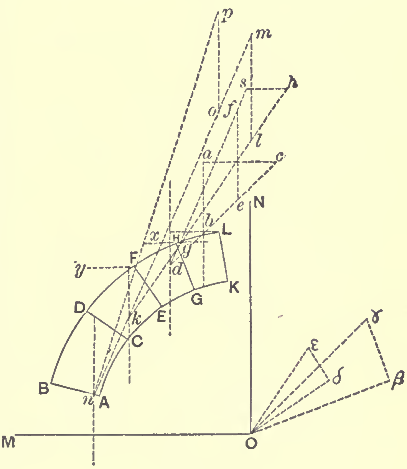


Fig. 293.

pressures exerted upon the two lateral faces of any one stone by the other stones of the course is a horizontal radial force, which, combined with the weight of the stone, gives, as the resultant of the two, a force which cuts the joint between G and H. Moreover, sufficient pressure will be developed to accomplish this result, as a failure to reach the result will only increase the pressure upon the lateral faces.

Moreover, if, when sufficient pressure has been developed to bring the resultant of the weight of the stone and the above-described horizontal radial force within the joint, it should make an angle with the normal to the joint greater than the angle of friction, the tendency of the stone to slide will increase the lateral pressure, and this in turn will increase the outward horizontal force till the angle made by the resultant with the normal to the joint is no greater than the angle of friction of the material of the voussoirs.

This will be made plain by reference to the figure (Fig. 293), where ab represents the weight of the stone $HLKG$, and where $O\beta$ is perpendicular to HG and $O\gamma$ is drawn so that $\gamma O\beta = \phi$, the angle of friction. Now, since ab produced passes outside of HG, horizontal thrust must be developed. And, moreover, were only sufficient horizontal thrust furnished to make the resultant cut HG at G, the angle between this resultant and the normal to the joint would be greater than ϕ; therefore we proceed as follows: assuming the horizontal thrust to act through L, the upper edge of the stone, we lay off from b, the intersection of the horizontal through L with a vertical line drawn through the centre of gravity of the stone, the weight ab to scale, then from b draw bc parallel to $O\gamma$, and draw through a a horizontal line to meet bc. Then will ac be the horizontal force that will be furnished by the other stones of the course to keep this stone in place; and the pressure upon joint HG is bc, and acts at the intersection of bc and HG.

Now prolong bc to meet the vertical drawn through the centre of gravity of the next stone, *HGFE*, at *d*. Combine it with the weight of this stone; this is done by laying off $de = bc$, and from e drawing ef vertical, and equal to the weight of *FHGE*. The resultant fd makes an angle with the normal to *FE* greater than ϕ: hence draw $O\delta$ perpendicular to *FE* and $O\epsilon$, so that $\epsilon O\delta = \phi$; then from g, the intersection of df with a horizontal line through *H*, the top of *FHGE*, lay off $gs = df$, through g draw gh parallel to $O\epsilon$, and through s draw sh horizontal. Then is sh the horizontal thrust that will be furnished at *H* to keep the stone *HGEF* in place; and hg is the pressure upon joint *FE*, and acts at the intersection of *FE* with hg.

Next, prolong hg to meet the vertical through the centre of gravity of stone *FEDC* at k; lay off $kl = gh$, and from l lay off lm = weight of stone *FEDC*; draw km, which cuts the joint within the joint itself, and needs no horizontal thrust to bring it inside; hence mk is the pressure on joint *DC*.

Then draw mk to meet the vertical through the centre of gravity of *ABCD* at n, and lay off $no = km$; draw op = weight of *ABCD*, and draw pn, which will be the pressure on the joint *BA*.

It is necessary, for stability, that all these forces should cut the joint inside of the joint if the stones are reckoned incompressible; or we may adopt the middle third, or other limits, as our criterion of stability.

As long as it is outward thrust that is required to produce stability, it is possible to furnish it; but, if we should reach a joint where inward thrust would be required, this could not be furnished, and the dome would be unstable. Moreover, the resultant pressure on the springing gives us the pressure exerted upon the support of the dome; and it must not cut any joint of the support outside of that joint, as otherwise the support would not stand.

In determining the numerical value and direction of this pressure on the support, we may either construct it graphically, or we may compute it as follows: (1°) Compound all the vertical forces, i.e., the weights, and find the magnitude and line of action of the resultant of these. (2°) Compound all the horizontal forces, and find the magnitude and line of action of their resultant (in this case the horizontal forces are two; viz., ac applied at L, and sh applied at H); then compound these two resultants. The graphical and analytical method should check if no mistake has been made in the work.

In the above calculation, it has been assumed that the figure represents the portion of a dome included between two meridional planes.

If we desire to ascertain the pressure exerted upon the lateral face of the stone by its neighbors in the same ring-course, we only need to know the angle made by the two meridional planes containing the lateral faces of the stone in question, then resolve the horizontal thrust upon that stone into two equal components, which make with each other an angle equal to the supplement of the angle of the planes; i.e., resolve the outward horizontal thrust into two components normal to the lateral faces.

In regard to the assumption that the outward thrust acts at the top of the stone, it should be said that this is Scheffler's custom, his reason being that less thrust will be required if he assumes it at the top than if he assumes it nearer the middle. The true position of this thrust is probably much nearer the middle of the stone.

An example will next be solved, giving Scheffler's method of working.

Example. — Given the dome shown in the figure, surmounted by a lantern at the top; determine whether it is stable, and what should be the thickness of the support in order that the resultant pressure may not pass outside any joint of the pier.

The dimensions are as follows: —

Diameter of outer vertical circle = 20 feet.

Diameter of inner vertical circle = 18 feet.

Angle made by springing-radius with vertical = 75° = angle *AOB*.

The inner edge of the upper voussoir subtends 18° on the lower circle; the width of the load of the lantern is 0.6; the voussoirs below that, each subtend 18°.

Assume 36 stones in a horizontal course. The width of the lowest will, then, be 1.51; the width of the others are determined from their lever arms.

Given height of pier = 8 feet.

Height of the centre of the sphere above base of pier = 8′ — 10 sin 15° = 5.41′.

The figure may be taken to represent the portion of the dome included between two vertical planes passing through the axis of the dome: hence it shows one vertical series of stones.

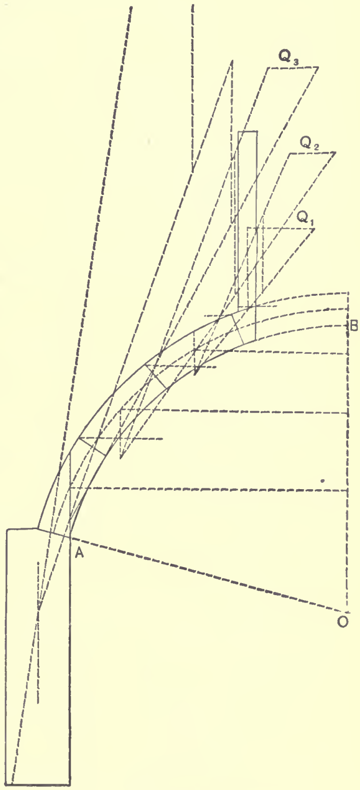


Fig. 294.

We first construct a table giving the weights of the different voussoirs with any superincumbent load, their centres of gravity, and the moments of their weights about an axis passing through *O*, and perpendicular to the central plane of the portion shown; and we so choose our unit of weight that the volumes of the voussoirs shall represent their weights. The work is arranged as follows :—

Elementary Forces.						Horizontal Forces.		
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
No. of Voussoir.	Area of Lateral Face.	Thickness.	Product.	Lever Arms.	Moment.	Horizontal Forces.	Lever Arms.	Moment.
1	0.6×6.680	0.53	2.124	3.07	6.521	1.74	9.60	16.104
2	2.985	0.74	2.209	4.75	10.493	1.26	9.33	11.756
3	2.985	1.23	3.672	7.05	25.888	1.32	7.78	10.273
4	2.985	1.51	4.507	8.68	39.121	–	–	–
–	–	–	12.512	–	82.023	4.32	–	38.730

Column (1) contains the numbers of the voussoirs, counting from the top.

Column (2) contains the areas of the lateral faces of the stones shown in the figure. For the three lower stones, the area of a ring subtending 18° at the centre, and of the dimensions given, is calculated. For the first, the height is 6.68 and the width 0.6.

Column (3) contains the thicknesses of the voussoirs; i.e., the length of arc between their two lateral faces measured on a horizontal circle through the centre of gravity of the voussoir, which is here taken at the middle point of the arc subtended by this voussoir on its middle vertical circle, i.e., one which has a radius 9.5 feet.

Hence, the thickness of the lower stone being 1.51 feet, that of the others will be

$$(1.51)\frac{3.07}{8.68} = 0.53, \qquad (1.51)\frac{4.25}{8.68} = 0.74,$$

$$(1.51)\frac{7.05}{8.68} = 1.23.$$

Column (4) gives the weights of the voussoirs and their loads: it is obtained by multiplying together the numbers in columns (2) and (3).

Column (5) gives the distances of the centres of gravity of the different voussoirs from the axis of the dome: it may be determined graphically or by calculation.

Column (6) gives the moments of the weights about a horizontal axis through O perpendicular to the central plane of this series of voussoirs. The graphical construction for determining the horizontal thrusts required is next made, and the results are recorded in column (7). It will be seen that no thrust is required on voussoir No. 4.

Column (8) contains the lever arms of these forces about the same axis.

Column (9) contains their moments about the same axis.

The construction thus far has shown no case where horizontal tension instead of horizontal thrust is required to cause the thrust on any joint to pass within the joint: hence thus far the dome is stable; and the question comes next as to what should be the width of the pier in order that the line of resistance, if continued down, may remain within it.

For this purpose we proceed as follows: —

Let t = thickness required.

Let breadth be equal to that of the lowest voussoir.

Height = 8 feet.

Take moments about the outer edge of the base of the pier.

We shall then have, —

1°. Moment of vertical load on dome, and of weight of dome sector about inner edge of springing, =

$$(12.512)(8.69 - 6.56) = 26.52.$$

2°. Moment of same about outer edge of springing of pier =

$$26.52 + (12.512)t.$$

3°. Moment of horizontal forces about the same axis =

$$38.730 + (4.32)(5.41) = 62.101.$$

4°. Moment of weight of pier about outer edge =

$$\{8(1.51)t\}\tfrac{1}{2}t = 6.04t^2.$$

Hence we have

$$6.04t^2 + 12.51t + 26.52 = 62.101$$

$$\therefore \quad t^2 + 2.07t = 5.89 \qquad \therefore \quad t = 1.60 \text{ feet.}$$

This is the thickness required in order that the line of resistance may remain within the lower joint.

If, on the other hand, while pursuing the same method with the dome itself, we require that the line of resistance shall remain within the middle third of the pier, we take moments about a point in the springing of the pier at a distance $\frac{2}{3}t$ from its inner edge, we should then have

$$\tfrac{2}{3}t^2 + \tfrac{2}{3}(2.07)t = 5.89$$

$$\therefore \quad t^2 + 2.07t = 8.84 \qquad \therefore \quad t = 2.10 \text{ feet.}$$

On the other hand, we could proceed in a similar way to the above, if we desired to keep the line of resistance in the dome within the middle third, by merely assuming the horizontal thrusts to act at two-thirds the thickness of a joint from the lower edge, and using a point two-thirds the thickness from the top, instead of the lower edge, as the lower limiting-point for the pressure to pass through.

This will not be done here, however.

Example. — As an example, St. Peter's dome will be given, with the dimensions as given by Scheffler reduced to English measures. The dome consists in its upper part, as will be evident from the figure, of two domes; the lantern resting on the two is assumed to have one-third of its weight resting on the upper, and two-thirds on the lower dome.

Diameter of dome = diameter at the base = 144 feet.

Up to a point 28.48 feet above the point *C* it is formed of a single dome 11.84 feet thick. In its upper part, on the other hand, it is composed of two domes whose normal distance apart is 5.15 feet; the exterior having a thickness of 2.56 feet, and the inner of 4.13 feet at the top and 5.15 feet at the springing. At the top of these two domes is an opening 12.24 feet radius, surmounted by a cylindrical lantern. The magnitude of the load of the lantern on the dome is represented on the figure by 1.82 feet width and 56.66 feet height.

Height of the entablature $ABCD = 23.69$.

Width of $ABCD$ normal to plane of paper = 1.02 feet.

Thickness of $ABCD = 10.30$ feet.

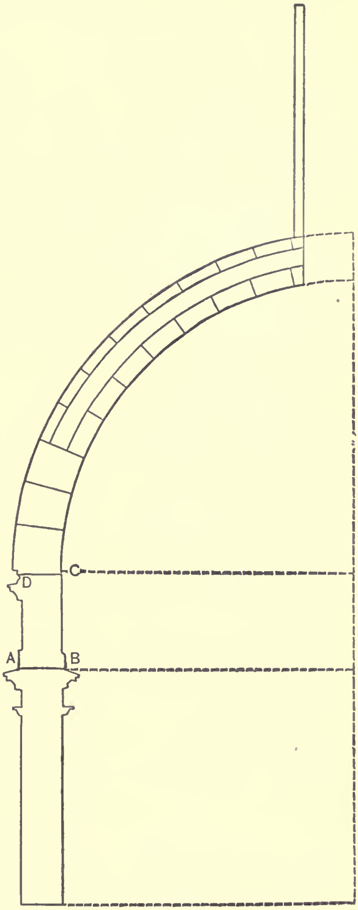


FIG. 295.

Divide the exterior dome into nine parts, the interior into eight of a uniform circumferential width of 10.08 feet, except the first, which has a width of only 1.82 feet.

Determine whether this thickness of $ABCD$ is sufficient to keep the line of resistance within joint AB.

CHAPTER X.

THEORY OF ELASTICITY, AND APPLICATIONS.

§ 275. **Strains.** — When a body is subjected to the action of external forces, and in consequence of this undergoes a change of form, it will be found that lines drawn within the body are changed, by the action of these external forces, in length, in direction, or in both; and the entire change of form of the body may be correctly described by describing a sufficient number of these changes.

If we join two points, A and B, of a body before the external forces are applied, and find, that, after the application of the external forces, the line joining the same two points of the body has undergone a change of length $\Delta(AB)$, then is the limit of the ratio $\frac{\Delta(AB)}{AB}$, as AB approaches zero, called the *strain* of the body at the point A in the direction AB.

If $AB + \Delta(AB) > AB$, the strain is one of tension; whereas, if $AB + \Delta(AB) < AB$, the strain is one of compression.

In order to study the changes of form of the body, let us assume a point O within the body when there are no external forces acting, and let us draw through this point three rectangular axes, OX, OY, and OZ, and assume a small rectangular parallelopipedical particle whose three edges are OA, OB, and

OC, and let us examine the form of this particle after the loads are applied; it will be found that the edges OA, OB, and OC will be of different lengths from what they were before, and that the angles AOB, AOC, and BOC will no longer be right angles, but will differ slightly from 90°. Let the parallelopiped $oabc$-$gdef$ represent the form and dimensions of the particle after the external forces are applied. Then we shall have,

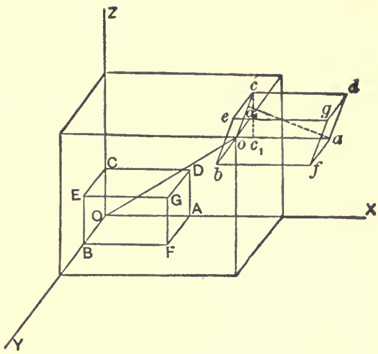


FIG. 296.

if ϵ_x, ϵ_y, and ϵ_z represent the strains in the directions OX, OY, and OZ respectively, that

$$\epsilon_x = \text{limit of } \frac{\text{proj. } og \text{ on } OX - OA}{OA} \text{ as } OA \text{ approaches zero,}$$

$$\epsilon_y = \text{limit of } \frac{\text{proj. } og \text{ on } OY - OB}{OB} \text{ as } OB \text{ approaches zero,}$$

$$\epsilon_z = \text{limit of } \frac{\text{proj. } og \text{ on } OZ - OC}{OC} \text{ as } OC \text{ approaches zero.}$$

In the figure, ϵ_x and ϵ_z are tensile strains, and ϵ_y is a compressive strain.

But these strains do not represent completely the distortion of the particle; for the plane $CEGD$ has slid by the plane $OABF$ through the distance oc_1, the distance apart of these planes being OC, and the plane halfway between the two has slid just half as far, so that the amount of shearing, or the *shearing-strain* of planes parallel to XOY in the direction OX, may be represented by $\frac{oc_1}{OC} = \frac{oc_1}{cc_1}$ nearly, or the distortion divided by the distance apart of these planes. This, moreover, is the tangent of the angle occ_1, or the tangent of the angle by which aoc differs from a right angle.

If, now, we let

γ_{zx} = shearing-strain in a plane perpendicular to OZ in the direction OX,

γ_{zy} = shearing-strain in a plane perpendicular to OZ in the direction OY,

γ_{yx} = shearing-strain in a plane perpendicular to OY in the direction OX,

γ_{yz} = shearing-strain in a plane perpendicular to OY in the direction OZ,

γ_{xz} = shearing-strain in a plane perpendicular to OX in the direction OZ,

γ_{xy} = shearing-strain in a plane perpendicular to OX in the direction OY,

and let $boc = \frac{\pi}{2} - \phi$, $aoc = \frac{\pi}{2} - \psi$, $aob = \frac{\pi}{2} - \chi$, then we shall have

$$\gamma_{zx} = \frac{oc_1}{cc_1} = \tan\psi, \qquad \gamma_{yz} = \tan\phi,$$

$$\gamma_{zy} = \tan\phi, \qquad \gamma_{xz} = \tan\psi,$$

$$\gamma_{yx} = \tan\chi, \qquad \gamma_{xy} = \tan\chi.$$

We thus have

$$\gamma_{zy} = \gamma_{yz} = \tan\phi,$$

$$\gamma_{xz} = \gamma_{zx} = \tan\psi,$$

$$\gamma_{xy} = \gamma_{yx} = \tan\chi,$$

three very important equations.

We thus have to determine six strains, in order to define completely the state of strain in a body at a given point; viz., if we assume three rectangular axes, we must know ϵ_x, ϵ_y, ϵ_z, $\gamma_{zy} = \gamma_{yz}$, $\gamma_{zx} = \gamma_{xz}$, $\gamma_{xy} = \gamma_{yx}$, three normal and three tangential strains.

§ 276. **Strains in Terms of Distortions.**—For the sake of clearness, we will consider first only the strains that are parallel to the z plane; hence, will use only two co-ordinate axes, OX and OY, as shown in Fig. 297. In this case let us assume a small rectangular particle, *acbd*, the co-ordinates of one corner of which are x, y, and of the other, $x + dx$, $y + dy$; this being

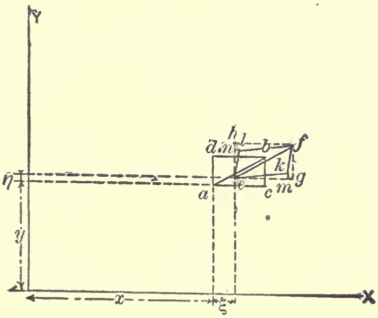


FIG. 297.

the case before the load is applied. Let the effect of the load be to move the point a to e, and b to f, transforming the rectangle *acbd* into *ekfl*, and thus changing x, y respectively into $x + \xi$, $y + \eta$, and changing $x + dx$, $y + dy$ respectively into $x + \xi + dx + d\xi$, $y + \eta + dy + d\eta$. Then are dx, dy the sides of the particle before the load is applied. Then from what has preceded we shall have

$$\epsilon_x = \frac{d\xi}{dx}; \qquad \epsilon_y = \frac{d\eta}{dy};$$

$$\gamma_{xy} = \gamma_{yx} = \frac{d\xi}{dy} + \frac{d\eta}{dx}.$$

The first two are evident at once. To prove the third, observe that the shearing-strain, γ_{xy} is the tangent of the angle by which the angle *kel* differs from a right angle; hence it is the tangent of the sum of the angles *kem* and *len*. Now, since these angles are small, we may take the sum of the tangents as nearly equal to the tangent of the sum. But

$$\tan kem = \frac{km}{em} = \frac{d\eta}{dx} \text{ nearly, and}$$

$$\tan len = \frac{ln}{en} = \frac{d\xi}{dy} \text{ nearly.}$$

Hence

$$\gamma_{xy} = \gamma_{yx} = \frac{d\xi}{dy} + \frac{d\eta}{dx}.$$

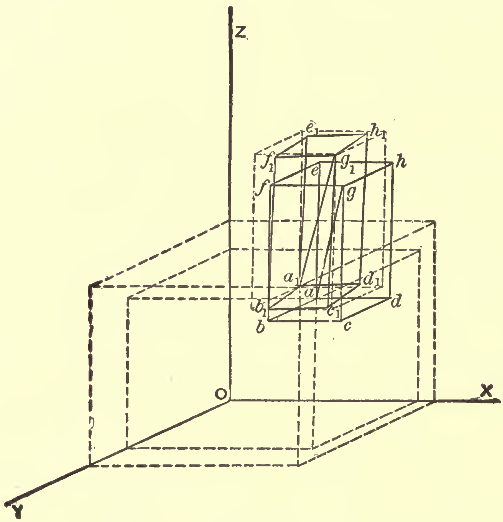


FIG. 298.

In the general case, Fig. 298, a rectangular parallelopipedical particle, the co-ordinates of one corner of which are x, y, z, and of the other, $x + dx$, $y + dy$, $z + dz$; this being the case before the load is applied.

Let the effect of the load be to change x, y, z, respectively, into $x + \xi$, $y + \eta$, $z + \zeta$, and to change $x + dx$, $y + dy$, $z + dz$, into $(x + \xi) + (dx + d\xi)$, $(y + \eta) + (dy + d\eta)$, $(z + \zeta) + (dz + d\zeta)$. Then are dx, dy, dz, the edges of the particle before the load is applied.

Then, from what has preceded, we shall have

$$\epsilon_x = \frac{d\xi}{dx}, \qquad \epsilon_y = \frac{d\eta}{dy}, \qquad \epsilon_z = \frac{d\zeta}{dz};$$

$$\gamma_{xy} = \gamma_{yx} = \frac{d\xi}{dy} + \frac{d\eta}{dx}, \quad \gamma_{xz} = \gamma_{zx} = \frac{d\xi}{dz} + \frac{d\zeta}{dx}, \quad \gamma_{yz} = \gamma_{zy} = \frac{d\eta}{dz} + \frac{d\zeta}{dy}.$$

The first three will be evident at once. As to the last three, the proof is similar to that just used in the case of two co-ordinate axes.

$$\frac{d\xi}{dy} + \frac{d\eta}{dx} = \tan\chi, \quad \frac{d\xi}{dz} + \frac{d\zeta}{dx} = \tan\psi, \quad \frac{d\eta}{dz} + \frac{d\zeta}{dy} = \tan\phi.$$

§ 277. **Determination of the Strain in any Given Direction.** — Suppose we are required, knowing the strains ϵ_x, ϵ_y, ϵ_z, γ_{xy}, γ_{xz}, γ_{yz}, to determine the strain in a direction making angles α, β, γ, with OX, OY, OZ respectively. Assume our rectangular parallelopipedical particle in such a way that the diagonal from (x, y, z) to $(x + dx, y + dy, z + dz)$ shall be in the required direction, and call the length of this diagonal ds (Fig. 298); then we shall have

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2, \qquad (1)$$

$$\cos\alpha = \frac{dx}{ds}, \qquad (2)$$

$$\cos\beta = \frac{dy}{ds}, \qquad (3)$$

$$\cos\gamma = \frac{dz}{ds}. \qquad (4)$$

Let ϵ be the strain in the required direction; then length of diagonal after load is applied will be

$$ds(1 + \epsilon),$$

and we shall have

$$(ds)^2(1 + \epsilon)^2 = (dx + d\xi)^2 + (dy + d\eta)^2 + (dz + d\zeta)^2,$$

or

$$(ds)^2 + 2\epsilon(ds)^2 + \epsilon^2(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 + 2(dxd\xi + dyd\eta + dzd\zeta) + (d\xi)^2 + (d\eta)^2 + (d\zeta)^2. \qquad (5)$$

Now, subtracting (1) from (5), and neglecting $\epsilon^2(ds)^2$, $(d\xi)^2$, $(d\eta)^2$, and $(d\zeta)^2$ as being very small compared with the rest, we have

$$2\epsilon(ds)^2 = 2dxd\xi + 2dyd\eta + 2dzd\zeta$$

$$\therefore \quad \epsilon ds = \frac{dx}{ds}d\xi + \frac{dy}{ds}d\eta + \frac{dz}{ds}d\zeta, \qquad (6)$$

or

$$\epsilon ds = d\xi\cos\alpha + d\eta\cos\beta + d\zeta\cos\gamma. \qquad (7)$$

But

$$d\xi = \frac{d\xi}{dx}dx + \frac{d\xi}{dy}dy + \frac{d\xi}{dz}dz, \qquad (8)$$

$$d\eta = \frac{d\eta}{dx}dx + \frac{d\eta}{dy}dy + \frac{d\eta}{dz}dz, \qquad (9)$$

$$d\zeta = \frac{d\zeta}{dx}dx + \frac{d\zeta}{dy}dy + \frac{d\zeta}{dz}dz. \qquad (10)$$

Hence, substituting these, we have, after dividing by ds, and observing (2), (3), and (4),

$$\epsilon = \frac{d\xi}{dx}\cos^2\alpha + \frac{d\eta}{dy}\cos^2\beta + \frac{d\zeta}{dz}\cos^2\gamma + \left(\frac{d\eta}{dz} + \frac{d\zeta}{dy}\right)\cos\beta\cos\gamma$$
$$+ \left(\frac{d\xi}{dz} + \frac{d\zeta}{dx}\right)\cos\alpha\cos\gamma + \left(\frac{d\eta}{dx} + \frac{d\xi}{dy}\right)\cos\alpha\cos\beta; \qquad (11)$$

or, making use of § 276, we have

$$\epsilon = \epsilon_x\cos^2\alpha + \epsilon_y\cos^2\beta + \epsilon_z\cos^2\gamma + \gamma_{yz}\cos\beta\cos\gamma$$
$$+ \gamma_{xz}\cos\alpha\cos\gamma + \gamma_{xy}\cos\alpha\cos\beta, \qquad (12)$$

which gives us the strain in any direction.

It can be shown that there are three directions, at right angles to each other, that give the maximum strains or minimum strains: and we might deduce the ellipsoid of strains, in which semi-diameters of the ellipsoid represent the strains; but we will pass on to the consideration of the stresses.

§ 278. **Stresses.** — When a body is subjected to the action of external forces, if we imagine a plane section dividing the body into two parts, the force with which one part of the body acts upon the other at this plane is called the *stress* on the plane; and, in order to know it completely, we must know its distribution and its direction at each point of the plane. If we consider a small area in this plane, including the point O, and represent the stress on this area by p, whereas the area itself is represented by a, then will the limit of $\frac{p}{a}$, as a approaches zero, be the intensity of the stress on the plane under consideration at the point O. Observe that we cannot speak of the stress at a certain point of a body unless we refer it to a certain plane of action: thus, if a body be in a state of strain, we do not attempt to analyze all the molecular forces with which any one particle is acted on by its neighbors: but, when we assume a certain plane of section through the point, the stress on this plane at the point becomes recognizable in magnitude and direction; and what the magnitude and direction of the stress at the given point is, depends upon the direction of the plane section chosen, the magnitude and direction differing for different plane sections through the point.

§ 279. **Simple Stress.** — A simple stress is merely a pull or a thrust. Assume a prismatic body, with sides parallel to OX, subjected to a pull in the direction of its length; the magnitude of the pull being P. Assume first a plane section AA normal to the direction of P, and let area of AA be A. Then, if p_x represent the intensity of stress at any point of this plane,

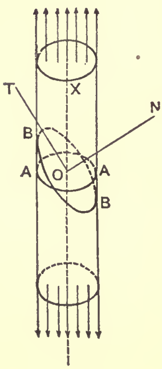


FIG. 299.

$$p_x = \frac{P}{A}.$$

This, which is the intensity of the stress as distributed over a plane normal to its direction, may be called its normal intensity.

On the other hand, if we desire to ascertain the intensity of the stress on the oblique plane BB, making an angle θ with AA, we shall have

$$\text{Area } BB = \frac{A}{\cos\theta}.$$

Hence, if p_r represent the intensity of the stress on this plane in the direction OX, we shall have

$$p_r = \frac{P}{\left(\dfrac{A}{\cos\theta}\right)} = \frac{P}{A}\cos\theta = p_x \cos\theta. \tag{1}$$

If we resolve this into two components, acting respectively nor mal and tangential to BB, and if we denote the normal intensity by p_n, and the tangential by p_t, we shall have

$$p_n = p_r \cos\theta = p_x \cos^2\theta, \qquad (2)$$

$$p_t = p_r \sin\theta = p_x \cos\theta \sin\theta. \qquad (3)$$

If, now, we assume another oblique plane section, perpendicular to the first, we shall obtain the normal p_n' and the tangential p_t' stress on this plane by substituting for θ, $\frac{\pi}{2} - \theta$; hence we obtain

$$p_n' = p_x \sin^2\theta, \qquad (4)$$

$$p_t' = p_x \cos\theta \sin\theta. \qquad (5)$$

Hence follows

$$p_t' = p_t;$$

or, the tangential components of a simple stress on a pair of planes at right angles to each other are equal.

§ 280. **Compound Stress.** — A compound stress may be accounted to be the resultant of a set of simple stresses, and may be analyzed into different groups of simple stresses.

Proposition. — *Whatever be the external forces applied to a body, if through any point we pass three planes of section at right angles to each other, the tangential components of the stress on any two of these planes in directions parallel to the third must be of equal intensity.*

To prove this proposition, assume three rectangular axes, origin at O, and assume a rectangular parallelopipedical particle, as shown in the figure, so small that we may without appreciable error assume the stress on any one of the faces to be the same as that on the opposite face; resolve these stresses,

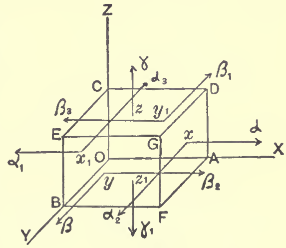


Fig. 300.

i.e., the forces exerted upon the faces of the particle by the other parts of the body, into components parallel to the axes.

Let σ_x = intensity of normal stress on the x plane,

σ_y = intensity of normal stress on the y plane,

σ_z = intensity of normal stress on the z plane,

τ_{xy} = intensity of shearing-stress on x plane in direction OY,

τ_{xz} = intensity of shearing-stress on x plane in direction OZ,

τ_{yx} = intensity of shearing-stress on y plane in direction OX,

τ_{yz} = intensity of shearing-stress on y plane in direction OZ,

τ_{zx} = intensity of shearing-stress on z plane in direction OX,

τ_{zy} = intensity of shearing-stress on z plane in direction OY.

We have thus apparently nine stresses, which must be given, in order to define the stress at the point O completely; but we will now proceed to prove that

$$\tau_{xy} = \tau_{yx}, \quad \tau_{xz} = \tau_{zx}, \quad \tau_{zy} = \tau_{yz}.$$

In the figure, the only ones of these stresses that are represented are the following: —

$$\begin{aligned} x\alpha &= x_1\alpha_1 = \sigma_x, \\ y\beta &= y_1\beta_1 = \sigma_y, \\ x\alpha_2 &= x_1\alpha_3 = \tau_{xy}, \\ y\beta_2 &= y_1\beta_3 = \tau_{yx}, \\ z\gamma &= z_1\gamma_1 = \sigma_z. \end{aligned}$$

The other four are omitted, in order not to complicate the figure.

Now, it is evident that the total normal force on the face $AFGD$ and the normal force on the face $OBEC$ balance each other independently, and likewise with the other normal forces.

The only forces tending to cause rotation around OZ are the equal and opposite parallel forces τ_{xy} (area $AFGD$), one acting on the face $AFGD$, and the other on the face $OBEC$; and the equal and opposite forces τ_{yx} (area $FBEG$), one acting on the face $FBEG$, and the other on the face $COAD$.

The first pair forms a couple whose moment is τ_{xy} (area $AFGD$) (xx_1), and the second has the moment τ_{yx} (area $FBEG$) (yy_1).

But

$$\text{Area } AFGD = (FA)(zz_1), \quad \text{area } FBEG = (FB)(zz_1)$$

$$\therefore \quad \tau_{xy}(FA)(zz_1)(xx_1) = \tau_{yx}(FB)(zz_1)(yy_1).$$

Cancelling zz_1, we have

$$\tau_{xy}(FA)(xx_1) = \tau_{yx}(FB)(yy_1).$$

But

$$FA = yy_1 \quad \text{and} \quad FB = xx_1$$

$$\therefore \quad \tau_{xy}(xx_1)(yy_1) = \tau_{yx}(xx_1)(yy_1)$$

$$\therefore \quad \tau_{xy} = \tau_{yx},$$

Q. E. D.

In a similar manner we can prove

$$\tau_{xz} = \tau_{zx},$$

$$\tau_{yz} = \tau_{zy}.$$

GENERAL REMARKS.

From what precedes, it follows, that, when we have the six stresses

$$\sigma_x, \quad \sigma_y, \quad \sigma_z, \quad \tau_{xy}, \quad \tau_{xz}, \quad \tau_{yz},$$

or, in other words, the normal and tangential components of the stresses on three planes at right angles to each other, given, the state of stress at that point is entirely determined; and, when these are given, it is possible to determine the direction and intensity of the stress on any given plane.

Moreover, if three rectangular axes, OX, OY, and OZ, be assumed, and the direct strains along these axes be given, and also the shearing-strain about these axes, then the direct strain in any given direction can be determined, and also the shearing-strain around this direction as an axis.

The two above-stated propositions furnish two of the fundamental propositions of the theory of elasticity, the third being the determination of the relation between the stresses and the strains.

§ 281. **Relations Governing the Variation of the Stresses at Different Points of a Body.** — If we assume a point whose co-ordinates are (x, y, z), and a small parallelopipedical particle having this point and the point $(x + dx, y + dy, z + dz)$ for the extremities of its diagonal, we shall have, for the edges of this particle, dx, dy, dz, respectively.

Now let the stresses at (x, y, z) be

$$\sigma_x, \quad \sigma_y, \quad \sigma_z, \quad \tau_{xy}, \quad \tau_{xz}, \quad \tau_{yz};$$

i.e, σ_x denotes the normal stress on a plane perpendicular to OX, and passing through the point (x, y, z), etc. Then, for the planes passing through $(x + dx, y + dy, z + dz)$, we shall have the stresses

$$\sigma_x + d\sigma_x, \quad \sigma_y + d\sigma_y, \quad \sigma_z + d\sigma_z, \quad \tau_{xy} + d\tau_{xy}, \quad \tau_{xz} + d\tau_{xz}, \quad \tau_{yz} + d\tau_{yz}.$$

We may also have outside forces acting upon the particle in question: if such is the case, let the components of the resultant external force along the axes be respectively

$$Xdxdydz, \qquad Ydxdydz, \qquad Zdxdydz.$$

Now impose the conditions of equilibrium between all the forces acting on the particle. To do this, place equal to zero the algebraic sum of all the forces parallel to each of the axes respectively, the moment equations having already been incorporated in our demonstration that

$$\tau_{xy} = \tau_{yx}, \qquad \tau_{xz} = \tau_{zx}, \qquad \tau_{yz} = \tau_{zy}.$$

Hence we have three conditions of equilibrium, as follows: —

$$(\sigma_x + d\sigma_x - \sigma_x)dydz + (\tau_{xy} + d\tau_{xy} - \tau_{xy})dxdz + (\tau_{xz} + d\tau_{xz} - \tau_{xz})dxdy + Xdxdydz = 0,$$

$$(\sigma_y + d\sigma_y - \sigma_y)dxdz + (\tau_{xy} + d\tau_{xy} - \tau_{xy})dydz + (\tau_{yz} + d\tau_{yz} - \tau_{yz})dydx + Ydxdydz = 0,$$

$$(\sigma_z + d\sigma_z - \sigma_z)dxdy + (\tau_{yz} + d\tau_{yz} - \tau_{yz})dxdz + (\tau_{xz} + d\tau_{xz} - \tau_{xz})dzdy + Zdxdydz = 0.$$

Hence, reducing, and dividing by $dxdydz$, we have

$$\frac{d\sigma_x}{dx} + \frac{d\tau_{xy}}{dy} + \frac{d\tau_{xz}}{dz} + X = 0, \qquad (1)$$

$$\frac{d\tau_{xy}}{dx} + \frac{d\sigma_y}{dy} + \frac{d\tau_{yz}}{dz} + Y = 0, \qquad (2)$$

$$\frac{d\tau_{xz}}{dx} + \frac{d\tau_{yz}}{dy} + \frac{d\sigma_z}{dz} + Z = 0. \qquad (3)$$

If the particle is in the interior of the body, and we disregard its weight, then $X = Y = Z = 0$.

Equations (1), (2), and (3) give the necessary relations which the variations of stress from point to point must satisfy in order that the conditions of equilibrium may be fulfilled.

§ 282. **Relations between the Stresses and Strains.** — Before proceeding to the general problems of composition of stresses, i.e., of determining from a sufficient number of data the stress upon any plane, we will first discuss the relations between the stresses and the strains; and we will confine ourselves to those bodies that are homogeneous, and of the same elasticity throughout.

From what we have already seen, if to a straight rod whose cross-section is A there be applied a pull P in the direction of its length, the intensity of the stress on the cross-section wi be

$$\sigma = \frac{P}{A};$$

and, if E be the tensile modulus of elasticity of the material of the rod, the strain in a direction at right angles to the cross-section, or, in other words, in the direction of the pull, will be

$$\epsilon = \frac{\sigma}{E}.$$

Now, another fact, which we have thus far taken no account of, is, that although there is no stress in a direction at right angles to the pull, or, in other words, although a section at right angles to the above-stated cross-section will have no stress upon it, yet there will be a strain in all directions at right angles to the direction of the pull: and this strain will be, for any direction at right angles to the pull,

$$\epsilon_1 = -\frac{\epsilon}{m},$$

being of the opposite kind from ϵ; thus, if ϵ is extension, ϵ_1 is compression, and *vice versa.*

Hence, if, at any point O of such a rod, we assume three rectangular axes, of which OX is in the direction of the pull, and we use the notation already adopted, we shall have

$$\sigma_x = \frac{P}{A}, \quad \sigma_y = \sigma_z = \tau_{xy} = \tau_{xz} = \tau_{yz} = 0,$$

$$\epsilon_x = \frac{\sigma_x}{E}, \quad \epsilon_y = \epsilon_z = -\frac{1}{m}\frac{\sigma_x}{E} = -\frac{\epsilon_x}{m},$$

$$\gamma_{xy} = \gamma_{xz} = \gamma_{yz} = 0.$$

MODULUS OF SHEARING ELASTICITY.

In the case of direct tension or compression, when only a simple stress is applied, we have defined the modulus of elasticity as the ratio of the stress to the strain in its own direction.

Adopting a similar definition in the case of shearing, we shall have

$$\frac{\tau_{xy}}{\gamma_{xy}} = \frac{\tau_{xz}}{\gamma_{xz}} = \frac{\tau_{yz}}{\gamma_{yz}} = G,$$

where G is the modulus of shearing elasticity.

GENERAL RELATIONS BETWEEN STRESSES AND STRAINS.

Whenever a compound stress acts on a body at a given point, let the stresses be

$$\sigma_x, \quad \sigma_y, \quad \sigma_z, \quad \tau_{xy}, \quad \tau_{xz}, \quad \tau_{yz};$$

then we shall have, for the strains.

$$\epsilon_x = \frac{\sigma_x}{E} - \frac{1}{m}\frac{\sigma_y}{E} - \frac{1}{m}\frac{\sigma_z}{E}, \qquad \gamma_{xy} = \frac{\tau_{xy}}{G},$$

$$\epsilon_y = \frac{\sigma_y}{E} - \frac{1}{m}\frac{\sigma_x}{E} - \frac{1}{m}\frac{\sigma_z}{E}, \qquad \gamma_{xz} = \frac{\tau_{xz}}{G},$$

$$\epsilon_z = \frac{\sigma_z}{E} - \frac{1}{m}\frac{\sigma_x}{E} - \frac{1}{m}\frac{\sigma_y}{E}, \qquad \gamma_{yz} = \frac{\tau_{yz}}{G}.$$

This enables us to determine the strains in terms of the stresses, as soon as the values of E, G, and m are known from experiment, for the material under consideration.

If, on the other hand, the stresses be required in terms of the strains, we can consider ϵ_x, ϵ_y, ϵ_z, γ_{xy}, γ_{xz}, γ_{yz}, as known, and determine σ_x, σ_y, σ_z, τ_{xy}, τ_{yz}, τ_{xz}, from the above equations.

We thus obtain

$$E\epsilon_x = \sigma_x - \frac{\sigma_y + \sigma_z}{m}, \qquad (1)$$

$$E\epsilon_y = \sigma_y - \frac{\sigma_x + \sigma_z}{m}, \qquad (2)$$

$$E\epsilon_z = \sigma_z - \frac{\sigma_x + \sigma_y}{m}; \qquad (3)$$

and, by solving these equations for the stresses, we have

$$\sigma_x = \frac{m}{m+1}E\left(\epsilon_x + \frac{\epsilon_x + \epsilon_y + \epsilon_z}{m-2}\right), \qquad (4)$$

$$\sigma_y = \frac{m}{m+1}E\left(\epsilon_y + \frac{\epsilon_x + \epsilon_y + \epsilon_z}{m-2}\right), \qquad (5)$$

$$\sigma_z = \frac{m}{m+1}E\left(\epsilon_z + \frac{\epsilon_x + \epsilon_y + \epsilon_z}{m-2}\right), \qquad (6)$$

and also

$$\tau_{xy} = G\gamma_{xy}, \quad (7) \qquad \tau_{xz} = G\gamma_{xz}, \quad (8) \qquad \tau_{yz} = G\gamma_{yz}. \quad (9)$$

These equations express the stresses in terms of the strains.

The three last might be written as follows (see § 276) : —

$$\tau_{xy} = G\left(\frac{d\xi}{dy} + \frac{d\eta}{dx}\right), \qquad (10)$$

$$\tau_{xz} = G\left(\frac{d\xi}{dz} + \frac{d\zeta}{dx}\right), \qquad (11)$$

$$\tau_{yz} = G\left(\frac{d\eta}{dz} + \frac{d\zeta}{dy}\right), \qquad (12)$$

as these forms are often convenient.

§ 283. **Case when $\sigma_z = 0$.** — Inasmuch as there are many cases in practice where the stress is all parallel to one plane, and where, consequently, the stress on any plane parallel to this plane has no normal component, it will be convenient to have the reduced forms of equations (4), (5), and (6) which apply in this case.

Let the plane to which the stresses are parallel be the Z plane; then $\sigma_z = 0$. Then equation (6) becomes

$$\epsilon_z + \frac{\epsilon_x + \epsilon_y + \epsilon_z}{m - 2} = 0$$

$$\therefore \quad \epsilon_z = -\frac{\epsilon_x + \epsilon_y}{(m - 1)};$$

and, substituting this value of ϵ_z in (4) and (5), and reducing, we obtain

$$\sigma_x = \frac{m}{m + 1}E\left(\epsilon_x + \frac{\epsilon_x + \epsilon_y}{m - 1}\right), \qquad (1)$$

$$\sigma_y = \frac{m}{m + 1}E\left(\epsilon_y + \frac{\epsilon_x + \epsilon_y}{m - 1}\right), \qquad (2)$$

which are the required forms.

The other three equations, viz., —

$$\tau_{xy} = G\gamma_{xy}, \quad \tau_{xz} = G\gamma_{xz}, \quad \tau_{yz} = G\gamma_{yz},$$

remain the same as before.

§ 284. **Values of** E, G, **and** m. — These three constants need to be known, to use the relations developed above.

1°. As to E, this is the modulus of elasticity for tension, and has been determined experimentally for the various materials, as has been already explained. Moreover, it has also been shown experimentally, that, with moderate loads, the modulus of elasticity for compression is nearly identical with that for tension in cast-iron, wrought-iron, and steel.

2°. As to m, in those few applications that Professor Rankine gives of his theory of internal stress, such as the case of combined twisting and bending, he determines the greatest intensity of the stress acting; and his criterion is, that this shall be kept within the working-strength of the material. This is equivalent to assuming $m = \infty$. The more modern writers, such as Grashof and others, take account of the fact that m has a finite value, and make their criterion that the greatest strain shall be kept within the quotient obtained by dividing the working-strength by the modulus of elasticity of the material.

Thus, if f is the working-strength, and σ_1 the greatest stress, and ϵ_1 the greatest strain, Rankine's criterion of safety is

$$\sigma_1 \leqq f;$$

whereas the more modern criterion is

$$E\epsilon_1 \leqq f.$$

The resulting formulæ differ in each case; and, as has been stated, those of Rankine could be derived from the more general ones by making

$$\frac{1}{m} = 0 \quad \text{or} \quad m = \infty,$$

which is never the case.

As to the value of m, but few experiments have been made. Those of Wertheim give, for brass, 2.94; for wrought-iron, 3.64.

The values $m = 3$ and $m = 4$ are those most commonly adopted, so that

$$\frac{1}{m} = \frac{1}{3} \quad \text{or} \quad \frac{1}{m} = \frac{1}{4}.$$

3°. The value of G, the shearing-modulus of elasticity, i.e., the ratio of the stress to the strain for shearing, has been determined experimentally, and has generally been found to be about two-fifths that for tension.

According to the theory of elasticity, we must have

$$G = \frac{1}{2}\frac{m}{m+1}E,$$

as may be proved as follows:—

Assume a square particle whose side is a, and let a simple normal stress σ be applied at the face AB; then we shall have, on the planes BD and AC, a shearing-stress (§ 279)

$$\tau = \sigma \sin 45^\circ \cos 45^\circ = \tfrac{1}{2}\sigma.$$

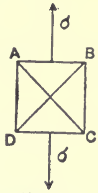


FIG. 301.

On the other hand, if we let

$$\epsilon = \frac{\sigma}{E},$$

the strain of the particle in the direction AD will be ϵ, while that in the direction AB will be $-\dfrac{\epsilon}{m}$; hence the particle will become a rectangle, the side AD changing its length from a to $a + a\epsilon$, and side AB changing from a to $a - \dfrac{a\epsilon}{m}$.

The diagonals will no longer be at right angles to each other; and, if we denote by α the angle by which their angle differs from a right angle, we shall have, for the shearing-strain on the planes AC and BD,

$$\gamma = \tan \alpha.$$

But, after the distortion, the angle ADB will become

$$\frac{1}{2}\left(\frac{\pi}{2} - \alpha\right)$$

$$\therefore \quad \tan\left(\frac{\pi}{4} - \frac{\alpha}{2}\right) = \frac{1 - \tan\dfrac{\alpha}{2}}{1 + \tan\dfrac{\alpha}{2}} = \frac{a - \dfrac{a\epsilon}{m}}{a + a\epsilon} = \frac{1 - \dfrac{\epsilon}{m}}{1 + \epsilon};$$

therefore, dividing, and carrying the division only to terms of the first degree, we have

$$1 - 2\tan\frac{\alpha}{2} = 1 - \left(1 + \frac{1}{m}\right)\epsilon$$

$$\therefore \quad 2\tan\left(\frac{\alpha}{2}\right) = \frac{m + 1}{m}\epsilon.$$

But

$$\gamma = \tan\alpha = 2\tan\frac{\alpha}{2} \text{ nearly}$$

$$\therefore \quad \gamma = \frac{m+1}{m}\epsilon$$

$$\therefore \quad \frac{\tau}{\gamma} = \frac{\frac{1}{2}\sigma}{\frac{m+1}{m}\epsilon} = \frac{1}{2}\frac{m}{m+1}\left(\frac{\sigma}{\epsilon}\right);$$

but

$$\frac{\tau}{\gamma} = G \quad \text{and} \quad \frac{\sigma}{\epsilon} = E$$

$$\therefore \quad G = \frac{1}{2}\frac{m}{m+1}E.$$

§ 285. **Conjugate Stresses.** — If the stress on a given plane at a given point of a body be in a given direction, the stress at the same point on a plane parallel to that direction will be parallel to the given plane. Let YOY represent, in section, a given plane, and let the stress on that plane be in the direction XOX.

Consider a small prism $ABCD$ within a body, the sides of whose base are parallel respectively to XOX and YOY. The forces on the plane AB are counterbalanced by the forces on the plane DC; the resultants of each of these sets being equal and opposite, and acting along a line passing through O. Hence the forces acting on the planes AD and BC must be balanced entirely independently of any of the forces on AB or DC: and this can be the case only when their direction is parallel to YOY; for otherwise their resultants, though equal in magnitude and opposite in direction, would not be directly opposite, but would form a couple, and, as there is no equal and opposite couple furnished by the forces on the other faces, equilibrium could not exist under this supposition.

§ 286. **Composition of Stresses.** — The general problem of the composition of stresses may be stated as follows: —

Knowing the stresses at a given point of a strained body on three planes passing through that point, to find the stress at the same point on any other plane, also passing through the same point. The stresses on the three given planes are not entirely independent; in other words, we could not give the stresses on these three planes, in magnitude and direction, at random, and expect to find the problem a possible one. Thus, suppose that the planes are at right angles to each other, we have already seen that we have the right to give their three normal components, σ_x, σ_y, and σ_z, and the three tangential, τ_{xy}, τ_{yz}, and τ_{xz}, and that $\tau_{yx} = \tau_{xy}$, etc. We will now proceed to special cases.

§ 287. **Problem.** — Given the three planes of action of the stress as the x, y, and z plane respectively, and given the normal and tangential components of the stresses on these planes, viz., σ_x, σ_y, σ_z, τ_{xy}, τ_{xz}, and τ_{yz}, to find the intensity and direction of the stress on a plane whose normal makes with OX, OY, and OZ the angles α, β, and γ respectively, where, of course, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

Draw the line ON, making angles α, β, and γ with OX, OY, and OZ respectively; then draw near O the plane ABC perpendicular to ON. It has the direction of the required plane, and cuts off intercepts OA, OB, and OC on the axes; and, moreover, we shall have, from trigonometry, the relations,

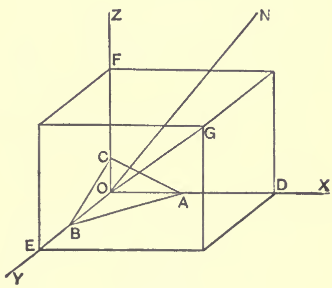


FIG. 302.

$$\text{Area } BOC = (ABC)\cos\alpha,$$
$$\text{Area } AOC = (ABC)\cos\beta,$$
$$\text{Area } AOB = (ABC)\cos\gamma.$$

Now consider the conditions of equilibrium of the tetrahedron $OABC$. The stress on ABC must be equal and directly opposed to the resultant of the stresses on the three faces AOC, BOC, and AOB. Now let us proceed to find this resultant.

In the direction OX we have the force

$$\sigma_x(BOC) + \tau_{xz}(AOB) + \tau_{xy}(AOC)$$
$$= (ABC)(\sigma_x \cos\alpha + \tau_{xy}\cos\beta + \tau_{xz}\cos\gamma).$$

Lay off OD to represent this quantity. In the same way represent the force in the direction OY by

$$OE = \sigma_y(AOC) + \tau_{yz}(BOA) + \tau_{xy}(BOC)$$
$$= (ABC)(\sigma_y \cos\beta + \tau_{xy}\cos\alpha + \tau_{yz}\cos\gamma),$$

and that in the direction OZ by

$$OF = \sigma_z(AOB) + \tau_{xz}(BOC) + \tau_{yz}(AOC)$$
$$= ABC(\sigma_z \cos\gamma + \tau_{xz}\cos\alpha + \tau_{yz}\cos\beta).$$

Now compound these three forces, and we have, as resultant force,

$$R = OG = \sqrt{OD^2 + OE^2 + OF^2},$$

and as resultant intensity

$$\sigma = \frac{R}{ABC} = \frac{\sqrt{OD^2 + OE^2 + OF^2}}{ABC}$$
$$= \sqrt{\{(\sigma_x \cos\alpha + \tau_{xy}\cos\beta + \tau_{xz}\cos\gamma)^2}$$
$$+ (\sigma_y \cos\beta + \tau_{xy}\cos\alpha + \tau_{yz}\cos\gamma)^2$$
$$\overline{+ (\sigma_z \cos\gamma + \tau_{xz}\cos\alpha + \tau_{yz}\cos\beta)^2\}}$$
$$= \sqrt{\{\sigma_x^2 \cos^2\alpha + \sigma_y^2\cos^2\beta + \sigma_z^2\cos^2\gamma}$$
$$+ \tau_{xy}^2(\cos^2\alpha + \cos^2\beta) + \tau_{xz}^2(\cos^2\alpha + \cos^2\gamma)$$
$$+ \tau_{yz}^2(\cos^2\beta + \cos^2\gamma) + 2\sigma_x(\tau_{xy}\cos\beta + \tau_{xz}\cos\gamma)\cos\alpha$$
$$+ 2\sigma_y(\tau_{xy}\cos\alpha + \tau_{yz}\cos\gamma)\cos\beta$$
$$+ 2\sigma_z(\tau_{xz}\cos\alpha + \tau_{yz}\cos\beta)\cos\gamma + 2\tau_{xy}\tau_{xz}\cos\beta\cos\gamma$$
$$\overline{+ 2\tau_{xy}\tau_{yz}\cos\alpha\cos\gamma + 2\tau_{yz}\tau_{xz}\cos\alpha\cos\beta\}};$$

the direction being given by the angles, α_r, β_r, and γ_r, where

$$\cos \alpha_r = \frac{OD}{R}, \quad \cos \beta_r = \frac{OE}{R}, \quad \cos \gamma_r = \frac{OF}{R}.$$

§ 288. **Stresses Parallel to a Plane.** — To solve the same problem when there is no stress in the direction OZ, and when the new plane is perpendicular to XOY, or, in other words, in the case when the planes of action are all perpendicular to one plane, to which the stresses are all parallel: we then have

$$\sigma_z = \tau_{xz} = \tau_{yz} = 0 \quad \text{and} \quad \beta = 90^\circ - \alpha,$$

and hence

$$\sigma = \sqrt{\sigma_x^2 \cos^2 \alpha + \sigma_y^2 \sin^2 \alpha + \tau_{xy}^2 + 2(\sigma_x + \sigma_y)\tau_{xy} \cos \alpha \sin \alpha}.$$

Or we may proceed as follows: —

Let the normal intensity of the stress on the x plane (i.e., that perpendicular to OX) be σ_x, that on the y plane σ_y, and the tangential intensity τ_{xy}. Let ON be the direction of the normal to the plane on which the stress is to be determined, and let the angle $XON = \alpha$. Then let the plane AB be drawn perpendicular to ON, and let us consider the equilibrium of the forces exerted by the other parts of the body upon the triangular prism whose base is ABO and altitude unity.

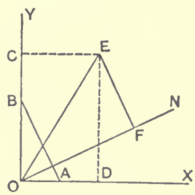


Fig. 303.

If we compound the forces acting on the faces AO and OB, we shall have, in their resultant, the total force on the face AB in magnitude and direction. Moreover, we have the relations,

Area OB = area $AB \cos \alpha$ and Area OA = area $AB \sin \alpha$.

Force acting on OB in direction $OX = \sigma_x(OB)$,
Force acting on OB in direction $OY = \tau_{xy}(OB)$,
Force acting on OA in direction $OX = \tau_{xy}(OA)$,
Force acting on OA in direction $OY = \sigma_y(OA)$.

Hence, if we lay off

$$OD = \sigma_x(OB) + \tau_{xy}(OA) \quad \text{and} \quad OC = \sigma_y(OA) + \tau_{xy}(OB),$$

then will OD represent the total force acting in the direction OX, and OC will represent the total force acting in the direction OY.

Compounding these, we shall have OE as the resultant total force on the face AB, and $\frac{OE}{AB}$ will represent its intensity.

To deduce the analytical values, we have

$$OD = \sigma_x(OB) + \tau_{xy}(OA) = (AB)(\sigma_x \cos\alpha + \tau_{xy}\sin\alpha),$$
$$OC = \sigma_y(OA) + \tau_{xy}(OB) = (AB)(\sigma_y \sin\alpha + \tau_{xy}\cos\alpha)$$

$$\begin{aligned} \therefore\ OE &= \sqrt{OD^2 + OC^2} \\ &= AB\sqrt{(\sigma_x \cos\alpha + \tau_{xy}\sin\alpha)^2 + (\sigma_y \sin\alpha + \tau_{xy}\cos\alpha)^2} \\ &= AB\sqrt{\{\sigma_x^2\cos^2\alpha + \sigma_y^2\sin^2\alpha + 2\tau_{xy}\cos\alpha\sin\alpha(\sigma_x + \sigma_y)} \\ &\qquad\qquad \overline{+ \tau_{xy}^2(\cos^2\alpha + \sin^2\alpha)\}}. \end{aligned}$$

Or, if σ_r represent the resultant intensity on the plane AB, and α_r the angle this resultant makes with OX, we shall have

$$\sigma_r = \sqrt{\{\sigma_x^2\cos^2\alpha + \sigma_y^2\sin^2\alpha + 2\tau_{xy}(\sigma_x + \sigma_y)\cos\alpha\sin\alpha + \tau_{xy}^2\}}, \quad (1)$$

and

$$\cos\alpha_r = \frac{OD}{OE} \quad \text{and} \quad \sin\alpha_r = \frac{OC}{OE}.$$

Moreover, it is sometimes desirable to resolve the stress into normal and tangential components. If this be done, and if σ and τ represent respectively the normal and tangential components, we shall have

$$\sigma = \frac{OF}{AB} \quad \text{and} \quad \tau = \frac{EF}{AB};$$

but

$$OF = OD\cos\alpha + ED\sin\alpha \quad \text{and} \quad EF = ED\cos\alpha - OD\sin\alpha$$

$$\therefore \quad \sigma = \frac{OD}{AB}\cos\alpha + \frac{OC}{AB}\sin\alpha$$

$$= \sigma_x\cos^2\alpha + \sigma_y\sin^2\alpha + 2\tau_{xy}\cos\alpha\sin\alpha \quad (2)$$

and

$$\tau = \frac{OC}{AB}\cos\alpha - \frac{OD}{AB}\sin\alpha$$

$$= (\sigma_y - \sigma_x)\cos\alpha\sin\alpha + \tau_{xy}(\cos^2\alpha - \sin^2\alpha)$$

$$= \left(\frac{\sigma_y - \sigma_x}{2}\right)\sin 2\alpha + \tau_{xy}\cos 2\alpha. \quad (3)$$

§ 289. **Principal Stresses.** — It will next be shown, that, whatever be the state of stress in a body, provided the stresses are all parallel to one plane, the planes of action being all taken perpendicular to this plane, there are always two planes, at right angles to each other, on which there is no tangential stress; these two planes being called the planes of principal stress, the stress on one of these planes being greater, and the other less, than that on any other plane through the same point.

To prove the above, it will be necessary only in the last case, which is a perfectly general one, to determine for what values of α the value of τ is zero, and whether these values of α are always possible. We have

$$\tau = \frac{\sigma_y - \sigma_x}{2}\sin 2\alpha + \tau_{xy}\cos 2\alpha:$$

and, if we put this equal to zero, we have

$$\frac{\sin 2\alpha}{\cos 2\alpha} = \tan 2\alpha = \frac{2\tau_{xy}}{\sigma_x - \sigma_y};$$

and this gives us, for all values of σ_x, σ_y, and τ_{xy}, two possible values for 2α, differing from each other by 180°, hence two values for α differing by 90°. Hence follows the first part of the proposition.

The latter part — that these are the planes of the greatest and least stresses — will be shown by differentiating the value of σ_r^2, and putting the first differential co-efficient equal to zero; and, as this gives us

$$\begin{aligned}- 2\sigma_x^2 \cos\alpha \sin\alpha + 2\sigma_y^2 \cos\alpha \sin\alpha \\ + 2\tau_{xy}(\sigma_x + \sigma_y)(\cos^2\alpha - \sin^2\alpha) \\ = 2(\sigma_x + \sigma_y)\{(\sigma_y - \sigma_x)\cos\alpha\sin\alpha + \tau_{xy}(\cos^2\alpha - \sin^2\alpha)\} \\ = 0,\end{aligned}$$

therefore we have the same condition for the maximum and minimum stresses as we have for the planes of no tangential stress.

It follows that the determination of the greatest and least stresses at any one point of a body is identical with the determination of the principal stresses; and it will be necessary, whenever the stresses on any two planes are given, to be able to determine the principal stresses, as one of these is the greatest stress at that point of the body, and the other the least.

§ 290. **Determination of Principal Stresses.** — When the stress is all parallel to one plane, viz., the z plane, and when the stresses on two planes at right angles to each other are given, i.e., their normal and tangential components, we may be required to determine the principal stresses. Proceed as follows: Given normal stresses on X and Y planes respectively, σ_x and σ_y, and tangential stress on each plane τ_{xy}, to find principal stresses.

From § 288 we have, for a plane whose normal makes an angle α with OX,

$$\sigma_r = \sqrt{\sigma_x^2 \cos^2\alpha + \sigma_y^2 \sin^2\alpha + 2\tau_{xy}(\sigma_x + \sigma_y)\cos\alpha\sin\alpha + \tau_{xy}^2}, \quad (1)$$

$$\sigma_n = \sigma_x \cos^2\alpha + \sigma_y \sin^2\alpha + 2\tau_{xy}\cos\alpha\sin\alpha, \quad (2)$$

$$\tau = \tau_{xy}(\cos^2\alpha - \sin^2\alpha) - (\sigma_x - \sigma_y)\cos\alpha\sin\alpha, \quad (3)$$

or

$$\tau = \tau_{xy}\cos 2\alpha - \frac{\sigma_x - \sigma_y}{2}\sin 2\alpha. \qquad (4)$$

Now, the condition that the plane shall be a plane of principal stress is, that $\tau = 0$. Hence write

$$\tau_{xy}(\cos^2\alpha - \sin^2\alpha) - (\sigma_x - \sigma_y)\cos\alpha\sin\alpha = 0,$$

find α, and substitute its value in (2), and we shall have the principal stresses. The operation may be performed as follows; viz., —

From (3) we have

$$(a) \quad 2\cos^2\alpha - 1 = \frac{\sigma_x - \sigma_y}{\tau_{xy}}\cos\alpha\sin\alpha$$

$$\therefore \quad \cos^2\alpha = \frac{1}{2}\left\{1 + \frac{\sigma_x - \sigma_y}{\tau_{xy}}\cos\alpha\sin\alpha\right\}.$$

$$(b) \quad 1 - 2\sin^2\alpha = \frac{\sigma_x - \sigma_y}{\tau_{xy}}\cos\alpha\sin\alpha$$

$$\therefore \quad \sin^2\alpha = \frac{1}{2}\left\{1 - \frac{\sigma_x - \sigma_y}{\tau_{xy}}\cos\alpha\sin\alpha\right\}.$$

Hence

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\cos\alpha\sin\alpha}{2}\left\{\frac{\sigma_x^2 - 2\sigma_x\sigma_y + \sigma_y^2}{\tau_{xy}} + 4\tau_{xy}\right\}$$

or

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\cos\alpha\sin\alpha}{2\tau_{xy}}\left\{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2\right\}.$$

But we have, since (4) equals zero,

$$\tan 2\alpha = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

$$\therefore \quad \sin 2\alpha = 2\sin\alpha\cos\alpha = \frac{\pm 2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}.$$

Hence substitute for $\cos\alpha \sin\alpha$ its value, and

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}, \qquad (5)$$

which gives us the magnitudes of the principal stresses; the plus sign corresponding to the greater, and the minus sign to the less.

EXAMPLES.

1. Let, in the last section, $\sigma_y = 0$, and find the principal stresses. Here we have

$$\tan 2\alpha = \frac{2\tau_{xy}}{\sigma_x}$$

and

$$\sigma_n = \frac{\sigma_x}{2} \pm \frac{1}{2}\sqrt{\sigma_x^2 + 4\tau_{xy}^2}.$$

2. Given two principal stresses, to find the stress on a plane whose normal makes an angle α with OX.

In this case $\tau_{xy} = 0$.

Hence we have the case of § 288, with the reduction of making $\tau_{xy} = 0$. We may therefore obtain the result by substitution in the results of § 288, or we may proceed as follows: —

(*a*) Find stress on new plane in direction OX; this will be, § 279,

$$\sigma_x \cos\alpha.$$

(*b*) Find stress on new plane in direction OY; this will be, § 279,

$$\sigma_y \sin\alpha.$$

(*c*) Compound the two, and the resultant is

$$\sigma_r = \sqrt{\sigma_x^2 \cos^2\alpha + \sigma_y^2 \sin^2\alpha}. \qquad (1)$$

(*d*) Normal component of $\sigma_x \cos\alpha$ is

$$\sigma_x \cos^2\alpha.$$

(*e*) Normal component of $\sigma_y \sin\alpha$ is

$$\sigma_y \sin^2\alpha.$$

(*f*) Add, and we have, for normal stress,

$$\sigma_n = \sigma_x \cos^2 \alpha + \sigma_y \sin^2 \alpha. \qquad (2)$$

(*g*) Tangential component of $\sigma_x \cos \alpha$ is

$$-\sigma_x \cos \alpha \sin \alpha.$$

(*h*) Tangential component of $\sigma_y \sin \alpha$ is

$$+\sigma_y \cos \alpha \sin \alpha.$$

(*k*) Add, and we have, for tangential stress,

$$\tau = (\sigma_y - \sigma_x) \cos \alpha \sin \alpha. \qquad (3)$$

§ 291. **Ellipse of Stress.** — In the case above, i.e., when the two principal stresses are σ_x and σ_y respectively, if we represent them graphically by $OA = \sigma_x$ and $OB = \sigma_y$, and let CD be the plane on which the stress is required, its normal making with OX the angle $XON = \alpha$, then, from what has been shown, if OR represent the intensity of the resultant stress on this plane, we shall have

$$OR = \sigma_r = \sqrt{\sigma_x^2 \cos^2 \alpha + \sigma_y^2 \sin^2 \alpha};$$

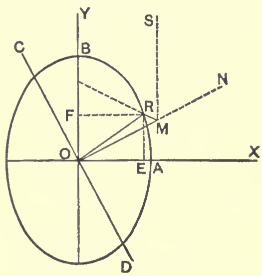


FIG. 304.

and, moreover,

$$OE = \sigma_x \cos \alpha, \qquad OF = \sigma_y \sin \alpha.$$

If we denote these by x and y respectively, letting (x, y) be the point R, i.e., the extremity of the line representing the stress on AB, then

$$x = \sigma_x \cos \alpha, \qquad y = \sigma_y \sin \alpha,$$

$$\therefore \left(\frac{x}{\sigma_x}\right)^2 = \cos^2 \alpha \quad \text{and} \quad \left(\frac{y}{\sigma_y}\right)^2 = \sin^2 \alpha$$

$$\therefore \frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} = 1,$$

which is the equation of an ellipse whose semi-axes are σ_x and σ_y respectively; hence the stress on any plane will be represented by some semi-diameter of the ellipse.

SPECIAL CASES.

I. When the two given stresses are equal, or $\sigma_x = \sigma_y$, then

$$\sigma_r = \sqrt{\sigma_x^2 \cos^2 \alpha + \sigma_y^2 \sin^2 \alpha} = \sigma_x,$$

and

$$\cos \alpha_r = \frac{\sigma_x \cos \alpha}{\sigma_x} = \cos \alpha \qquad \text{and} \qquad \sin \alpha_r = \sin \alpha;$$

therefore the stress is of the same intensity on all planes, and always normal to the plane.

II. When the two given stresses are equal in magnitude but opposite in sign, or $\sigma_y = -\sigma_x$, then

$$\sigma_r = \sigma_x.$$

But

$$\cos \alpha_r = \cos \alpha \qquad \text{and} \qquad \sin \alpha_r = -\sin \alpha,$$

hence

$$\alpha_r = -\alpha;$$

therefore the stress on any plane whose normal makes an angle α with OX is of the same intensity σ_x, but makes an angle equal to α with OX on the side opposite to that of the normal to the plane.

Problem. — A pair of principal stresses being given, to find the positions of the planes on which the shear is greatest.

Solution. — Let $\tau = (\sigma_y - \sigma_x) \sin \alpha \cos \alpha = \max.$

Therefore differentiate, and

$$\cos^2 \alpha - \sin^2 \alpha = 0$$

$$\therefore \quad \cos \alpha = \pm \sin \alpha \qquad\qquad \therefore \quad \alpha = 45^\circ \text{ or } 135^\circ.$$

§ 292. **Some Special Modes of Solution of some Problems.** — The case where two principal stresses, σ_x and σ_y, are given, to find the stress on any plane whose normal makes an angle α with OX, may be solved as follows, graphically:—

Let, Fig. 304, $\sigma_x = OA$, and $\sigma_y = OB$. Let $XON = \alpha$.

Now,

$$\sigma_y = \frac{\sigma_y + \sigma_x}{2} + \frac{\sigma_y - \sigma_x}{2},$$

$$\sigma_x = \frac{\sigma_y + \sigma_x}{2} - \frac{\sigma_y - \sigma_x}{2}.$$

Hence, instead of proceeding at once to find the resultant stress on CD due to the action of σ_x and σ_y, we may first find that due to the action of the two equal principal stresses of the same kind,

$$\frac{\sigma_y + \sigma_x}{2},$$

then that due to the pair

$$\frac{\sigma_y - \sigma_x}{2} \quad \text{and} \quad -\frac{\sigma_y - \sigma_x}{2},$$

and then the resultant of these two resultants.

The first resultant will be evidently laid off on ON, and equal in magnitude to $\frac{\sigma_y + \sigma_x}{2}$; hence let $OM = \frac{\sigma_y + \sigma_x}{2}$, and OM will be the first resultant.

The second resultant will be of magnitude $\frac{\sigma_y - \sigma_x}{2}$, and will have a direction MR such that the angle $NMS = SMR$.

Hence, laying off this angle, and making $MR = \frac{\sigma_y - \sigma_x}{2}$, we shall have for the final resultant, OR, as before.

This construction will be useful in the following case:—

To find the most oblique stress, we must find for what value of α the angle MOR is greatest. This will be made evident if we observe, that, for all positions of the plane, the triangle OMR has always $OM = \frac{\sigma_y + \sigma_x}{2}$, and $MR = \frac{\sigma_y - \sigma_x}{2}$; both of constant length. Hence, if, with M as a centre and MR as a radius, a circle were described, and a tangent were drawn from O to this circle, the point of tangency being taken for R, then will OR be the most oblique stress; i.e., the stress is most oblique when $ORM = 90°$. Therefore greatest obliquity =

$$\sin^{-1}\frac{\sigma_y - \sigma_x}{\sigma_y + \sigma_x}.$$

§ 293. **Converse of the Ellipse of Stress.** — The converse of the ellipse of stress would be the following problem: Given any two planes passing through the point in question; given the intensities and directions of the stresses on these planes, — to find the principal stresses in magnitude and in direction.

The first step to be taken is, to assure ourselves that the conditions are not incompatible, as they are liable to be if the planes and stresses are taken at random. The test of this question is, to resolve each stress into two components, respectively parallel to the two planes; and, if the conditions are not inconsistent, the components of each stress along the plane on which it acts must be equal. The proof of this statement can be made in a similar way to that used in proving that the intensities of the shearing-stresses on two planes at right angles to each other are equal. If, upon applying this test, we find that the conditions are not inconsistent, we may proceed as follows: —

Suppose CD (Fig. 304) were the given plane, and OR the stress upon it, and suppose the position of the principal axes, OX and OY, and, indeed, all the rest of the figure, were absent, i.e., not known. Now, we can easily draw the normal ON; and, if we could determine upon it the point M such that OM

should be one-half the sum of the principal stresses, we should be able to reproduce the whole figure. Hence we will devote ourselves to the determination of the position of the point M.

Let $OR = p =$ stress on plane CD.

Let stress on the other given plane be p_1.

Let $NOR = \theta =$ obliquity of p.

Let $\theta_1 =$ obliquity of p_1.

Then we have

$$MR^2 = OR^2 + OM^2 - 2OM \cdot OR \cos\theta;$$

or, if σ_x and σ_y denote the (unknown) magnitudes of the principal stresses,

$$\left(\frac{\sigma_y - \sigma_x}{2}\right)^2 = p^2 + \left(\frac{\sigma_x + \sigma_y}{2}\right)^2 - 2\left(\frac{\sigma_x + \sigma_y}{2}\right)p\cos\theta. \quad (1)$$

From the triangle constructed in the same way, with the stress on the other plane, we should have

$$\left(\frac{\sigma_y - \sigma_x}{2}\right)^2 = p_1^2 + \left(\frac{\sigma_x + \sigma_y}{2}\right)^2 - 2\left(\frac{\sigma_x + \sigma_y}{2}\right)p_1\cos\theta_1. \quad (2)$$

Hence, by subtraction,

$$p^2 - p_1^2 = 2\left(\frac{\sigma_x + \sigma_y}{2}\right)(p\cos\theta - p_1\cos\theta_1) \quad (3)$$

$$\therefore \quad \frac{\sigma_x + \sigma_y}{2} = \frac{p^2 - p_1^2}{2(p\cos\theta - p_1\cos\theta_1)}. \quad (4)$$

Having thus found $\frac{\sigma_x + \sigma_y}{2}$, we can next find, from either (1) or (2), the value of $\frac{\sigma_y - \sigma_x}{2}$.

Now, therefore, we know OM and MR, and hence we can lay off this value of OM, and complete the triangle OMR; then bisect the angle NMR, and the line MS is parallel to the axis of greater principal stress. Hence draw OY parallel to MS, and OX perpendicular to OY, and lay off on OY

$$OB = \sigma_y = OM + MR,$$

and on OX

$$OA = \sigma_x = OM - MR,$$

and the problem is solved.

§ 294. **Rankine's Graphical Solution.**—The following is Rankine's graphical solution of the preceding problem :

Draw the straight line ON, Fig. 305, and then lay off angle $NOR = \theta$, and angle $NOR_1 = \theta_1$, also $OR = p$ and $OR_1 = p_1$. Then join R and R_1, and bisect RR_1 by a perpendicular SM. From the point M, where this meets ON, draw MR and MR_1. Then will

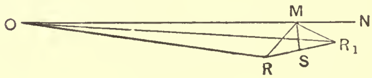


Fig. 305.

$$\frac{\sigma_y + \sigma_x}{2} = OM, \qquad (1) \qquad \frac{\sigma_y - \sigma_x}{2} = MR = MR_1; \quad (2)$$

$$\therefore \ \sigma_x = OM - MR, \ (3) \ \text{ and } \ \sigma_y = OM + MR; \quad (4)$$

and the angles made by OY with ON and ON_1, respectively, will be

$$90^\circ - \alpha = \frac{NMR}{2}, \qquad (5) \quad \text{and} \quad 90^\circ - \alpha_1 = \frac{NMR_1}{2}.$$

A comparison of this figure with the triangle OMR of § 291 will show that this is merely a graphical construction for the analytical solution given in § 293; and the equations of that article can readily be deduced from the figure given above.

SPECIAL CASES.

(*a*) *When the two given planes are at right angles to each other.*—In this case the tangential components of OR and OR_1 are equal, and hence (Fig. 306) $aR_1 = bR$.

Hence the figure becomes that shown where RR_1 is parallel to ON; and if we let $Ob = p_n$, $Oa = p_{n1}$, and $bR = aR_1 = p_t$, we shall have

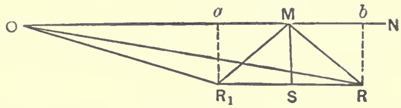


FIG. 306.

$$\frac{\sigma_y + \sigma_x}{2} = OM = \frac{p_n + p_n'}{2};$$

$$\frac{\sigma_y - \sigma_x}{2} = MR = \sqrt{(Mb)^2 + p_t^2} = \sqrt{\left(\frac{p_n - p_n'}{2}\right)^2 + p_t^2};$$

whence we readily obtain σ_x and σ_y, and then α and α_1, just as before.

(*b*) *When the two given stresses are conjugate* (see § 285).—In this case the obliquities of the two stresses are the same, and the figure becomes Fig. 307.

We then obtain

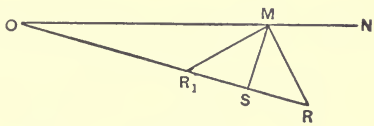


FIG. 307.

$$\frac{\sigma_x + \sigma_y}{2} = OM = \frac{OS}{\cos\theta}$$

$$= \frac{p + p_1}{2\cos\theta}; \qquad (9)$$

$$\frac{\sigma_y - \sigma_x}{2} = MR = \sqrt{(SR)^2 + (MS)^2}$$

$$= \sqrt{\left(\frac{p - p_1}{2}\right)^2 + OS^2 \tan^2 \alpha} = \sqrt{\left(\frac{p - p_1}{2}\right)^2 + \left(\frac{p + p_1}{2}\right)^2 \tan^2 \theta}$$

$$= \sqrt{\left(\frac{p + p_1}{2}\right)^2 (1 + \tan^2 \theta) - pp_1} = \sqrt{\frac{(p + p_1)^2}{4 \cos^2 \theta} - pp_1}. \qquad (10)$$

(*c*) The following proposition, due to Rankine, will next be proved:

The stress in every direction being a thrust, and the greatest obliquity being given, it is required to find the ratio of two conjugate thrusts whose common obliquity is given.

Let ϕ denote the greatest obliquity, then we shall have (see last equation of § 292)

$$\frac{\sigma_y - \sigma_x}{\sigma_y + \sigma_x} = \sin \phi. \qquad (11)$$

Now let θ denote the common obliquity of two conjugate stresses whose intensities are p and p_1. Moreover, $\theta < \phi$, and we will consider $p_1 < p$. Then from equations (9) and (10) we deduce

$$1 - \frac{4pp_1 \cos^2 \theta}{(p + p_1)^2} = \left(\frac{\sigma_y - \sigma_x}{\sigma_y + \sigma_x}\right)^2;$$

and combining this with (11) we obtain

$$\frac{(p + p_1)^2}{4pp_1} = \frac{\cos^2 \theta}{\cos^2 \phi}; \qquad (2)$$

$$\therefore \left(\frac{p - p_1}{p + p_1}\right)^2 = \frac{\cos^2\theta - \cos^2\phi}{\cos^2\theta};$$

$$\therefore \frac{p_1}{p} = \frac{\cos\theta - \sqrt{\cos^2\theta - \cos^2\phi}}{\cos\theta + \sqrt{\cos^2\theta - \cos^2\phi}},$$

which is the ratio shown.

§ 295. **Case of any Stresses in Space.** — In the case of stress which is not all parallel to one plane, we should find that it is always possible, no matter how complicated the state of stress in a body, to find three planes at right angles to each other on which the stress is wholly normal, these being the principal stresses; and a number of propositions follow analogous to those for stresses all parallel to one plane. The discussions of these cases become very complex, and will not be treated here.

§ 296. **Some Applications.** — The following are some of the practical cases which require the theory of elasticity for their solution.

§ 297. **Combined Twisting and Bending.** — This is the case very generally in shafting, as the twist is necessary for the transmission of power, and the bending is due to the weight of the pulleys and shafting, and the pull of the belts, this being especially so when there are pulleys elsewhere than close to the hangers; also in overhanging shafts, in crank-shafts, etc.

Thus far we have no tests of shafting under combined twisting and bending, and therefore the methods used for calculating such shafts vary. With many it is the practice to compute their proper size from the twisting-moment only, but to make up for the bending by using a large factor of safety, the magnitude of this factor depending upon how much the computer imagines the shaft will be weakened by the particular bending to which it is subjected.

With others it is customary to compute the deflections, under the greatest belt-pulls that can come upon it, by the principles of transverse stress, without any reference to the torsion, and to so determine it that the deflection computed in this way should not exceed $\frac{1}{1200}$ or $\frac{1}{1600}$ of the span.

On the other hand, Unwin and some others give the formulæ, which will be developed here for combined twisting and bending, as deduced by the theory of elasticity. This formula has not, as yet, been very extensively used; and its constants are taken from experiments on tension or torsion alone, and not on a combination of the two. It is to be hoped that we may some time have some experiments on such a combination. We will now proceed to deduce a formula for the greatest intensity of the stress at any point of the shaft.

For this purpose

Let M_1 = bending-moment at any section.

M_2 = twisting-moment at the same section.

I_1 = moment of inertia about neutral axis for bending.

I_2 = moment of inertia about axis of shaft.

r = distance from axis to outside fibre.

Then, if we denote by σ the greatest intensity of the stress due to bending, and by τ the greatest intensity of the stress due to twisting, we have,

$$\sigma = \frac{M_1 r}{I_1}. \quad (1) \qquad \tau = \frac{M_2 r}{I_2}. \quad (2)$$

For a circular or hollow circular shaft,

$$I_2 = 2I_1;$$

hence

$$\sigma = \frac{M_1 r}{I_1}. \quad (3) \qquad \tau = \frac{M_2 r}{2I_1}. \quad (4)$$

Then, at a point at the outside of the shaft in the section under consideration, we shall have, —

1°. On a plane normal to the axis,

(*a*) a normal stress σ,

(*b*) a shearing-stress τ.

2°. On a plane in the direction of the axis,

(*a*) a normal stress 0.

(*b*) a shearing-stress τ.

We thus have the case solved in Example I., § 290.

If, therefore, the greatest and least principal stresses be denoted by σ_1 and σ_2 respectively, we shall have

$$\sigma_1 = \frac{\sigma}{2} + \sqrt{\frac{\sigma^2}{4} + \tau^2}, \tag{5}$$

$$\sigma_2 = \frac{\sigma}{2} - \sqrt{\frac{\sigma^2}{4} + \tau^2}. \tag{6}$$

But, if ϵ_1 and ϵ_2 denote the strains in the directions of the principal stresses, we have

$$E\epsilon_1 = \sigma_1 - \frac{\sigma_2}{m}, \qquad E\epsilon_2 = \sigma_2 - \frac{\sigma_1}{m};$$

Hence, substituting for σ_1 and σ_2 their values, we have

$$E\epsilon_1 = \frac{m-1}{2m}\sigma + \frac{m+1}{2m}\sqrt{\sigma^2 + 4\tau^2}, \tag{7}$$

$$E\epsilon_2 = \frac{m-1}{2m}\sigma - \frac{m+1}{2m}\sqrt{\sigma^2 + 4\tau^2}. \tag{8}$$

We then have, for the greatest stress on any fibre, the greater of the two quantities (7) and (8); and this should not at any section of the shaft exceed the working-strength of the material for tension.

The greater of the two is $E\epsilon_1$: hence we should have, if $f =$ greatest stress,

$$\frac{m-1}{2m}\sigma + \frac{m+1}{2m}\sqrt{\sigma^2 + 4\tau^2} = f. \qquad (9)$$

If, now, we let $m = 4$, as is commonly done, we have

$$\tfrac{3}{8}\sigma + \tfrac{5}{8}\sqrt{\sigma^2 + 4\tau^2} = f, \qquad (10)$$

this being the formula given by Grashof and others for combined twisting and bending.

On the other hand, Rankine puts the value of σ_1 in (5) equal to f, and hence Rankine's formula is

$$\frac{\sigma}{2} + \frac{1}{2}\sqrt{\sigma^2 + 4\tau^2} = f. \qquad (11)$$

This might be derived from (9) by making $m = \infty$ instead of $m = 4$.

The formulæ developed above are applicable to any section.

APPLICATION TO CIRCULAR AND HOLLOW CIRCULAR SHAFTS.

Substituting for σ and τ in (10) the values from (3) and (4), we should obtain

$$\frac{r}{I_1}\left\{\frac{3}{8}M_1 + \frac{5}{8}\sqrt{M_1^2 + M_2^2}\right\} = f, \qquad (12)$$

which is Grashof's formula, and is given by Unwin and others; and, substituting in (11) instead, we should have

$$\frac{r}{2I_1}(M_1 + \sqrt{M_1^2 + M_2^2}) = f. \qquad (13)$$

Equation (12) is equivalent to the following **rule:—**

Calculate the shaft as though it were subjected to a bending-moment

$$M_0 = \tfrac{3}{8}M_1 + \tfrac{5}{8}\sqrt{M_1^2 + M_2^2};$$

and equation (13) is equivalent to the following rule:—

Calculate the shaft as though it were subjected to a bending-moment

$$M_0 = \frac{M_1}{2} + \frac{1}{2}\sqrt{M_1^2 + M_2^2}.$$

Now, if, as is usually the case, the section where the greatest bending-moment acts is also subjected to the greatest twisting-moment, it will only be necessary to put for M_1 the greatest bending-moment, and for M_2 the greatest twisting-moment.

§ 298. **Thick Hollow Cylinders subjected to a Uniform Normal Pressure.** — Let inside radius $= r$, outside radius $= r_1$, length of portion under consideration = unity, intensity of internal normal pressure $= P$, of external normal pressure $= P_1$.

1°. Divide the cylinder into a series of concentric rings; let radius of any ring be ρ, and thickness $d\rho$, these being the dimensions before the pressure is applied.

Let ρ become $\rho + \xi$, and $d\rho$, $d\rho + d\xi$, after the pressure is applied.

Then at any point of this ring we shall have, for the strain in the radial direction,

$$\frac{d\xi}{d\rho}; \tag{1}$$

and, since the length of the ring before the application of the pressure is $2\pi\rho$, and after is $2\pi(\rho + \xi)$, hence the strain in a direction at right angles to the radius is

$$\frac{2\pi\xi}{2\pi\rho} = \frac{\xi}{\rho}. \tag{2}$$

2°. Impose, now, the conditions of equilibrium upon the forces exerted by the rest of the cylinder upon the upper half-ring. For this purpose let

p = intensity of normal pressure on inside; i.e., at distance ρ from the axis.

$p + dp$ = intensity of normal pressure on outside; i.e., at distance $\rho + d\rho$ from the axis.

Then we shall have for these forces, —

(*a*) Upward force due to internal pressure,

$$2p(\rho + \xi).$$

(*b*) Downward force due to external pressure,

$$2(p + dp)(\rho + \xi + d\rho + d\xi).$$

(*c*) Upward force at right angles to radius acting at division line between the two half-rings,

$$2t(d\rho + d\xi),$$

where t = intensity of hoop-tension per square unit; i.e., of tension in a circumferential direction. Then we have

$$2(p + dp)(\rho + \xi + d\rho + d\xi) - 2p(\rho + \xi) - 2t(d\rho + d\xi) = 0;$$

and, if this be reduced, and the terms

$$2pd\xi, \quad 2\xi dp, \quad 2dpd\rho, \quad 2dpd\xi, \quad \text{and} \quad 2td\xi$$

be omitted, all of which are very small compared with the remaining ones, we shall have

$$\frac{dp}{d\rho} + \frac{p - t}{\rho} = 0. \tag{3}$$

Now, the two stresses p and t are principal stresses, since there are no shearing-stresses on these planes. Hence we have, from equations (1) and (2), § 282,

$$E\frac{d\xi}{d\rho} = p - \frac{t}{m}, \tag{4}$$

$$E\frac{\xi}{\rho} = t - \frac{p}{m}. \tag{5}$$

Now eliminate p and t between (3), (4), and (5), and obtain a differential equation between ρ and ξ.

Proceed as follows: —

From (4) and (5),

$$p = \frac{Em^2}{m^2 - 1}\left(\frac{d\xi}{d\rho} + \frac{\xi}{m\rho}\right)$$

$$\therefore \quad \frac{dp}{d\rho} = \frac{Em^2}{m^2 - 1}\left(\frac{d^2\xi}{d\rho^2} + \frac{1}{m\rho}\frac{d\xi}{d\rho} - \frac{\xi}{m\rho^2}\right).$$

From (4) and (5) also,

$$\frac{p - t}{\rho} = \frac{Em}{m + 1}\left(\frac{d\xi}{d\rho} - \frac{\xi}{\rho}\right)\frac{1}{\rho}.$$

Hence, substituting in (3), and reducing, we obtain

$$\frac{d^2\xi}{d\rho^2} + \frac{1}{\rho}\frac{d\xi}{d\rho} - \frac{\xi}{\rho^2} = 0 \tag{6}$$

$$\therefore \quad \frac{d^2\xi}{d\rho^2} = -\left(\frac{1}{\rho}\frac{d\xi}{d\rho} - \frac{\xi}{\rho^2}\right) = \frac{-d\left(\frac{\xi}{\rho}\right)}{d\rho}.$$

Hence, by integration,

$$\frac{d\xi}{d\rho} = -\frac{\xi}{\rho} + 2a; \tag{7}$$

$2a$ being an arbitrary constant, to be determined from the conditions of the problem.

From (7) we obtain

$$\rho\frac{d\xi}{d\rho} + \xi = 2a\rho \quad \text{or} \quad \frac{d(\xi\rho)}{d\rho} = 2a\rho.$$

Hence, integrating, we have

$$\xi\rho = a\rho^2 + b, \tag{8}$$

b being another arbitrary constant.

From (8) we obtain

$$\xi = a\rho + \frac{b}{\rho}, \tag{9}$$

which gives us, for the two strains,

$$\frac{d\xi}{d\rho} = a - \frac{b}{\rho^2}, \tag{10}$$

$$\frac{\xi}{\rho} = a + \frac{b}{\rho^2}, \tag{11}$$

Hence, substituting these values in (4) and (5), and solving for p and t successively, we obtain

$$p = \frac{Em}{m-1}a - \frac{Em}{m+1}\frac{b}{\rho^2}, \tag{12}$$

$$t = \frac{Em}{m-1}a + \frac{Em}{m+1}\frac{b}{\rho^2}. \tag{13}$$

Now, to determine a and b, we have the conditions, that, when $\rho = r$, $p = P$, and when $\rho = r_1$, $p = -P_1$.

Hence

$$P = \frac{Em}{m-1}a - \frac{Em}{m+1}\frac{b}{r^2}, \quad P_1 = \frac{Em}{m-1}a - \frac{Em}{m+1}\frac{b}{r_1^2},$$

$$\therefore \quad a = \frac{m-1}{Em}\frac{P_1r_1^2 - Pr^2}{r_1^2 - r^2}, \quad b = \frac{m+1}{Em}\frac{P_1 - P}{r_1^2 - r_2^2}r^2r_1^2,$$

$$\therefore \quad t = \frac{P_1r_1^2 - Pr^2}{r_1^2 - r^2} + \frac{1}{\rho^2}\frac{(P_1 - P)r^2r_1^2}{r_1^2 - r^2}. \tag{14}$$

The greatest value of t, and hence the greatest intensity of the hoop-tension, occurs when $\rho = r$; and hence we obtain

$$\text{Max}\, t = \frac{2P_1 r_1^2 - P(r_1^2 + r^2)}{r_1^2 - r^2}, \tag{15}$$

this value of t being negative when there is hoop-tension, because the signs were so chosen as to make t positive when denoting compression.

If $P_1 = 0$, i.e., if there is no external pressure, we have

$$\text{Max}\, t = -P\left(\frac{r_1^2 + r^2}{r_1^2 - r^2}\right); \tag{16}$$

and, according to Professor Rankine's method, we should determine the proper dimensions by keeping max t within the working-strength of the material.

On the other hand, if we decide that we will keep the value of $E\left(\frac{\xi}{\rho}\right)$ within the working-strength, we shall find for this, when we make $\rho = r$,

$$\text{Max}\, E\left(\frac{\xi}{\rho}\right) = \frac{[2P_1 r_1^2 - P(r_1^2 + r^2)] - \frac{1}{m}P(r_1^2 - r^2)}{(r_1^2 - r^2)}; \tag{17}$$

and, if $m = 4$,

$$\text{Max}\, E\left(\frac{\xi}{\rho}\right) = \frac{2P_1 r_1^2 - P(r_1^2 + r^2) - \frac{1}{4}P(r_1^2 - r^2)}{(r_1^2 - r^2)}. \tag{18}$$

When $P_1 = 0$,

$$\text{Max}\, E\left(\frac{\xi}{\rho}\right) = \frac{-P(\frac{5}{4}r_1^2 + \frac{3}{4}r^2)}{r_1^2 - r^2}. \tag{19}$$

Practical cases of thick, hollow cylinders subjected to a uniform normal pressure occur in hydraulic presses and in ordnance.

§ 299. **Rankine's Theory of Earthwork.**—For a mass of earth bounded above by a plane surface, either horizontal or sloping, his theory assumes the following proportions to be true, viz.:

"1°. The pressure on a plane parallel to the upper plane surface (which may be called a conjugate plane) is vertical, and proportional to the depth.

"2°. The pressure on a vertical plane is parallel to the upper plane surface, and conjugate to the vertical pressure.

"3°. The state of stress at a given depth is uniform."

If we let w denote the weight of the earth per unit of volume, x the depth of a given conjugate plane below the surface, θ the inclination of the conjugate plane, then is the intensity of the vertical pressure on the conjugate plane

$$p = wx \cos \theta. \qquad (1)$$

Moreover, if ϕ is the angle of repose of the earth, then is ϕ the angle of greatest obliquity. If now we denote by p the pressure against the vertical plane, then we have that

$$p \geqq wx \cos \theta \frac{\cos \theta - \sqrt{\cos^2 \theta - \cos^2 \phi}}{\cos \theta + \sqrt{\cos^2 \theta - \cos^2 \phi}}, \qquad (2)$$

and
$$p \leqq wx \cos \theta \frac{\cos \theta + \sqrt{\cos^2 \theta - \cos^2 \phi}}{\cos \theta - \sqrt{\cos^2 \theta - \cos^2 \phi}}; \qquad (3)$$

i.e., p may have any value between these two.

When the problem is to determine the pressure exerted by a mass of earth against the vertical face ab of a retaining-wall, we determine the intensity of the pressure p against the face (acting along cd) at a depth x below the upper surface ac by means of equation (2); inasmuch as Rankine claims to prove by means of Moseley's principle of least resistance that the pressure against the vertical plane is the lesser of the two conjugate pressures.

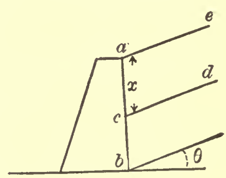


FIG. 308.

Hence, for the entire pressure against the wall we have a distributed pressure whose intensity is zero at a, and which varies uniformly as we go downwards, the direction of the pressure being parallel to the upper surface ae, and the point of application of the resultant being at a depth below a equal to $\frac{2}{3}(ab)$.

When the upper surface ae is horizontal, we have $\cos\theta = 1$; and hence (2) becomes

$$p \geq wx \frac{1 - \sin\phi}{1 + \sin\phi}, \tag{4}$$

and (3) becomes

$$p = wx \frac{1 + \sin\phi}{1 - \sin\phi}. \tag{5}$$

SUPPORTING POWER OF EARTH FOUNDATION ACCORDING TO RANKINE.

Let the surface of the ground be horizontal.

Then Rankine says that the conjugate pressure may be increased beyond the least amount by the application of some external pressure, as the weight of a building founded upon the earth; that, in this case, the conjugate pressure will be the least which is consistent with the vertical pressure due to the weight of the building, and if that conjugate pressure does not exceed the greatest conjugate pressure consistent with the weight of the earth above the same stratum on which the building rests, the mass of earth will be stable.

Moreover, the greatest horizontal pressure at the depth x, consistent with stability, is

$$p = wx \frac{1 + \sin\phi}{1 - \sin\phi}. \tag{6}$$

The greatest vertical pressure, consistent with this horizontal pressure, is

$$p' = p \frac{1 + \sin\phi}{1 - \sin\phi} = wx \left(\frac{1 + \sin\phi}{1 - \sin\phi}\right)^2; \tag{7}$$

and this is the greatest intensity of the pressure consistent with stability, of a building founded on a horizontal stratum of earth at a depth x, the angle of repose being ϕ.

§ 300. **Strength of Flat Plates.** — In this regard, the formulæ that will be deduced are those of Professor Grashof, the reasoning followed being substantially that given by him in his "Festigkeitslehre."

ROUND PLATES.

Let the curved line CA be a meridian curve of the middle layer of the plate after it is bent. Take the origin at O; let axis OZ be vertical, and axis OX horizontal, and let the axis at right angles to ZOX be $O\Phi$, so that z, x, and ϕ are the co-ordinates of any point in the middle layer of the plate.

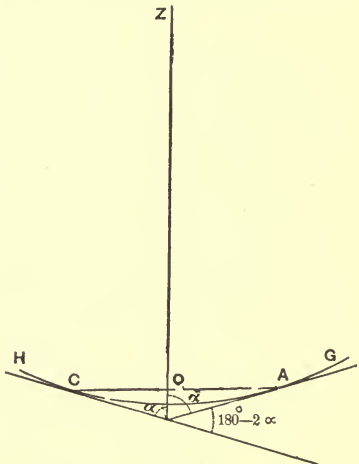


FIG. 309.

Let y denote the (vertical) distance of any horizontal layer from the middle layer of the plate.

Let R = radius of curvature of meridian line at any point (x, z).

Let R_1 = radius of curvature of section of middle layer normal to meridian line.

Then we should have, from the differential calculus,

$$\frac{1}{R} = - \frac{\frac{d^2z}{dx^2}}{\left(1 + \left(\frac{dz}{dx}\right)^2\right)^{\frac{3}{2}}} = - \frac{d^2z}{dx^2} \text{ nearly,}$$

$$\frac{1}{R_1} = - \frac{1}{x}\frac{dz}{dx}.$$

Hence, reasoning in the same way as in the common theory of beams, we should have, for the strains of the layer whose distance from neutral layer is y at point (x, z), provided there is no stress in the plane of the neutral layer,

$$\epsilon_x = \pm\frac{y}{R}, \quad \epsilon_\phi = \pm\frac{y}{R_1}.$$

When there is such a stress, let the strains due to that stress be ϵ_{x_0} and ϵ_{ϕ_0}.

Then we shall have

$$\epsilon_x = \epsilon_{x_0} - y\frac{d^2z}{dx^2}, \tag{1}$$

$$\epsilon_\phi = \epsilon_{\phi_0} - \frac{y}{x}\frac{dz}{dx}. \tag{2}$$

Hence, substituting in (1) and (2) of § 283, we have

$$\sigma_x = \frac{mE}{m^2 - 1}\left[m\epsilon_{x_0} + \epsilon_{\phi_0} - y\left(m\frac{d^2z}{dx^2} + \frac{1}{x}\frac{dz}{dx}\right)\right], \tag{3}$$

$$\sigma_\phi = \frac{mE}{m^2 - 1}\left[\epsilon_{x_0} + m\epsilon_{\phi_0} - y\left(\frac{d^2z}{dx^2} + \frac{m}{x}\frac{dz}{dx}\right)\right]. \tag{4}$$

Now let us suppose the plate to be subjected, before loading, to a uniform pull in its own plane, and normal to its circumference; and let the intensity of this pull be p_1. Then

$$\sigma_{x_0} = \sigma_{\phi_0} = p_1;$$

and hence, we have,

$$\epsilon_{x_0} = \epsilon_{\phi_0} = \frac{p_1(m - 1)}{mE}. \tag{5}$$

Therefore, substituting in (3) and (4), and reducing,

$$\sigma_x = p_1 - \frac{mEy}{m^2 - 1}\left(m\frac{d^2z}{dx^2} + \frac{1}{x}\frac{dz}{dx}\right), \tag{6}$$

$$\sigma_\phi = p_1 - \frac{mEy}{m^2 - 1}\left(\frac{d^2z}{dx^2} + \frac{m}{x}\frac{dz}{dx}\right). \tag{7}$$

These equations express the stresses in terms of the co-ordinates of the points.

Now impose the conditions of equilibrium upon the forces acting on any half-ring of thickness $dx = d\phi$. These forces are —

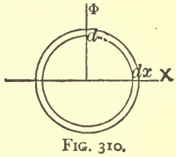


FIG. 310.

1°. Force exerted upon it by the outer part of the plate,

$$\{2x\sigma_x + 2d(x\sigma_x)\}dz.$$

2°. Force exerted by the inner part of the plate,

$$-2x\sigma_x dz.$$

3°. Force exerted upon it by the other half-ring,

$$-2\sigma_\phi dx dz.$$

4°. Force exerted by resistance to shear on top and bottom,

$$\{(\tau + d\tau) - \tau\}2x dx.$$

Hence, equating to zero the algebraic sum of these, and reducing, we obtain

$$\frac{d\tau}{dz} = \frac{d\tau}{dy} = \frac{\sigma_\phi}{x} - \frac{1}{x}\frac{d(x\sigma_x)}{dx}. \tag{8}$$

Now substitute for σ_x and σ_ϕ their values, and reduce, and we have

$$\frac{d\tau}{dy} = \frac{m^2Ey}{m^2 - 1}\left(\frac{d^3z}{dx^3} + \frac{1}{x}\frac{d^2z}{dx^2} - \frac{1}{x^2}\frac{dz}{dx}\right). \tag{9}$$

Integrate with regard to y, and we have, since the quantity in brackets is not a function of y,

$$\tau = \frac{m^2Ey^2}{2(m^2 - 1)}\left(\frac{d^3z}{dx^3} + \frac{1}{x}\frac{d^2z}{dx^2} - \frac{1}{x^2}\frac{dz}{dx}\right) + c.$$

But, when $y = \frac{h}{2}$ (h being the thickness of the plate), $\tau = 0$, since there is no shearing-force at top or bottom;

$$\therefore \quad c = -\frac{m^2Eh^2}{8(m^2-1)}\left(\frac{d^3z}{dx^3} + \frac{1}{x}\frac{d^2z}{dx^2} - \frac{1}{x^2}\frac{dz}{dx}\right)$$

$$\therefore \quad \tau = -\frac{m^2E(h^2-4y^2)}{8(m^2-1)}\left(\frac{d^3z}{dx^3} + \frac{1}{x}\frac{d^2z}{dx^2} - \frac{1}{x^2}\frac{dz}{dx}\right). \qquad (10)$$

This gives us the intensity of the shearing-stress at any point (x, z) at distance y from middle layer; and this is the intensity of the shear at that point between two horizontal layers, and hence also along a vertical plane through the point (x, z).

Now let us take the case of a centre load P combined with a distributed load p per unit of area. Then shearing-force at distance x from centre =

$$\pi x^2 p + P,$$

this tending to shear out a circular piece of radius x. Hence we must have this balanced by the whole shearing resistance on the surface subjected to shear;

$$\therefore \quad 2\pi x \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau dy = \pi x^2 p + P$$

$$\therefore \quad \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau dy = \frac{1}{2}\left(px + \frac{P}{\pi x}\right). \qquad (11)$$

Now substitute the value of τ from equation (10), integrate, and reduce, and we obtain

$$\frac{d^3z}{dx^3} + \frac{1}{x}\frac{d^2z}{dx^2} - \frac{1}{x^2}\frac{dz}{dx} = -\frac{6(m^2-1)}{m^2Eh^3}\left(px + \frac{P}{\pi x}\right). \qquad (12)$$

Hence, for the intensity of the shearing-force, we have

$$\tau = \frac{3}{4}\frac{h^2 - 4y^2}{h^3}\left(px + \frac{P}{\pi x}\right). \tag{13}$$

This gives the intensity of the shearing-force at any point of the plate.

Next, to find its deflection, or the equation of the meridian line, we have, from (12),

$$\frac{d}{dx}\left(\frac{d^2z}{dx^2} + \frac{1}{x}\frac{dz}{dx}\right) = -\frac{6(m^2 - 1)}{m^2Eh^3}\left(px + \frac{P}{\pi x}\right)$$

$$\therefore \quad \frac{d^2z}{dx^2} + \frac{1}{x}\frac{dz}{dx} = -\frac{6(m^2 - 1)}{m^2Eh^3}p\frac{x^2}{2} - \frac{6(m^2 - 1)}{m^2Eh^3}\frac{P}{\pi}\log_e x + c$$

$$\therefore \quad x\frac{d^2z}{dx^2} + \frac{dz}{dx} = -\frac{6(m^2 - 1)}{m^2Eh^3}p\frac{x^3}{2} - \frac{6(m^2 - 1)}{m^2Eh^3}\frac{P}{\pi}x\log_e x + cx.$$

But

$$x\frac{d^2z}{dx^2} + \frac{dz}{dx} = \frac{d}{dx}\left(x\frac{dz}{dx}\right);$$

hence, integrating, we have

$$x\frac{dz}{dx} = -\frac{6(m^2 - 1)}{m^2Eh^3}p\frac{x^4}{8}$$
$$- \frac{6(m^2 - 1)}{m^2Eh^3}\frac{P}{\pi}\frac{x^2}{2}\log_e x + \frac{6(m^2 - 1)}{m^2Eh^3}\frac{P}{\pi}\frac{x^2}{4} + \frac{cx^2}{2} + d. \tag{14}$$

Hence, dividing through by x, and integrating,

$$z = -\frac{6(m^2 - 1)}{m^2Eh^3}p\frac{x^4}{32}$$
$$- \frac{6(m^2 - 1)}{m^2Eh^3}\frac{P}{\pi}\frac{x^2}{4}(\log_e x - 1) + \frac{cx^2}{4} + d\log_e x + e; \tag{15}$$

and this is the meridian line of the surface, the constants c, d, and e being as yet undetermined.

This is as far as we can proceed before taking up special cases.

(*a*) *Full Plate.* — When the plate is full, the slope becomes zero, for $x = 0$; therefore (14) gives us

$$d = 0,$$

and in this case (15) becomes

$$z = -\frac{6(m^2-1)}{m^2Eh^3}p\frac{x^4}{32} - \frac{6(m^2-1)}{m^2Eh^3}\frac{P}{\pi}\frac{x^2}{4}(\log_e x - 1) + \frac{cx^2}{4} + e. \quad (16)$$

(*a*) *Uniformly Loaded, no Centre Load.* — $P = 0$.

$$\therefore \quad z = -\frac{6(m^2-1)}{m^2Eh^3}p\frac{x^4}{32} + \frac{cx^2}{4} + e. \quad (17)$$

But when $x = r$, $z = 0$;

$$\therefore \quad e = \frac{6(m^2-1)}{m^2Eh^3}p\frac{r^4}{32} - \frac{cr^2}{4}$$

$$\therefore \quad z = \left(\frac{6(m^2-1)}{m^2Eh^3}p\frac{x^2+r^2}{8} - c\right)\left(\frac{r^2-x^2}{4}\right). \quad (18)$$

And, substituting for z, $\frac{dz}{dx}$ and $\frac{d^2z}{dx^2}$, their values in (1) and (2), we obtain

$$E\epsilon_x = \frac{m-1}{m}\sigma_{x_0} + E\left(\frac{3}{8}\frac{6(m^2-1)}{m^2Eh^3}px^2 - \frac{c}{2}\right)y, \quad (19)$$

$$E\epsilon_\phi = \frac{m-1}{m}\sigma_{x_0} + E\left(\frac{1}{8}\frac{6(m^2-1)}{m^2Eh^3}px^2 - \frac{c}{2}\right)y; \quad (20)$$

and (13) gives

$$\tau = \frac{3}{4}\frac{h^2-4y^2}{h^3}px. \quad (21)$$

(β) *Supported all around.*— When $x = r$, $\sigma_x = \sigma_{x_0} = p_1$ for all values of y: therefore, from (6),

$$m\left(\frac{d^2z}{dx^2}\right)_r + \frac{1}{r}\left(\frac{dz}{dx}\right)_r = 0;$$

and, substituting the values of $\frac{dz}{dx}$ and $\frac{d^2z}{dx^2}$ as determined by differentiating (18), we have, after reducing,

$$c = \frac{3}{2}\frac{(m-1)(3m+1)}{m^2}\frac{pr^2}{Eh^3}.$$

Hence equation of meridian line is

$$z = \frac{3}{16}\frac{m^2-1}{m^2}\frac{p}{Eh^3}\left\{\frac{5m+1}{m+1}r^2 - x^2\right\}(r^2 - x^2). \quad (22)$$

Hence we have maximum deflection by making $x = 0$;

$$\therefore \quad z_0 = \frac{3}{16}\frac{(m-1)(5m+1)}{m^2}\frac{pr^4}{Eh^3}. \quad (23)$$

And, substituting in (19) and (20), we obtain, after reduction,

$$E\epsilon_x = \frac{m-1}{m}p_1 + \frac{3}{4}\frac{m^2-1}{m^2}\frac{p}{h^3}\left\{\frac{3m+1}{m+1}r^2 - 3x^2\right\}y, \quad (24)$$

$$E\epsilon_\phi = \frac{m-1}{m}p_1 + \frac{3}{4}\frac{m^2-1}{m^2}\frac{p}{h^3}\left\{\frac{3m+1}{m+1}r^2 - x^2\right\}y. \quad (25)$$

But, in a plate supported all around, $p_1 = 0$; and then the maximum value of either one occurs when $y = \frac{h}{2}$, and hence

$$E\epsilon_0 = \frac{3}{8}\frac{(m-1)(3m+1)}{m^2}\frac{pr^2}{h^2}. \quad (26)$$

On the other hand, τ becomes greatest when $x = r$ and $y = 0$. Hence

$$\text{Max}\,\tau = \frac{3}{4}\frac{r}{h}p;$$

and, if ϵ_1 represent the maximum strain due to this shearing force, we have

$$\text{Max}\,(E\epsilon_1) = \left(\frac{m+1}{m}\right)(\text{max}\,\tau) = \frac{3}{4}\frac{m+1}{m}\frac{r}{h}p. \quad (27)$$

RESULTING FORMULÆ FOR PLATE SUPPORTED ALL ROUND.

$$\text{Max}\,E\epsilon_0 = \frac{3}{8}\frac{(m-1)(3m+1)}{m^2}\frac{r^2}{h^2}p \qquad \text{or} \qquad \frac{3}{4}\frac{m+1}{m}\frac{r}{h}p,$$

whichever is greatest.

$$z_0 = \frac{3}{16}\frac{(m-1)(5m+1)}{m^2}\frac{pr^4}{Eh^3}.$$

PLATE FIXED AT ENDS.

Equation (17) applies to this case also.

Now, when $x = r$, $\dfrac{dz}{dx} = 0$;

$$\therefore \quad c = \frac{3}{2}\frac{m^2-1}{m^2Eh^3}pr^2$$

$$\therefore \quad z = \frac{3}{16}\frac{m^2-1}{m^2}\frac{p}{Eh^3}(r^2 - x^2)^2. \quad (28)$$

Hence greatest deflection is

$$z_0 = \frac{3}{16}\frac{m^2-1}{m^2}\frac{pr^4}{Eh^3},$$

and

$$E\epsilon_x = \frac{m-1}{m}p_1 + \frac{3}{4}\frac{m^2-1}{m^2}\frac{p}{h^3}(r^2 - 3x^2)y, \qquad (29)$$

$$E\epsilon_\phi = \frac{m-1}{m}p_1 + \frac{3}{4}\frac{m^2-1}{m^2}\frac{p}{h^3}(r^2 - x^2)y. \qquad (30)$$

When p_1 is positive or zero, then $E\epsilon_x$ is maximum for $x = 0$ $y = \frac{h}{2}$, and for $x = r$, $y = -\frac{h}{2}$; and $E\epsilon_\phi$ is maximum for $x = 0$, $y = \frac{h}{2}$: and the maximum value of $E\epsilon_\phi$ is equal to first maximum of $E\epsilon_x$. We have

$$\text{First max } E\epsilon_x = \frac{m-1}{m}p_1 + \frac{3}{8}\frac{m^2-1}{m^2}\frac{r^2}{h^2}p, \qquad (31)$$

$$\text{Second max } E\epsilon_x = \frac{m-1}{m}p_1 + \frac{3}{4}\frac{m^2-1}{m^2}\frac{r^2}{h^2}p. \qquad (32)$$

Hence the second is the real maximum.

RESULTING FORMULÆ FOR PLATES FIXED AT THE ENDS.

$$\text{Max } E\epsilon_0 = \frac{m-1}{m}p_1 + \frac{3}{4}\frac{m^2-1}{m^2}\frac{r^2}{h^2}p,$$

$$z_0 = \frac{3}{16}\frac{m^2-1}{m^2}\frac{pr^4}{Eh^3}.$$

For $p_1 = 0$,

$$\text{Max } E\epsilon_0 = \frac{3}{4}\frac{m^2-1}{m^2}\frac{r^2}{h^2}p.$$

§ 301. **Thickness of Plates.** — Grashof advises the use of 3 as value of m. If this be adopted, we should have, for the proper thickness of round plates,

Supported.	Fixed.
$h = r\sqrt{\frac{5p}{6f}}$	$h = r\sqrt{\frac{2p}{3f}}$

where h = thickness, r = radius, p = pressure per square inch, and f = working-strength per square inch. If, now, we use a factor of safety 8, and use as tensile strength of cast-iron 20000, of wrought-iron, 48000, and of steel 80000, we should have:—

	Supported.	Fixed.
Cast-iron . . .	$h = 0.0182570 r\sqrt{p}$	$h = 0.0163300 r\sqrt{p}$
Wrought-iron .	$h = 0.0117850 r\sqrt{p}$	$h = 0.0105410 r\sqrt{p}$
Steel	$h = 0.0091287 r\sqrt{p}$	$h = 0.0081649 r\sqrt{p}$

§ 302. **Rectangular Plates.** — Refer the plate to rectangular axes, as before, OZ, OX, $O\Phi$; the origin being at the middle of its middle layer.

Let y = distance of any point in the plate from the middle layer.

Let ρ_x be the radius of curvature of a normal section parallel to OX at the point (x, z, ϕ).

Let ρ_ϕ be the radius of curvature of a normal section parallel to $O\Phi$ at the point (x, z, ϕ).

Then we shall have, by the principles of the common theory of beams,

$$\epsilon_x = \epsilon_{x_0} \pm \frac{y}{\rho_x}, \qquad \epsilon_\phi = \epsilon_{\phi_0} \pm \frac{y}{\rho_\phi},$$

where ϵ_{x_0} and ϵ_{ϕ_0} are the strains of the middle layer in the directions OX and $O\Phi$ respectively.

Moreover, from the Differential Calculus, we have

$$\rho_x = \frac{\sqrt{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{d\phi}\right)^2}}{\frac{d^2z}{dx^2}\cos^2\lambda + 2\frac{d^2z}{dxd\phi}\cos\lambda\cos\mu + \frac{d^2z}{d\phi^2}\cos^2\mu},$$

where λ = angle between normal and z axis, and μ = angle between normal and x axis. But $\frac{dz}{dx}$ and $\frac{dz}{d\phi}$ being the slopes, and hence small, we shall have nearly

$$\cos\lambda = 1, \qquad \cos\mu = 0,$$

$$\frac{1}{\rho_x} = \mp\frac{d^2z}{dx^2}, \qquad \frac{1}{\rho_\phi} = \mp\frac{d^2z}{d\phi^2}$$

$$\therefore \quad \epsilon_x = \epsilon_{x_0} - y\frac{d^2z}{dx^2}, \tag{1}$$

$$\epsilon_\phi = \epsilon_{\phi_0} - y\frac{d^2z}{d\phi^2}. \tag{2}$$

Hence (1) and (2) of § 283 give us

$$\sigma_x = \frac{mE}{m^2-1}(m\epsilon_{x_0} + \epsilon_{\phi_0}) - y\frac{mE}{m^2-1}\left\{m\frac{d^2z}{dx^2} + \frac{d^2z}{d\phi^2}\right\},$$

$$\sigma_\phi = \frac{mE}{m^2-1}(\epsilon_{x_0} + m\epsilon_{\phi_0}) - y\frac{mE}{m^2-1}\left\{\frac{d^2z}{dx^2} + m\frac{d^2z}{d\phi^2}\right\}.$$

And, if σ_{x_0}, σ_{ϕ_0}, denote the stresses in the middle layer, we shall have, since

$$\sigma_{x_0} = \frac{mE}{m^2-1}(m\epsilon_{x_0} + \epsilon_{\phi_0}), \qquad \sigma_{\phi_0} = \frac{mE}{m^2-1}(\epsilon_{x_0} + m\epsilon_{y_0}),$$

$$\sigma_x = \sigma_{x_0} - \frac{m^2E}{m^2-1}y\left\{\frac{d^2z}{dx^2} + \frac{1}{m}\frac{d^2z}{d\phi^2}\right\}, \tag{3}$$

$$\sigma_\phi = \sigma_{\phi_0} - \frac{m^2E}{m^2-1}y\left\{\frac{1}{m}\frac{d^2z}{dx^2} + \frac{d^2z}{d\phi^2}\right\}. \tag{4}$$

Now, if ξ and η denote the increments in x and ϕ respectively due to the load, we shall have

$$\xi = \int_0^x \epsilon_x dx = x\epsilon_{x_0} - y\frac{dz}{dx} \qquad \therefore \quad \frac{d\xi}{d\phi} = -y\frac{d^2z}{dxd\phi},$$

$$\eta = \int_0^\phi \epsilon_\phi d\phi = \phi\epsilon_{\phi_0} - y\frac{dz}{d\phi} \qquad \therefore \quad \frac{d\eta}{dx} = -y\frac{d^2z}{dxd\phi}.$$

But

$$\tau_{x\phi} = G\gamma_{x\phi} = G\left(\frac{d\xi}{d\phi} + \frac{d\eta}{dx}\right);$$

hence

$$\tau_{x\phi} = -2Gy\frac{d^2z}{dxd\phi}. \tag{5}$$

Equations (3), (4), and (5) are the expressions giving the stresses on two planes at right angles to each other, parallel to OX and $O\Phi$ respectively. Hence we have a case of stress on two planes at right angles to each other, and we are to find the principal stresses: we thus have —

1°. Normal stress on x plane, σ_x.

2°. Shearing-stress on x plane, $\tau_{x\phi}$.

3°. Normal stress on ϕ plane, σ_ϕ.

4°. Shearing-stress on ϕ plane, $\tau_{\phi x}$.

Hence, if we denote by σ_1 and σ_2 the maximum and minimum principal stress, we have (§ 290)

$$\sigma_1 = \tfrac{1}{2}(\sigma_x + \sigma_\phi) + \tfrac{1}{2}\sqrt{(\sigma_x - \sigma_\phi)^2 + 4\tau_{x\phi}^2}, \tag{6}$$

$$\sigma_2 = \tfrac{1}{2}(\sigma_x + \sigma_\phi) - \tfrac{1}{2}\sqrt{(\sigma_x - \sigma_\phi)^2 + 4\tau_{x\phi}^2}; \tag{7}$$

and hence, if ϵ_1 and ϵ_2 denote the strains in the directions of the principal stresses,

$$E\epsilon_1 = \sigma_1 - \frac{\sigma_2}{m} = \frac{m-1}{2m}(\sigma_x + \sigma_\phi) + \frac{m+1}{2m}\sqrt{(\sigma_x - \sigma_\phi)^2 + 4\tau_{x\phi}^2}, \tag{8}$$

$$E\epsilon_2 = \sigma_2 - \frac{\sigma_1}{m} = \frac{m-1}{2m}(\sigma_x + \sigma_\phi) - \frac{m+1}{2m}\sqrt{(\sigma_x - \sigma_\phi)^2 + 4\tau_{x\phi}^2}; \tag{9}$$

and for the strain ϵ_3, parallel to OZ, we have

$$E\epsilon_3 = -\frac{\sigma_x + \sigma_\phi}{m}. \qquad (10)$$

In order to use (8), (9), and (10), however, we must know σ_x, σ_ϕ, and $\tau_{x\phi}$; and for this purpose we must know the equation of the middle layer after bending. For this purpose, apply the equations (1), (2), (3), of § 281 to any particle $dxd\phi dz$ in the interior of the body. We have then, $X = Y = Z = 0$. Therefore

$$\frac{d\sigma_x}{dx} + \frac{d\tau_{xz}}{dy} + \frac{d\tau_{x\phi}}{d\phi} = 0 \qquad \therefore \quad \frac{d\tau_{xz}}{dy} = -\left(\frac{d\sigma_x}{dx} + \frac{d\tau_{x\phi}}{d\phi}\right),$$

$$\frac{d\sigma_\phi}{d\phi} + \frac{d\tau_{x\phi}}{dx} + \frac{d\tau_{\phi z}}{dy} = 0 \qquad \therefore \quad \frac{d\tau_{\phi z}}{dy} = -\left(\frac{d\sigma_\phi}{d\phi} + \frac{d\tau_{x\phi}}{dx}\right),$$

$$\frac{d\sigma_z}{dy} + \frac{d\tau_{\phi z}}{d\phi} + \frac{d\tau_{xz}}{dx} = 0.$$

Therefore, making use of (3), (4), and (5) with the above conditions, we deduce

$$\frac{d\sigma_x}{dx} = -\frac{m^2 Ey}{m^2 - 1}\left(\frac{d^3z}{dx^3} + \frac{1}{m} \cdot \frac{d^3z}{dxd\phi^2}\right), \qquad (11)$$

$$\frac{d\sigma_\phi}{d\phi} = \frac{-m^2 Ey}{m^2 - 1}\left(\frac{d^3z}{d\phi^3} + \frac{1}{m}\frac{d^3z}{dx^2d\phi}\right), \qquad (12)$$

$$\frac{d\tau_{x\phi}}{dx} = -\frac{mEy}{m + 1}\frac{d^3z}{dx^2d\phi}, \qquad (13)$$

$$\frac{d\tau_{x\phi}}{d\phi} = -\frac{mEy}{m + 1}\frac{d^3z}{dxd\phi^2}, \qquad (14)$$

$$\frac{d\tau_{xz}}{dy} = \frac{m^2 Ey}{m^2 - 1}\left(\frac{d^3z}{dx^3} + \frac{d^3z}{dxd\phi^2}\right), \qquad (15)$$

$$\frac{d\tau_{\phi z}}{dy} = \frac{m^2 Ey}{m^2 - 1}\cdot\left(\frac{dz^3}{d\phi^3} + \frac{d^3z}{dx^2d\phi}\right). \qquad (16)$$

Hence, by integrating (15) and (16), we have

$$\tau_{xz} = \frac{m^2Ev^2}{2(m^2-1)}\left(\frac{d^3z}{dx^3} + \frac{d^3z}{dxd\phi^2}\right) + c_1,$$

$$\tau_{\phi z} = \frac{m^2Ev^2}{2(m^2-1)}\left(\frac{d^3z}{d\phi^3} + \frac{d^3z}{dx^2d\phi}\right) + c_2.$$

But when $v = \frac{h}{2}$, $\tau_{\phi z} = \tau_{xz} = 0$;

$$\therefore \quad c_1 = -\frac{m^2Eh^2}{8(m^2-1)}\left(\frac{d^3z}{dx^3} + \frac{d^3z}{dxd\phi^2}\right)$$

and

$$c_2 = -\frac{m^2Eh^2}{8(m^2-1)}\left(\frac{d^3z}{d\phi^3} + \frac{d^3z}{dx^2d\phi}\right)$$

$$\therefore \quad \tau_{\phi z} = \frac{m^2E}{m^2-1}\left(\frac{d^3z}{dx^2d\phi} + \frac{d^3z}{d\phi^3}\right)\left(\frac{v^2}{2} - \frac{h^2}{8}\right)$$

and

$$\tau_{xz} = \frac{m^2E}{m^2-1}\left(\frac{d^3z}{dx^3} + \frac{d^3z}{dxd\phi^2}\right)\left(\frac{v^2}{2} - \frac{h^2}{8}\right).$$

Hence

$$\frac{d\tau_{\phi z}}{d\phi} = \frac{m^2E}{m^2-1}\left(\frac{d^4z}{dx^2d\phi^2} + \frac{d^4z}{d\phi^4}\right)\left(\frac{v^2}{2} - \frac{h^2}{8}\right),$$

$$\frac{d\tau_{xz}}{dx} = \frac{m^2E}{m^2-1}\left(\frac{d^4z}{dx^4} + \frac{d^4z}{dx^2d\phi^2}\right)\left(\frac{v^2}{2} - \frac{h^2}{8}\right).$$

Now we have $\sigma_z = p$, where p is the intensity of the load; therefore the third equation gives us, on integrating between the limits $\frac{h}{2}$ and $-\frac{h}{2}$,

$$\sigma_z + \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{d\tau_{\phi z}}{d\phi}dy + \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{d\tau_{xz}}{dx}dy = 0$$

$$\therefore \quad p + \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{d\tau_{\phi z}}{d\phi}dy + \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{d\tau_{xz}}{dx}dy = 0$$

$$\therefore\quad p + \frac{m^2E}{m^2 - 1}\left\{\left(\frac{v^3}{6} - \frac{h^2v}{8}\right)\left(\frac{d^4z}{dx^4} + 2\frac{d^4z}{dx^2d\phi^2} + \frac{d^4z}{d\phi^4}\right)_{-\frac{h}{2}}^{\frac{h}{2}}\right\} = 0$$

$$\therefore\quad p + \frac{m^2E}{m^2 - 1}\left(\frac{d^4z}{dx^4} + 2\frac{d^4z}{dx^2d\phi^2} + \frac{d^4z}{d\phi^4}\right)\left(\frac{h^3}{24} - \frac{h^3}{8}\right) = 0$$

$$\therefore\quad \frac{d^4z}{dx^4} + 2\frac{d^4z}{dx^2d\phi^2} + \frac{d^4z}{d\phi^4} = \frac{12(m^2 - 1)p}{m^2Eh^3}; \qquad (17)$$

and this is the differential equation of the surface, and should be integrated in each special case.

INDEFINITE PLATES WHICH ARE FIRMLY HELD AT A SYSTEM OF POINTS DIVIDING THEM INTO RECTANGULAR PANELS.

Let the sides of the panels be $2a$ and $2b$. Assume the origin at the middle of the panel, the axis of x being parallel to $2a$, and the axis of y parallel to $2b$. We shall in this case have the following conditions; viz., —

(*a*) $\dfrac{dz}{dx} = 0$ for $x = \pm a$ and all values of ϕ.

(*b*) $\dfrac{dz}{d\phi} = 0$ for $\phi = \pm b$ and all values of x.

(*c*) $z \;= 0$ when $x = \pm a$, $\phi = \pm b$.

(*d*) If we develop the value of z in powers of x and ϕ, there must enter only even powers of x and ϕ, since the value of z remains the same when we put $-x$ for x, or $-\phi$ for ϕ.

Now, if we write

$$z = A + Bx^2 + C\phi^2 + Dx^2\phi^2 + Ex^4 + F\phi^4 + Gx^6 + Hx^4\phi^2 + Kx^2\phi^4 + L\phi^6 + Mx^8, \text{ etc.,}$$

the above conditions will be fulfilled:—

1°. By making all the co-efficients after the fourth, each zero.

2°. By making $D = 0$, therefore writing

$$z = A + Bx^2 + C\phi^2 + Ex^4 + F\phi^4.$$

Now

$$\frac{dz}{dx} = 2Bx + 4Ex^3, \qquad \frac{dz}{d\phi} = 2C\phi + 4F\phi^3,$$

$$\therefore \quad 2Ba + 4Ea^3 = 0, \qquad 2Cb + 4Fb^3 = 0,$$

and

$$0 = A + Ba^2 + Cb^2 + Ea^4 + Fb^4$$

$$\therefore \quad B = -2Ea^2, \qquad C = -2Fb^2,$$

$$\therefore \quad A = 2Ea^4 + 2Fb^4 - Ea^4 - Fb^4 = Ea^4 + Fb^4.$$

Hence the equation becomes

$$z = Ea^4 + Fb^4 - 2Ea^2x^2 - 2Fb^2\phi^2 + Ex^4 + F\phi^4$$
$$= E(a^2 - x^2)^2 + F(b^2 - \phi^2)^2$$

$$\therefore \quad \frac{dz}{dx} = -4Ex(a^2 - x^2) = 4Ex^3 - 4Ea^2x$$

$$\therefore \quad \frac{d^2z}{dx^2} = 12Ex^2 - 4Ea^2,$$

also

$$\frac{dz}{d\phi} = -4F\phi(b^2 - \phi^2) = 4F\phi^3 - 4Fb^2\phi$$

$$\therefore \quad \frac{d^2z}{d\phi^2} = 12F\phi^2 - 4Fb^2,$$

$$\frac{d^3z}{dx^2d\phi} = 0 \quad \therefore \quad \frac{d^4z}{dx^2d\phi^2} = 0, \qquad \frac{d^3z}{dxd\phi^2} = 0, \qquad \frac{d^4z}{dx^2d\phi^2} = 0,$$

$$\frac{d^3z}{dx^3} = 24Ex, \qquad \frac{d^3z}{d\phi^3} = 24F\phi, \qquad \frac{d^4z}{dx^4} = 24E, \qquad \frac{d^4z}{d\phi^4} = 24F$$

$$\therefore \quad 24(E + F) = \frac{12(m^2 - 1)p}{m^2Eh^3} \qquad \therefore \quad E + F = \frac{(m^2 - 1)p}{2m^2Eh^3}.$$

Hence equation of the middle layer is

$$z = E(a^2 - x^2)^2 + F(b^2 - \phi^2)^2, \text{ where } E + F = \frac{(m^2 - 1)p}{2m^2Eh^3}. \qquad (18)$$

Now, in the case of an ordinary beam fixed at both ends, and loaded uniformly with p lbs. per unit of area, if b is the breadth, we have:—

1°. The points of inflection are at a distance from the middle equal to $\frac{a}{\sqrt{3}}$, where a is the half-span; and

2°. The bending-moment at a section at a distance x from the middle is $\frac{pb}{2}\left(\frac{a^2}{3} - x^2\right)$ when $x < \frac{a}{\sqrt{3}}$, and $\frac{pb}{2}\left(x^2 - \frac{a^2}{3}\right)$ when $x > \frac{a}{\sqrt{3}}$; therefore the value of z is found from the formula

$$z = \frac{6p}{Eh^3}\int_x^a\int_x^a\left(x^2 - \frac{a^2}{3}\right)dx^2 \text{ when } \times > \frac{a}{\sqrt{3}};$$

or

$$z = \frac{6p}{Eh^3}\int_{\frac{a}{\sqrt{3}}}^a\int_x^a\left(x^2 - \frac{a^2}{3}\right)dx^2 + \frac{6p}{Eh^3}\int_0^x\int_0^x\left(\frac{a^2}{3} - x^2\right)dx^2 \text{ when } \times < \frac{a}{\sqrt{3}}.$$

Either one, when integrated, gives for z the value

$$z = \frac{p}{2Eh^3}(a^2 - x^2)^2.$$

Hence in the flat plate, if $b = 0$, the values of E and F must be such that the formula shall reduce to $z = \frac{p}{2Eh^3}(a^2 - x^2)^2$ when $b = 0$. Now, it does reduce to $z = E(a^2 - x^2)^2$. Therefore E must be such a function of a and b, that, when $b = 0$, it shall reduce to $\frac{p}{2Eh^3}$. So likewise F must be such a function of a and b, that, when $a = 0$, it shall reduce to $\frac{p}{2Eh^3}$. Suppose, then, we put

$$E = \frac{p}{2Eh^3} + b^nc \qquad \text{and} \qquad F = \frac{p}{2Eh^3} + a^nc,$$

since these functions fulfil the above conditions.

Now we have

$$E + F = \frac{(m^2 - 1)p}{2m^2Eh^3}$$

$$\therefore \qquad \frac{p}{Eh^3} + c(a^n + b^n) = \frac{(m^2 - 1)p}{2m^2Eh^3}$$

$$\therefore \quad (a^n + b^n)c = -\frac{(m^2 + 1)p}{2m^2Eh^3}$$

$$\therefore \qquad c = -\frac{(m^2 + 1)p}{2m^2Eh^3(a^n + b^n)}$$

$$E\epsilon_x = \sigma_{x_0} - \frac{1}{m}\sigma_{\phi_0} + yE\frac{d^2z}{dx^2},$$

$$E\epsilon_\phi = \sigma_{\phi_0} - \frac{1}{m}\sigma_{x_0} - yE\frac{d^2z}{d\phi^2}.$$

Hence, substituting for $\frac{d^2z}{dx^2}$ and $\frac{d^2z}{d\phi^2}$ their values, and observing that

$$\epsilon_x \text{ is greatest for } x = \pm a, \ y = \pm\frac{h}{2},$$

$$\epsilon_\phi \text{ is greatest for } \phi = \pm b, \ y = \pm\frac{h}{2},$$

we obtain

$$\max(E\epsilon_x) = \sigma_{x_0} - \frac{1}{m}\sigma_{\phi_0} \pm 2\frac{a^n - \frac{1}{m^2}b^n}{a^n + b^n}\frac{a^2}{h^2}p, \qquad (19)$$

$$\max(E\epsilon_\phi) = \sigma_{\phi_0} - \frac{1}{m}\sigma_{x_0} \pm 2\frac{b^n - \frac{1}{m^2}a^n}{a^n + b^n}\frac{b^2}{h^2}p. \qquad (20)$$

These may be written as follows:

$$\max E\epsilon_x = \sigma_{x_0} - \frac{1}{m}\sigma_{\phi_0} \pm 2\,\frac{1 - \dfrac{1}{m^2}\left(\dfrac{b}{a}\right)^n}{1 + \left(\dfrac{b}{a}\right)^n}\,\frac{a^2}{h^2}p, \qquad (21)$$

$$\max E\epsilon_\phi = \sigma_{\phi_0} - \frac{1}{m}\sigma_{x_0} \pm 2\,\frac{1 - \dfrac{1}{m^2}\left(\dfrac{a}{b}\right)^n}{1 + \left(\dfrac{a}{b}\right)^n}\,\frac{b^2}{h^2}p. \qquad (22)$$

We have also, by substituting for E and F their values in equation (18),

$$z = \frac{p}{2Eh^3}\left\{\frac{a^n - \dfrac{1}{m^2}b^n}{a^n + b^n}(a^2 - x^2)^2 + \frac{b^n - \dfrac{1}{m^2}a^n}{a^n + b^n}(b^2 - \phi^2)^2\right\}. \qquad (23)$$

In these results the exponent n is undetermined, and we have no means of determining it in the general case. We only know, that, since the deflection must increase for a decrease in x and ϕ, therefore we must have, whenever $a > b$,

$$\left(\frac{a}{b}\right)^n < m^2 \qquad \therefore \quad n < \frac{2\log m}{\log\left(\dfrac{a}{b}\right)}.$$

This leaves the general case indeterminate; but a common practical case is not subject to this indetermination, i.e., the case when $a = b$, for then

$$\left(\frac{a}{b}\right)^n = \left(\frac{b}{a}\right)^n = 1,$$

whatever the value of n; and hence equations (21), (22), and (23) give

$$\max (F\epsilon_x) = \sigma_{x_0} - \frac{1}{m}\sigma_{\phi_0} \pm \frac{m^2 - 1}{m^2}\frac{a^2}{h^2}p, \qquad (24)$$

$$\max E\epsilon_\phi = \sigma_{\phi_0} - \frac{1}{m}\sigma_{x_0} \pm \frac{m^2 - 1}{m^2}\frac{a^2}{h^2}p, \qquad (25)$$

$$z = \frac{m^2 - 1}{m^2} \frac{p}{4Eh^3} \{ (a^2 - x^2)^2 + (a^2 - \phi^2)^2 \}, \qquad (26)$$

and

$$\max z = \frac{m^2 - 1}{m^2} \frac{a^4}{2Eh^3} p. \qquad (27)$$

FORMULÆ FOR THE SHEETS OF A LOCOMOTIVE FIRE-BOX.

In this case we have $a = b$; hence (24), (25), and (27) apply: and if we write, with Grashof, $m = 3$, they become

$$\max (E\epsilon_x) = \sigma_{x_0} - \frac{1}{3}\sigma_{\phi_0} + \frac{8}{9}\frac{a^2}{h^2}p, \qquad (28)$$

$$\max (E\epsilon_\phi) = \sigma_{\phi_0} - \frac{1}{3}\sigma_{x_0} + \frac{8}{9}\frac{a^2}{h^2}p, \qquad (39)$$

$$\max (z) = \frac{4}{9}\frac{pa^4}{Eh^3}. \qquad (30)$$

Now, in the case of the horizontal sheets, $\sigma_{x_0} = \sigma_{\phi_0} = 0$, and we have

$$\max (E\epsilon_x) = \frac{8}{9}\frac{a^2}{h^2}p, \qquad (31)$$

$$\max (z) = \frac{4}{9}\frac{pa^4}{Eh^3}. \qquad (32)$$

In the case of the vertical walls, inasmuch as these have to resist the steam-pressure in a vertical direction, the inner one is called upon to bear compression, and the outer tension, in a vertical direction. If l is the length of the outside of the fire-box, and l_1 its breadth, we shall have for the outer plate, taking axis of x vertical,

$$\sigma_x = \frac{ll_1 p}{2(l + l_1)h}, \qquad \sigma_{\phi_0} = 0;$$

and for the inner plate, if l and l_1' are corresponding dimensions of inside of fire-box,

$$\sigma_{x_0} = \frac{ll_1 p}{2(l' + l_1')h'}, \qquad \sigma_{\phi_0} = 0.$$

And, by making these substitutions in (28), (29), and (30), we obtain our formulæ.

RECTANGULAR PLATE FIXED AT THE EDGES.

For this case Grashof deduces the equation of the middle layer as follows:

1°. This equation must be a function of x and ϕ.

2°. If $2a$ and $2b$ are the sides of the plate, this function must become

(α) When $b = \infty$ for all values of ϕ,

$$z = \frac{p}{2Eh^3}(a^2 - x^2)^2.$$

(β) When $a = \infty$ for all values of x,

$$z = \frac{p}{2Eh^3}(b^2 - \phi^2)^2,$$

because the plate then becomes a beam fixed at the ends.

The function that will satisfy these two conditions is

$$z = \frac{p}{2Eh^3} \frac{(a^2 - x^2)^2(b^2 - \phi^2)^2}{a^4 + b^4}. \tag{1}$$

From this he deduces for max z, when $x = \phi = 0$,

$$\max z = \frac{1}{2} \frac{p}{Eh^3} \frac{a^4 b^4}{a^4 + b^4}. \tag{2}$$

From (1) he deduces

$$\frac{d^2z}{dx^2} = -\frac{2p}{Eh^3}\frac{(a^2 - 3x^2)(b^2 - \phi^2)^2}{a^4 + b^4}, \tag{3}$$

$$\frac{d^2z}{d\phi^2} = -\frac{2p}{Eh^3}\frac{(a^2 - x^2)^2(b^2 - 3\phi^2)}{a^4 + b^4}, \tag{4}$$

$$\frac{d^2z}{dxd\phi} = \frac{8p}{Eh^3}\frac{(a^2 - x^2)x(b^2 - \phi^2)\phi}{a^4 + b^4}, \tag{5}$$

$$\max\frac{d^2z}{dx^2} = \frac{4p}{Eh^3}\frac{a^2b^4}{a^4 + b^4}, \quad \text{for} \quad x = \pm a, \quad \phi = 0, \tag{6}$$

$$\max\frac{d^2z}{d\phi^2} = \frac{4p}{Eh^3}\frac{a^4b^2}{a^4 + b^4}, \quad \text{for} \quad \phi = \pm b, \quad x = 0, \tag{7}$$

$$\max\frac{d^2z}{dxd\phi} = \frac{32}{27}\frac{p}{Eh^3}\frac{a^3b^3}{a^4 + b^4}, \quad \text{for} \quad x^2 = \frac{1}{3}a^2, \quad \phi^2 = \frac{1}{3}b^2, \tag{8}$$

these corresponding to the points of inflection of a loaded beam fixed at the ends.

Hence (1), (2), and (5) of § 300 give

$$\max E\epsilon_x = \sigma_{x_0} - \frac{1}{m}\sigma_{\phi_0} \pm \frac{2b^4}{a^4 + b^4}\frac{a^2}{h^2}p, \tag{9}$$

$$\max(E\epsilon_\phi) = \sigma_{\phi_0} - \frac{1}{m}\sigma_{x_0} \pm \frac{2a^4}{a^4 + b^4}\frac{b^2}{h^2}p, \tag{10}$$

$$\max(\tau_z) = \frac{8}{27}\frac{m}{m + 1}\frac{2a^2b^2}{a^4 + b^4}\frac{ab}{h^2}p. \tag{11}$$

At the places where ϵ_x and ϵ_ϕ are greatest,

$$\tau_z = 0.$$

At the place where τ_z is greatest,

$$\sigma_x = \sigma_{x_0}, \qquad \sigma_\eta = \sigma_{\eta_0}.$$

Hence it is either (9) or (10) that gives us the suitable formula to use in any special case.

EXAMPLES OF THEORY OF ELASTICITY.

1. It has been sometimes proposed to use oblique seams in a boiler-shell. Assume the seams at an angle of 45° with the axis of the boiler, a pressure of 100 lbs. per square inch of the steam, and a diameter of 4 feet. Find the tension per inch of length of seam, and its direction.

2. Given a shaft carrying 80 *HP*, and running at 250 revolutions per minute. Suppose the driving-pulley to be at the middle of the length, this being 6 feet, and given that the ratio of the tension on the tight side of the belt to that on the loose side is 3.75. Find the proper size of shaft, assuming 10000 lbs. per square inch as the working-strength of the iron.

3. What should be the thickness of a flat plate to bear 150 lbs. pressure per square inch, and stayed at points forming squares 8 inches on a side, the plate being of wrought-iron, working-strength 10000 lbs. per square inch.

4. Find inner radius of a hydraulic press to bear 1500 lbs. per square inch, given outer radius = 18 inches; material, cast-iron; tensile strength 20000 lbs. per square inch.

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